Isometries, rigidity, and universal covers

Benson Farb

joint work with Shmuel Weinberger

Banff, July 13, 2005
Outline of talk

I. Statement of main result

II. Some applications

III. Proof sketch

References (available on archive, and my web page)


Terminology

$M$ = a closed, smooth, $n$-manifold, $n > 1$.

$g_{\text{loc}}$ = locally symmetric metric, noncompact type (= some $\Gamma \backslash G/K$)

$\text{Riem}(M) = \{\text{Riem. metrics on } M\}$

Standing Assumptions Today:

1. $M$ is closed.

2. $M$ is aspherical.

3. $M$ is not finitely covered by a smooth product.
Locally symmetric manifolds

Bochner (1946): $\text{Isom}(M, g_{\text{loc}})$ is finite.

FW ('03):

1. $\text{Isom}(M, h) \hookrightarrow \text{Isom}(M, g_{\text{loc}})$ $\forall h \in \text{Riem}(M)$.

2. When $M$ arithmetic, characterized $g_{\text{loc}}$ among all $h \in \text{Riem}(M)$ via $\text{Isom}(M_i, h)$ where $M_i$ finite covers.

NOTE: Any isometry of any cover lifts to $\tilde{M}$.

Eberlein ('80, '82): If $\text{Isom}(\tilde{M}, h)$ is nondiscrete, and if $K(h) \leq 0$, then $h \sim g_{\text{loc}}$. 
Aspherical Riemannian manifolds

Basic question: For which Riemannian manifolds \((M, h)\) is \(\text{Isom}(\widetilde{M}, h)\) nondiscrete?

Theorem 1. [informal statement] Let \(M\) be any closed, aspherical Riemannian manifold. Then either \(\text{Isom}(\widetilde{M})\) is discrete or \(M\) has a finite cover which is isometric to a manifold on an explicitly given list.
Application I: New characterizations of locally symmetric manifolds

**Theorem 2.** For any Riemannian $M$, the following are equivalent:

1. $\pi_1(M)$ has no nontrivial normal abelian subgroup, and $\text{Isom}(\widetilde{M})$ is not discrete.

2. $M$ is isometric to some irreducible $\Gamma\backslash G/K$.

**Sample consequence:** If $\pi_1(M)$ word-hyperbolic and $\text{Isom}(\widetilde{M})$ non-discrete then $M$ is isometric to a rank one manifold (no irreducibility assumption needed!).
A quantitative version

**Conjecture 3.** The hypothesis “\( \text{Isom}(\widetilde{M}) \) is not discrete” in the above theorems can be replaced by: 
\[ [\text{Isom}(\widetilde{M}) : \pi_1(M)] > C, \text{ where } C \text{ depends only on } \pi_1(M). \]

Can prove in special case.

**Theorem 4.** Suppose \( M \) admits some \( g_{\text{loc}} \). Then there exists a constant \( C \), depending only on \( \pi_1(M) \), such that for any \( h \in \text{Riem}(M) \):

\[ [\text{Isom}(\widetilde{M}) : \pi_1(M)] > C \text{ if and only if } h \sim g_0 \]

where \( \sim \) denotes “up to homothety of direct factors”. 
Application II: Which contractible $X$ admit both compact and finite volume quotients? Irreducible quotients?

Answers to first question:

- Homogeneous $X$
  - Solvable: only compact quotients (Mostow)
  - Semisimple: both types (Borel)

- Nonpositively curved $X$: only symmetric spaces have both (Eberlein)

Theorem 5. If $X$ is contractible Riemannian, and if $X$ has compact and finite volume quotients, then $X$ is isometric to a warped product

$$X = Y \times X_0$$

with $X_0$ homogeneous and covering both types.
Application III: Complex manifolds

Conjecture 6. [Kazhdan] If a bounded complex domain $\Omega$ covers a compact manifold $M$, and if $\Omega$ has a 1-parameter family of holomorphic automorphisms, then $\Omega$ is biholomorphic to a symmetric domain.

Theorem 7. [Nadel ’90, Frankel ’95] Conjecture 6 is true; indeed only need $c_1(M) < 0$. 
A general theorem

Theorem 8. Let $M$ be a closed, aspherical Riemannian manifold. Then either $\text{Isom}(\widetilde{M})$ is discrete, or $M$ has a finite-sheeted Riemannian cover $M'$ which is isometric to an orbibundle

$$F \to M' \to B$$

where:

- $B$ is a good Riemannian orbifold.
- Each fiber $F'$, endowed with the induced metric, is isometric to a closed, aspherical, locally homogeneous Riemannian $n$-manifold, $n > 0$. 
A truly singular orbibundle

**Theorem 9.** There exists a closed, aspherical Riemannian manifold $M$ such that:

1. $M$ is a Riemannian orbibundle, fibers over a singular orbifold.
2. No finite cover of $M$ fibers over a manifold.
3. $\text{Isom}(\widetilde{M})$ is not discrete.

A key ingredient in this construction is:

**Theorem 10.** There exists a finitely presented group $\Gamma \in \text{Diff}(\mathbb{R}^n)$ acting properly and cocompactly, but $\Gamma$ is not virtually torsion-free.