1 Zeno’s arrow paradox

Wikipedia has a nice summary of Zeno’s arrow paradox:

   In the arrow paradox (also known as the fletcher’s paradox), Zeno states that for motion to occur, an object must change the position which it occupies. He gives an example of an arrow in flight. He states that in any one (durationless) instant of time, the arrow is neither moving to where it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every instant of time there is no motion occurring. If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

This more or less captures the central conceptual idea in differential calculus. When we have an object in motion, we’d like to be able to talk about how fast it is going at any given instant. But the essence of motion is moving from one position to another, whereas in a single, durationless instant, an object only occupies a single position. So how can we even think about the speed at a particular instant—or, to use slightly fancier terminology, the “instantaneous velocity”?  

   There are basically two ways to think about this. The more mathematically rigorous way is to use limits. The idea here is to say “since we can’t make the change in time zero, let’s make it arbitrarily small.” Since we can’t touch the (here) Zero beast, let’s handle it through the saddle of the Arbitrary.

   The other way—the “walking on clouds” approach that was used for the first two centuries or so after calculus was invented—is to say, in essence, “Let’s pretend that the instant at time $t_0$ actually does have an ‘infinitesimal’ duration, which we call $dt$, and see what happens.” This “infinitesimal” duration, $dt$, is bigger than zero, but smaller than any positive real number. The philosopher Berkeley called such infinitesimals “ghosts of recently departed quantities.”

   The textbook is of the opinion that the first, rigorous, approach is the only way to go. Personally, I find the second approach extremely useful, even if it is just “walking on clouds.” I also think you need to see it, since if you should
need calculus in applied science (physics, chemistry, atmospheric chemistry,...) this is most likely the language you will see. But I’m honestly not sure which approach is less confusing to see first, so I’m going to accept the following wisdom: When in doubt, follow the textbook. More or less.

2 Defining instantaneous velocity

As said above, the essence of motion is changing from one position to another. So, let’s suppose that an object changes its position over time. As Zeno pointed out, at any given time, it has only one position. Thus, if we denote the object’s position by \( x \), then \( x \) is a function of the time \( t \): there exists a function \( f \) such that

\[
x = f(t).
\]

Consider what happens near a fixed time \( t_0 \). As a small amount of time elapses, the object’s position changes by a small amount; the velocity is the change in position divided by the change in time. For some reason, it is customary to use the Greek letter \( \Delta \) (capital delta) to represent “change in.” Thus, with this notation, the above sentence states that

\[
\text{velocity} = \frac{\Delta x}{\Delta t}.
\]

There’s a bit of a problem here, though. If we specifically want the velocity at the instant \( t_0 \), then we don’t have any change in time to work with: \( \Delta t = 0 \). Likewise, within the single instant, there is no change in position: \( \Delta x = 0 \). So, the expression above would tell us that velocity = \( 0/0 \). Since \( 0/0 \) is undefined, this is not terribly helpful.

However, we have been studying a way to “fill in” such undefined values: use limits. Thus, we define the instantaneous velocity at \( t_0 \), denoted \( dx/dt|_{t=t_0} \), to be

\[
\left. \frac{dx}{dt} \right|_{t=t_0} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t},
\]

provided that this limit exists. The notation follows the convention that “when you take a limit, you should replace Greek letters by Roman letters.” In this case, we replace the Greek letter \( \Delta \) by the Roman letter \( d \).

Recall that the object starts at time \( t_0 \). If the time changes by \( \Delta t = h \), then the corresponding change in position is \( \Delta x = f(t_0 + h) - f(t_0) \). Thus, the above equation can also be written

\[
\left. \frac{dx}{dt} \right|_{t=t_0} = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.
\]

Finally, if we bring to bear all of the different notations we’re likely to use for this, we’ll get

\[
f'(t_0) = \frac{df}{dt}(t_0) = \left. \frac{dx}{dt} \right|_{t=t_0} = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.
\]
This quantity (when it exists) is called the **derivative of** $f$ at $t_0$.

