MATH 258 HOMEWORK #5
DUE MONDAY, FEBRUARY 9

(1) An inner product on a vector space $V$ over $F$ is a bilinear map $\langle \cdot, \cdot \rangle : V \times V \to F$ satisfying the extra conditions
- $\langle v, w \rangle = \langle w, v \rangle$, and
- $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.

(a) Show that the standard dot product on $\mathbb{R}^n$ is an inner product.
(b) Show that $(f, g) \mapsto \int f(x)g(x) \, dx$ is an inner product on $C^\infty([0, 1], \mathbb{R})$.
(c) Suppose that $F$ is ordered. Prove that for any $v, w \in V$,
$$\langle v, w \rangle^2 \leq \langle v, v \rangle \langle w, w \rangle.$$ When does equality hold? What standard inequality in trigonometry does this reflect when $V = \mathbb{R}^n$?
(d) We say that two vectors $v, w$ in $V$ are orthogonal if $\langle v, w \rangle = 0$. Suppose that $T : V \to V$ is a linear transformation satisfying $\langle Tv, w \rangle = \langle v, Tw \rangle$ for all $v, w$. Show that eigenvectors of $T$ with different eigenvalues are orthogonal.

(2) Show that $(F \oplus \infty)^* \cong F \times \infty$. Conclude that it is not the case that $V$ and $V^*$ are always isomorphic.

(3) Suppose that $F'$ is a field containing $F$ and $V$ is an $F$-vector space. If we consider $F'$ to be an $F$-vector space, we can form the tensor product $F' \otimes V$, which is naturally an $F$-vector space. Show that it is also an $F'$-vector space. This is called the change of base of $V$.

(4) Prove the universal property for tensor products. In other words, show that for any vector spaces $U$, $V$, and $W$ there is a bijection
$$\left\{ \text{bilinear maps} \quad U \times V \to W \right\} \leftrightarrow \left\{ \text{linear maps} \quad U \otimes V \to W \right\}.$$

(5) Let $\{u_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^n$ be bases of $U$ and $V$, respectively. Show that a general element $w = \sum_{i=1}^m \sum_{j=1}^n w_{ij} u_i \otimes v_j$ is the sum of $r$ pure tensors if and only if the $m \times n$ matrix $(w_{ij})$ has rank at most $r$.

(6) This next problem will investigate tensor products of modules. Note that the definition of “linear” and “bilinear” still work for rings instead of fields. Suppose that $R$ is a commutative ring with unit, and $M$ and $N$ are $R$-modules. We define the tensor product $M \otimes_R N$ to be the $R$-module such that there is a bijection
$$\left\{ \text{bilinear maps} \quad M \times N \to S \right\} \leftrightarrow \left\{ \text{linear maps} \quad M \otimes_R N \to S \right\}$$
for any $R$-module $S$.

(a) Suppose that $M = R^m$ and $N = R^n$. Show that $M \otimes_R N = R^{mn}$.
(b) Suppose that $R \to S$ is a homomorphism of rings. Show that $S$ is an $R$-module. Show that $S \otimes_R M$ is an $S$-module.
(c) Now suppose that $R = \mathbb{Z}$. As we discussed before, $\mathbb{Z}$-modules are just abelian groups. What is $\mathbb{Z} \otimes \mathbb{Z} \mathbb{Z}/n\mathbb{Z}$? What is $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q\mathbb{Z}$ for not necessarily distinct primes $p$ and $q$? Find a general description of $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$.