1. (*) Read Dummit and Foote, Sections 8.3–9.3.

2. (*) Dummit and Foote, Section 8.2, #2–4.

3. Dummit and Foote, Section 8.2, #5:
   Let \( R = \mathbb{Z}[\sqrt{-5}] \). Define the ideals \( I_2 = (2, 1 + \sqrt{-5}) \), \( I_3 = (3, 2 + \sqrt{-5}) \), and \( I_3' = (3, 2 - \sqrt{-5}) \).
   
   (a) Prove that \( I_2, I_3, \) and \( I_3' \) are not principal ideals.
   
   (b) Prove that the product of two non-principal ideals may be a principal ideal by showing that \( I_2^2 = (2) \).
   
   (c) Prove that \( I_2I_3 = (1 - \sqrt{-5}) \) and \( I_2I_3' = (1 + \sqrt{-5}) \) are principal. Conclude that \( I_2^2I_3I_3' = (6) \).

4. Dummit and Foote, Section 8.2, #6:
   Let \( R \) be an integral domain, and suppose that every prime ideal in \( R \) is principal. This exercise shows that \( R \) must be a P.I.D.
   
   (a) Assume that the set of ideals of \( R \) that are not principal is non-empty, and prove that this set has a maximal element under inclusion.
   
   (b) Let \( I \) be an ideal which is maximal with respect to being non-principal, and let \( a, b \in R \) with \( ab \in I \) but with \( a \notin I \) and \( b \notin I \). Let \( I_a = (I, a) \) be the ideal generated by \( I \) and \( a \), let \( I_b = (I, b) \) be the ideal generated by \( I \) and \( b \), and define \( J = \{ r \in R \mid rI_a \subseteq I \} \). Prove that \( I_a = (\alpha) \) and \( J = (\beta) \) are principal ideals in \( R \) with \( I \subseteq I_a \subseteq J \) and \( I_aJ = (\alpha\beta) \subseteq I \).
   
   (c) If \( x \in I \), show that \( x = sa \) for some \( s \in J \). Deduce that \( I = I_aJ \) is principal, a contradiction.

5. Suppose \( R \) is an integral domain with Euclidean norm \( N \) satisfying the following two conditions:
   
   - For any natural number \( n \), the set \( \{0\} \cup \{ a \in R \mid N(a) < n \} \) is a subgroup of the additive group of \( R \).
   - For \( ab \neq 0 \), \( N(ab) \geq \max\{N(a), N(b)\} \).
   
   Then, prove that Euclidean division is unique with respect to \( N \): in other words, prove that for any pair \( (a, b) \) with \( b \neq 0 \), there exists a unique pair \( (q, r) \) subject to the conditions \( a = bq + r \) and \( r = 0 \) or \( N(r) < N(b) \).

6. Let \( k \) be a field. Let \( R \) the formal power series ring \( k[[x]] \). Define \( N \) on \( R \setminus \{0\} \) as follows: \( N(f) \) is the smallest \( n \) for which the coefficient of \( x^n \) in \( f \) is nonzero.
   
   (a) Prove that \( R \) is a Euclidean domain with Euclidean norm \( N \).
   
   (b) For \( a, b \) elements of \( R \), prove that \( N(a + b) \) cannot be bounded as a function of \( N(a) \) and \( N(b) \).
   
   (c) Prove that if \( a \) and \( b \) are two power series such that \( b \) does not divide \( a \) (and \( b \neq 0 \)), there are infinitely many pairs \( (q, r) \) for which \( a = bq + r \) and \( N(r) < N(b) \).

7. Let \( R \) be a ring with 1. For \( a \) a unit in \( R \), consider the map:

\[ \varphi_a : x \mapsto axa^{-1} \]
(a) Prove that $\varphi_a$ is an automorphism of $R$.
(b) Prove that the map $a \mapsto \varphi_a$ is a homomorphism from the multiplicative group of units in $R$ to the automorphism group of $R$.
(c) Suppose the additive group of $R$ is generated by all the multiplicative units. Prove that if $L$ is a left ideal of $R$ with the property that $\alpha(L) \subseteq L$ for all automorphisms $\alpha$ of $R$, then $L$ is a two-sided ideal of $R$.

8. (a) Suppose $R$ is an integral domain that is a Noetherian ring (i.e., every ideal in $R$ is finitely generated). Prove that if $r$ is a nonzero non-unit of $R$, we can write $r = up_1^{k_1} \cdots p_n^{k_n}$ where $u$ is a unit and $p_i$ are irreducibles. (Hint: Imitate the proof for principal ideal domains).
(b) Suppose $R$ is an integral domain. Prove that if a nonzero non-unit $r \in R$ can be written as $up_1^{k_1} \cdots p_n^{k_n}$ where all the $p_i$ are prime and $u$ is a unit, then any two factorizations of $r$ into irreducibles are equal up to ordering and associates.
(c) Use parts (a) and (b) along with the fact that in a Bezout domain, every irreducible element is prime, to show that every principal ideal domain is a unique factorization domain.

9. Suppose $O$ is a quadratic integer ring, with $N$ the algebraic norm. Prove that if $a$ is a prime element of $O$, then $|N(a)|$ is either prime (as a natural number) or the square of a prime. Give examples where $|N(a)|$ is prime and examples where $|N(a)|$ is the square of a prime.

10. Dummit and Foote, Section 8.3, #5:
Let $R = \mathbb{Z}[\sqrt{-n}]$, where $n$ is a square-free integer greater than 3.
(a) Prove that 2, $\sqrt{-n}$, and $1 + \sqrt{-n}$ are irreducibles.
(b) Prove that $R$ is not a U.F.D. Conclude that the quadratic integer ring $O$ is not a U.F.D. when $D \equiv 2, 3 \pmod{4}$ and $D < -3$.
(c) Give an explicit ideal in $R$ that is not principal.

11. (*) Dummit and Foote, Section 9.1, #1–7, 9, and 16.

12. Dummit and Foote, Section 9.1, #10:
Prove that the ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1x_2, x_3x_4, x_5x_6, \ldots)$ contains infinitely many minimal prime ideals.


14. A combination of Dummit and Foote, Section 9.2, #10, 11:
Let $f(x), g(x) \in \mathbb{Q}[x]$ be two non-zero polynomials, and let $d(x)$ be their gcd.
(a) Given $h(x) \in \mathbb{Q}[x]$, show that there are polynomials $a(x), b(x) \in \mathbb{Q}[x]$ such that $a(x)f(x) + b(x)g(x) = h(x)$ if and only if $d(x)$ divides $h(x)$.
(b) If $a_0(x)$ and $b_0(x)$ are particular solutions to the equation in part (a), show that the full set of solutions is given by:
\[
\begin{align*}
a(x) &= a_0(x) + m(x) \frac{g(x)}{d(x)} \\
b(x) &= b_0(x) - m(x) \frac{f(x)}{d(x)}
\end{align*}
\]
as $m(x)$ ranges over all polynomials in $\mathbb{Q}[x]$.
(c) When $f(x) = x^3 + 4x^2 + x - 6$ and $g(x) = x^5 - 6x + 5$, find $d(x)$ and at least one pair of solutions for $a_0(x)$ and $b_0(x)$ when $h(x) = d(x)$.