Problems and progress in understanding the Torelli group

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Outline of talk

PART I: Combinatorial group theory
PART II: Pseudo-Anosov theory
PART III: Cohomology
The mapping class group

$\Sigma_g =$ closed, oriented surface of genus $g$

$\text{Mod}_g =$ the mapping class group of $\Sigma_g$

$= \pi_0(\text{Homeo}^+\Sigma_g)$

Some elements in $\text{Mod}_g$:

1. Dehn twists $T_\alpha$
   (Dehn, Humphries) $2g + 1$ of these generate $\text{Mod}_g$.

2. Pseudo-Anosov homeomorphisms $f$
   
   $f^n(\alpha) \not\sim \alpha, \forall n, \alpha$. 
The Torelli group

**Definition (Torelli group):** Let $\mathcal{I}_g$ be the subgroup of $\text{Mod}_g$ acting trivially on $H_1(\Sigma_g, \mathbb{Z})$:

$$1 \rightarrow \mathcal{I}_g \rightarrow \text{Mod}_g \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1$$

**Some elements in $\mathcal{I}_g$:**

1. Dehn twists:
   - $T_\gamma$ about separating $\gamma$
   - $T_\alpha T_\beta^{-1}$ with $\{\alpha, \beta\}$ bounding pair

2. Some pseudo-Anosovs (Thurston, Penner)
The bounding twist group

Definition (Bounding twist group): Let

\[ \mathcal{K}_g := \langle \{ T_\gamma : \gamma \text{ separates } \Sigma_g \} \rangle \]

Relation to homology 3-spheres (Morita):

- Every integral homology 3-sphere is of form: take out handlebody in \( S^3 \), reglue with \( f \in \mathcal{K}_g \).
- The Casson invariant

\[ \lambda : \mathcal{K}_g \rightarrow \mathbb{Z} \]

\[ f \mapsto \lambda(M_f) \]

is a homomorphism.
**\( \mathcal{I}_g \) versus \( \mathcal{K}_g \)**

**Question 1. [Birman, etc. '70’s]** Does \( \mathcal{I}_g = \mathcal{K}_g \), or at least \( [\mathcal{I}_g : \mathcal{K}_g] < \infty \)?

**Answer:**

- Powell '78: \( \mathcal{I}_2 = \mathcal{K}_2 \)

- D. Johnson '80-'83:

  Let \( H = H_1(\Sigma_g, \mathbb{Z}) \). Action of \( \mathcal{I}_g \) on \( \pi'/[\pi, \pi'] \) gives **Johnson homomorphism**

  \[
  \tau : \mathcal{I}_g \rightarrow \wedge^3 H/H
  \]

  surjective with \( \ker(\tau) = \mathcal{K}_g \).

  \[
  \Rightarrow [\mathcal{I}_g : \mathcal{K}_g] = \infty
  \]
Sidebar: $\mathcal{I}_g$ determines $\text{Mod}_g$

**Theorem 2. [Farb-Ivanov]** Let $g \geq 5$. Then

\[
\text{Comm}(\mathcal{I}_g) \approx \text{Aut}(\mathcal{I}_g) \approx \text{Mod}_g^{\pm}
\]

**Theorem 3. [Brendle-Margalit]** Let $g \geq 4$. Then

\[
\text{Comm}(\mathcal{K}_g) \approx \text{Aut}(\mathcal{K}_g) \approx \text{Mod}_g^{\pm}
\]

\text{Aut results extended to } g \geq 3 \text{ by McCarthy-Vautaw.}
Generation problem for $\mathcal{I}_g$

Prehistory: Nielsen (1924), Magnus (1934)

**Question 4. [Birman, '71]** Is $\mathcal{I}_g$ finitely generated?

**Difficulty:** Johnson’s $\tau$ shows that any genset for $\mathcal{I}_g$ must have $\geq O(g^3)$ elements.

[Contrast: Mod$_g$ can be generated by 2 elts!]

**Theorem 5. [D. Johnson, '83]** $\mathcal{I}_g, g \geq 3$, is finitely generated (by $O(2^g)$ twists about bounding pairs).

**Problem 6.** Find a generating set for $\mathcal{I}_g$ having $O(g^d)$ elements, for some $d \geq 3$. 
Generation problem for $\mathcal{K}_g$

**Question 7.** [Johnson '83, Birman '86, Morita '90, etc.]

Is $\mathcal{K}_g$ finitely generated?

**Answer:**

- (Birman-Craggs, Johnson, Morita): used Rochlin and Casson invariants to find large abelian quotients of $\mathcal{K}_g$.

- (McCullough-Miller '86): $\mathcal{K}_2$ is not finitely generated.