### 3 A few more bits on continuity

People generally learn stuff better if they see it over a period of time, rather than all at once. Thus, I've decided to distribute the subject of continuity over multiple lectures.¹

**Theorem.** Every polynomial or rational function is continuous on its natural domain. The same holds if you throw in $n$th roots.

The proof of this theorem is by repeated applications of the Main Limit Theorem. I won't try to give the complete proof, but I will give you an example.

**Example.** Show, using the Main Limit Theorem, that the function $f$ defined by

$$f(x) = \frac{1 + \sqrt{2x}}{x^3 - 13}$$

is continuous.

**Solution.** For every real number $c$ such that $f(c)$ is defined, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1 + \sqrt{2x}}{x^3 - 13} = \frac{\lim_{x \to c} (1 + \sqrt{2x})}{\lim_{x \to c} (x^3 - 13)} = \frac{1 + \sqrt{2c}}{c^3 - 13} = f(c).$$

By hypothesis, $f(c)$ is defined, i.e., the last line makes sense. By the Main Limit Theorem, the previous line makes sense and is equal to it, and so on all the way up. Thus, for every $c$ in the domain of $f$,

$$\lim_{x \to c} f(x) = f(c).$$

In other words, $f$ is continuous. \(\square\)

Finally, an example of using the Intermediate Value Theorem that hearkens back to the bonus problem from the first lecture:

**Example.** Show that $\sqrt[3]{31}$ exists. In other words, show that there is a positive real number $x_0$ such that $x_0^3 = 31$.

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¹Translation: I did not really get to say all I wanted on continuity last lecture, so I'm trying to cram in a few extra notes at the end of this one.
Let $f$ be the function defined by $f(x) = x^3$. Since $f$ is a polynomial function, $f$ is continuous on its domain $(-\infty, \infty)$, and in particular on the interval $[0, 4]$. Observe that

$$f(0) = 0 < 31 < 64 = f(4).$$

Hence, there exists some $x_0$ such that $0 < x_0 < 4$ and $f(x_0) = 31$. \hfill \square
Assignment 12 (due Monday, October 31, a.k.a. Halloween)

Section 1.3, Problems 3, 4, 7, and 8. Remember, these problems are about showing you understand how to use the Main Limit Theorem, not about finding the limits. Problems 4 and 8 will be graded carefully.

Section 1.6, Problems 2, 4, 6-8, 32, and 33. Problems 6-8 and 32 will be graded carefully. You do not need to give ε-δ proofs for these problems.

Bonus problem: Let \( f \) be a function (which you do not get to choose). Consider the statement

\[
\text{For all } x_0 \text{ in the domain of } f, \text{ for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that whenever } |x - x_0| < \delta, \text{ then } |f(x) - f(x_0)| < \varepsilon.
\]

Explain why this is equivalent to the statement that “\( f \) is continuous.” (One dilemma you may need to address: Why does it make no difference if we write \( |x - x_0| < \delta \) rather than \( 0 < |x - x_0| - \delta \)?)

Assignment 13 (due Wednesday, 2 November)

NOTE: From this assignment on, you no longer need to write anything about “By the Main Limit Theorem,...” when showing your work to take a limit. (You should, however, continue to show your work.)

Use the Intermediate Value Theorem to prove that, no matter what Diophantus\(^2\) of Alexandria might have thought, \( \sqrt{2} \) does, in fact, exist. (In other words, there exists a positive real number \( x_0 \) such that \( x_0^2 = 2 \).) This problem will be graded carefully.

Section 2.2, Problems 45-48 and 51, 52. Be sure to follow the instructions carefully on 51 and 52; these problems are as much about how you find the derivative, as what answer you get. Problems 46, 48, and 52 will be graded carefully.

Let \( a \neq 0 \) be a real number (which you don’t get to choose). Let \( f \) be the function defined by

\[ f(x) = ax. \]

Show that \( f \) is continuous using an ε-δ proof. This problem will be graded carefully.

\(^2\)See Lecture 3, page 2.