1 Definition: Limits as $x \to c$

Recall from last time the definition of the limit of $f(x)$ as $x \to c$:

**Definition.** We say

\[
\lim_{x \to c} f(x) = \ell
\]

if

For arbitrarily small $\varepsilon > 0$, when $x$ is sufficiently close to $c$, $f(x)$ is within $\varepsilon$ of $\ell$.

- **opponent’s move**
- **our move**
- **judge’s decision**

More formally,

\[
\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that if } |x - c| < \delta \text{ and } x \neq c, \text{ then } |f(x) - \ell| < \varepsilon.
\]

A couple of notes on this definition:

- When we wanted to compute $\lim_{x \to \infty} f(x)$, this depended only on the function $f$. If we want to compute $\lim_{x \to c} f(x)$, we will, quite probably, have a different limit for every different choice of $c$.

- We specifically exclude the “judge” from looking at the value of $f(x)$ when $x = c$, because the limit is supposed to detect what “should be” going on at $c$ without actually touching $c$. This was not an issue for $\lim_{x \to \infty} f(x)$, where we’re sort of “setting $c = \infty$,” because $f(\infty)$ does not make sense anyway.
2 Examples: Using the ε-δ definition

At this point, we’ll begin trying to understand the definition better by doing some examples of ε-δ proofs. We are doing this to help us understand the definition, and the concept, of limit, which is much more useful in more complicated situations.

Example. Consider the function $f$ defined by

$$f(x) = \begin{cases} 
2x - 1 & \text{if } x \neq 2, \\
4 & \text{if } x = 2.
\end{cases}$$

Its graph looks like this:

Let’s use the ε-δ definition of the limit to show that

$$\lim_{x \to 2} f(x) = 3.$$  

Note: When looking at this sort of example, we are not using the formal definition of the limit to better understand the function $f$. We are using the function $f$ to better understand the definition. The formal definition becomes really useful when we are dealing with functions $f$ for which we don’t have formulas.

Let $\varepsilon > 0$ be given (by our opponent; we can’t choose it). Before we go around choosing $\delta$ haphazardly to set what is “sufficiently close to 2,” let’s anticipate what the judge will say. In other words, let’s “solve” the inequality he cares about as best we can, without knowing $\varepsilon$:

$$|f(x) - 3| < \varepsilon.$$
Since the judge does not care what happens when \( x = c = 2 \), we can assume \( f(x) = 2x - 1 \). In this case, the judge’s “test” is whether

\[
\begin{align*}
|2x - 1 - 3| &< \varepsilon \\
|2x - 4| &< \varepsilon \\
2|x - 2| &< \varepsilon \\
|x - 2| &< \frac{1}{2}\varepsilon.
\end{align*}
\]

Now, we could proceed to finish “solving” the inequality; but in this case, that would be counterproductive. The condition we impose, by our choice of \( \delta \), is that

\[
|x - 2| < \delta.
\]

Thus, if we set \( \delta = \frac{1}{2}\varepsilon \), we are guaranteed that the judge will like all the “sufficiently small” values of \( x \) we allow him to look at.

**Important:** \( \delta \) may depend on \( \varepsilon \), and usually will. (Since our opponent has already chosen \( \varepsilon \), we’re allowed to use it.) But \( \delta \) cannot depend on \( x \). (The judge doesn’t choose \( x \) until after we’ve already chosen \( \delta \).)

If you recall the discussion of the “narrative” of a proof with quantifiers, the way you “tell” a proof is often quite different from the way you work it out. Now that we’ve worked out what the value of \( \varepsilon \) should be, let’s “tell” the story of what goes on in the courtroom—without trying to get into the characters’ heads (as we were, earlier, by anticipating the judge). Just the facts, ma’am.

**Solution.** Let \( \varepsilon > 0 \) be given. Set \( \delta = \frac{1}{2}\varepsilon \). Assume \( |x - 2| < \delta \) and \( x \neq 2 \).

Since \( x \neq 2 \), we know \( f(x) = 2x - 1 \). Hence,

\[
\begin{align*}
|f(x) - 3| &= |2x - 1 - 3| \\
&= |2x - 4| \\
&= 2|x - 2| \\
&< 2\delta \\
&= 2\left(\frac{1}{2}\varepsilon\right) \\
&= \varepsilon.
\end{align*}
\]

Thus, under these hypotheses, \( |f(x) - 3| < \varepsilon \), as desired.

Hence, \( \lim_{x \to 2} f(x) = 3 \). \( \square \)
Test Wednesday, 19 October

The test includes lectures through Wednesday, October 12, and assignments 1 through 7 (but not assignment 4.5). Note: The assignment numbers are one off from the lecture numbers, since I gave no (non-bonus) assignment on the first day of class. In particular, although the quiz does not include any new material from the Lectures 9 and 10, it does include the homework set due Monday, 17 October.

Anything that appeared on a quiz will probably show up in some form on the test. (Exception: no contrapositive questions.) Anything that appeared on a non-bonus homework question might show up on the test.

If I said something in a lecture that did not make it in any form into a quiz or homework question, then it will not be on the test.

Assignment 8 (due Friday, 21 October)

Give $\varepsilon$-$\delta$ proofs of the following facts:

$$\lim_{x \to 0} 7x = 0 \quad (1)$$
$$\lim_{x \to 1} 2x = 2 \quad (2)$$
$$\lim_{x \to \frac{1}{2}} 4x + 1 = -1 \quad (3)$$
$$\lim_{x \to \frac{1}{5}} \frac{1}{2}x - 2 = -\frac{1}{2} \quad (4)$$

They will all be graded carefully.