- (Mess '89, Putman '05): $\mathcal{K}_2$ is free on \{symplectic splittings of $H_1(\Sigma_2, \mathbb{Z})$\}.

**Note:** $\mathcal{K}_g$ is not free for $g \geq 3$. 
Theorem 8. [Biss-Farb '04] \( \mathcal{K}_g \) is not finitely generated for any \( g \geq 2 \).

Proof outline: Revisit strategy of McCullough-Miller.

1. Action of \( \mathcal{K}_g \) on certain (non-canonical) abelian cover \( Y \) gives representation

\[
\rho : \mathcal{K}_g \to \text{Aut}_\mathcal{L}(H_1(Y, \mathbb{Z}))
\]

with \( \mathcal{L} = \) group-ring of deck group.

2. Find codimension 2 \( \rho \)-trivial subspace, inducing

\[
\hat{\rho} : \mathcal{K}_g \to \text{GL}(2, \mathcal{L})
\]

3. Get action on Bruhat-Tits-Serre tree, group amalgam, etc. Analyze!
Finiteness properties

Question 9. [Morita] Is $H_1(K_g, \mathbb{Z})$ finitely generated?

Conjecture 10. $\mathcal{I}_g$ is finitely presented for $g \geq 4$.

Problem 11. Find an infinite presentation for $\mathcal{I}_g$ and for $K_g$.

Question 12. What is the cohomological dimension of $\mathcal{I}_g$?

Problem 13. Determine the maximal number $f(g)$ for which there is a $K(\mathcal{I}_g, 1)$ with finitely many cells in dimensions $\leq f(g)$.

\[ f(2) = 0 \text{ since } \mathcal{I}_2 \text{ not f.g.} \]
\[ f(3) \leq 3 \text{ (Johnson-Millson)} \]

For $g \geq 3$, combining Johnson and Akita gives:

\[ 1 \leq f(g) \leq 6g - 5 \]
Recall: \( f \) pseudo-Anosov if \( f^n(\alpha) \not\sim \alpha, \forall n, \alpha. \)

(Nielsen, Thurston): A pseudo-Anosov \( f \) has a \textit{stretch factor} \( \lambda(f) > 1 \):

- \( \exists \) measured foliation \( \mathcal{F} \) with \( f(\mathcal{F}) = \lambda(f)\mathcal{F} \)

- \( \log \lambda(f) = \text{Teichmüller translation length.} \)
The smallest stretch factor

Problem 14. [Penner] Determine the shortest geodesic in moduli space:

\[ s_g := \inf \{ \log \lambda(f) : f \text{ pseudo-Anosov} \} \]

(Penner, Baer, McMullen):

\[ s_g \asymp \frac{1}{g} \]

Theorem 15. [Farb-Leininger-Margalit]

\[ \lambda(f) > \sqrt{2} \text{ for any } f \in \mathcal{I}_g. \]

Key Lemma: For any such \( f \), and any simple closed curve \( \gamma \), we have

\[ i(\gamma, f(\gamma)) \geq 4 \text{ or } i(\gamma, f^2(\gamma)) \geq 4. \]
Part III: (Co)homology

**Goal:** Understand $H^*(\mathcal{I}_g)$ and $H^*(K_g)$.

**Note:** These are $\mathcal{S}_{2g}$-modules.
Infinite dimensionality

**Theorem 16. [Akita, '01]** $H_*(\mathcal{I}_g, \mathbb{Q})$ and $H_*(\mathcal{K}_g, \mathbb{Q})$ are infinite-dimensional for $g \geq 7$.

**Proof for $\mathcal{I}_g$ (a la Smillie-Vogtmann):**

If $\dim_{\mathbb{Q}}(H_*(\mathcal{I}_g, \mathbb{Q})) < \infty$ then

$$1 \to \mathcal{I}_g \to \text{Mod}_g \to \text{Sp}(2g, \mathbb{Z}) \to 1$$

gives

$$\chi(\mathcal{I}_g) = \chi(\text{Mod}_g)/\chi(\text{Sp}(2g, \mathbb{Z}))$$

and so

$$\chi(\mathcal{I}_g) = \frac{1}{2 - 2g} \prod_{k=1}^{g-1} \frac{1}{\zeta(1 - 2k)} \notin \mathbb{Z}$$

by Harer-Zagier, Harder. But $\mathcal{I}_g$ is torsion-free, so $\chi(\mathcal{I}_g) \in \mathbb{Z}$.

**Problem 17.** Find infinitely many linearly independent cycles in $H_*(\mathcal{I}_g, \mathbb{Q})$ and $H_*(\mathcal{K}_g, \mathbb{Q})$.  

