1 The derivative as the slope of the tangent line

In classical geometry, the tangent to a curve was the line that somehow “touched the curve without crossing it.” Euclid attempted to make this precise by describing the tangent as the line that intersected the curve in only one point. His definition works quite well for circles (and also ellipses, parabolas, and hyperbolas):

However, it can fail rather drastically for more complicated curves. In the curve below, the almost-vertical line is the one that intersects the curve in only one point, while the almost-horizontal line clearly “ought” to be the tangent line. (Intuitively, the almost-vertical line crosses the curve, while the almost-horizontal line does not—at least, not at the point in question.)
For another example, in the following picture, neither the vertical nor the horizontal line really “touches the curve without crossing it.” Each of them intersects the curve exactly once. But if one of them is the tangent line, it is the horizontal line rather than the vertical line.

Thus, we take another approach to defining what exactly the tangent line should be. An easier definition is to define a secant line—that is, a line that passes through two specified points on a curve. This is easy to specify, since two points determine a line. We want to think of a tangent line as a “secant line that passes through the same point twice.” Unfortunately, this does not actually make any sense.

To remedy the situation, we consider another way of specifying a line: a point \((x_0, y_0)\) together with a slope \(\Delta y/\Delta x\). Thus, the secant line through \((x_0, y_0)\) and \((x, y)\) is the line passing through \((x_0, y_0)\) with slope equal to

\[
\frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}.
\]

If we want to take the tangent line at \((x_0, y_0)\), we already have a point through which the line should pass. We just need to know what its slope ought to be. This is essentially the same problem we were faced with last lecture—we need a definition for “slope at a point,” in spite of the fact that slope is, inherently, a property relating two different points. And we solve it the same way: we take a limit. We say that the slope of the tangent line is

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]
The picture below shows the tangent line as a limit of secant lines:

Now, if you recall the previous definition of the derivative, you will see that, if $y$ is given as $y = f(x)$ for some function $f$, then in fact, we will have the slope of the tangent line equal to the derivative:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left. \frac{dy}{dx} \right|_{x=x_0} = f'(x_0).$$

This gives us the following

**Definition.** Let $f$ be a function defined at $x_0$. The **tangent line** to $f$ at $x_0$ is the line passing through the point $(x_0, f(x_0))$ and having slope equal to $f'(x_0)$, provided that this derivative exists.

Let’s do an example.

**Example.** Let the function $f$ be defined by

$$f(x) = x^2.$$  

Compute the derivative of $f$ at $x_0 = 1$. Plot the function and the line tangent to $f$ at $x_0$.

**Solution.** First, let’s solve for $\Delta y$ in terms of $\Delta x$:

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$
$$= (1 + \Delta x)^2 - 1^2$$
$$= 1 + 2\Delta x + (\Delta x)^2 - 1$$
$$= 2\Delta x + (\Delta x)^2.$$
Thus, we have

\[ f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2 + \Delta x = 2. \]

Now, we plot the function \( y = f(x) \), together with line passing through \((x_0, f(x_0)) = (1, 1)\) and having slope \( f'(x_0) = 2 \):

\[ y \]

\[ x \]

\[ (x_0, f(x_0)) \]

\[ \square \]

2 **Infinitesimals**

The idea of infinitesimals, as it relates to slopes of tangent lines, is to define the tangent line to \( f \) at \( x_0 \) as the line through \((x_0, y_0)\) and another point \((x_0 + dx, y_0 + dy)\) that is “infinitely close” to the first point. What this means, in this example, is that \( dx \) is “so small” that \( dx^2 = 0 \), even though \( dx \) is not zero.
This sort of makes sense, in that the square of a small number is a much smaller number; for instance,

\[ 0.001^2 = 0.000001 \]

It does not really make sense—no nonzero number can square to zero—but that’s why I called this “walking on clouds.”

To start with, we treat the “infinitesimal changes” \( dx \) and \( dy \) exactly as though they were more conventional changes \( \Delta x \) and \( \Delta y \). Our earlier computation of \( \Delta y \) in terms of \( \Delta x \) still holds:

\[
\begin{align*}
\Delta y &= 2\Delta x + \Delta x^2 \\
\Delta y &= 2\Delta x \\
\frac{dy}{dx} &= 2 
\end{align*}
\]

since \( \Delta x^2 = 0 \)

when evaluated at the point \( x_0 = 1 \).
Assignment 13 (due Wednesday, 2 November)

NOTE: From this assignment on, you no longer need to write anything about “By the Main Limit Theorem,...” when showing your work to take a limit. (You should, however, continue to show your work.)

Use the Intermediate Value Theorem to prove that, no matter what Diophantus\(^1\) of Alexandria might have thought, \(\sqrt{2}\) does, in fact, exist. (In other words, there exists a positive real number \(x_0\) such that \(x_0^2 = 2\).) This problem will be graded carefully.

Section 2.2, Problems 45-48 and 51, 52. Be sure to follow the instructions carefully on 51 and 52; these problems are as much about how you find the derivative, as what answer you get. Problems 46, 48, and 52 will be graded carefully.

Let \(a \neq 0\) be a real number (which you don’t get to choose). Let \(f\) be the function defined by

\[
f(x) = ax.
\]

Show that \(f\) is continuous using an \(\varepsilon-\delta\) proof. This problem will be graded carefully.

Assignment 14 (due Friday, 4 November)

Section 2.2, Problems 37–44. The even-numbered problems will be graded carefully.

For each of Problems 1–4 in Section 2.2, do the following steps:

(a) Find the indicated derivative using infinitesimals.

(b) Find the indicated derivative using the limit definition.

(c) Graph the function together with the tangent line at the indicated point.

Problems 2 and 4 will be graded carefully.

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\(^1\)See Lecture 3, page 2.