Morita-Mumford-Miller classes

\[ e_i \in H^{2i}(\text{Mod}_g, \mathbb{Z}) \] generate stable rational cohomology of \( \text{Mod}_g \). (Madsen-Weiss, 2002)

- Odd classes \( e_{2j+1} \) vanish on \( \mathcal{I}_g \).
- All classes \( e_i \) vanish on \( \mathcal{K}_g \) (Morita)

**Question 18.** Does \( e_{2j} = 0 \) on \( \mathcal{I}_g \)?

**Status:** Not even known for \( j = 1 \).
Known nontrivial classes

1. Morita '91: $H^1(K_g, \mathbb{Z})^{\text{Mod}_g} \cong \mathbb{Z} = \langle d_1 \rangle$

2. Johnson, '83-'85:

$$H^1(I_{g,1}, \mathbb{Z}) \approx \bigwedge^3 H \oplus B_2$$

Problem 19. **Determine the algebra generated by these classes.**
Rational cohomology from $\tau$

Recall Johnson homomorphism

$$\tau: \mathcal{I}_{g,1} \to \wedge^3(H_1(\Sigma_{g,1}, \mathbb{Z}))$$

- $\tau$ is $\text{Mod}_{g,1}$-equivariant.

- $\ker \tau = \mathcal{K}_g$ (Johnson)

\[\tau^*: H^*(\wedge^3 H, \mathbb{Q}) \to H^*(\mathcal{I}_{g,1}, \mathbb{Q})\]

$\text{Sp}(2g, \mathbb{Q})$-module homomorphism.

**Degree 1:** $\tau^*$ is an isomorphism. (Johnson, '85)

**Degree 2:** $\ker \tau^*$ computed using representation theory of $\text{Sp}(2g, \mathbb{Q})$. (Hain, '97)

**Degree 3:** $\ker \tau^*$ computed up to one irreducible summand. (Sakasai, '03)
The Birman-Craggs homomorphism

Fix an embedding $h : \Sigma_g \hookrightarrow S^3$. Define

$$\psi_h : \mathcal{I}_g \to \mathbb{Z}/2\mathbb{Z}$$

$$f \mapsto \mu(M(h, f))$$

where

$$M(h, f) = \text{split } S^3 \text{ along } h(S), \text{ reglue via } f$$

$$\mu(M) = \sigma(W)/8 \mod 2 \text{ (Rochlin invariant)}$$

**Johnson:** Vary $h$, combine all $\psi_h$ to give surjections

$$\sigma : \mathcal{I}_g \to B_3$$

$$\sigma|_{\mathcal{K}_g} : \mathcal{K}_g \to B_2$$

$$B_i = \text{deg } \leq i \text{ summand of graded } \mathbb{F}_2\text{-algebra } B$$

$$B_i \approx \sum_{j=0}^{i} \binom{2g}{j} \text{ copies of } \mathbb{Z}/2\mathbb{Z}.$$
Mod 2 cohomology from $\sigma$

**Theorem 20. [Brendle-Farb]**  Each of the images of

$$\sigma^* : H^2(B_3, \mathbb{F}_2) \to H^2(\mathcal{I}_g, \mathbb{F}_2)$$

$$(\sigma|_{\mathcal{K}_g})^* : H^2(B_2, \mathbb{F}_2) \to H^2(\mathcal{K}_g, \mathbb{F}_2)$$

has dimension at least $O(g^4)$.

- Not detectable rationally, or with $\tau$.
- Difficulty: modular representation theory too hard.
- On $\mathcal{K}_g$, whole picture lifts to Casson invariant, integral classes.

**Proof idea:** Evaluate on abelian cycles. Use: Johnson’s formula for $\sigma$; intuition.
Abelian cycles

Let \( f, g \in \mathcal{K}_{g,1} \) with \( fg = gf \).

\[
i : \langle f, g \rangle \cong \mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathcal{K}_{g,1}
\]

induces

\[
i_* : H_2(\mathbb{Z} \times \mathbb{Z}) \cong \langle t \rangle \rightarrow H_2(\mathcal{K}_g)
\]

giving the **abelian cycle**

\[
\{f, g\} = i_*(t) \in H_2(\mathcal{K}_g)
\]

**Remark:** All abelian cycles vanish in \( H_2(\text{Mod}_g, \mathbb{Z}) \).
Further problems

Problem 21.  Determine images of $\tau^*$ and $\sigma^*$ in all dimensions.

Problem 22.  Exhibit a single non-abelian cycle in $H^*(I_g)$ and $H^*(K_g)$. 