LOCALIZATION METHOD AS A UNIVERSAL APPROACH TO THE GROMOV-WITTEN THEORY

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Abstract. In this thesis, I illustrate a new approach to the Gromov-Witten theory through localization technique. This new approach allows us to derive recursion relations for any natural cohomology class on given moduli space of curves. As its applications, I present the localization-proof of the Witten conjecture, the recent proof of Marinõ-Vafa conjecture, recursion relations for Hodge integrals of any type.

1. Introduction

The localization technique proved itself to be a very useful tool in studying the Gromov-Witten invariants as was shown in a new proof of the Witten conjecture [21] and the proof of the Marinõ-Vafa formula [28]. More generally, it allows us to derive recursion relations for any given cohomology class on moduli space of curves. These include, for example, ψ-classes, λ-classes, the virtual class $c_{g,n}^{1/r}$ which appears in the generalized Witten conjecture. In the setting of relative stable moduli, the localization technique gives recursion relations for the Gromov-Witten invariants in terms of other invariants such as double Hurwitz numbers or previously well-known Gromov-Witten invariants. In this thesis, I illustrate a universal method to obtain recursion relations on the Gromov-Witten type invariants. The method I present strongly supports the intuition that the Gromov-Witten invariants are determined by degeneration types of the domain curve and the factorization rules for the class being integrated. This intuition is verified for the case of Witten conjecture.

The rest of this thesis are consisted as follows: In section 2, I briefly review the Gromov-Witten invariants, the moduli space of relative stable morphisms, and the virtual localization technique. In section 3, I present the universal approach to the Gromov-Witten invariants using localization technique. In sections 4 and 5, I present, as applications, a new proof of Witten conjecture, the proof of Marinõ-Vafa formula. In section 6, I list several recursion formulas for the Hodge integrals with any number of λ-classes involved in them. In section 7, I list several open problems to which we can apply this general approach. In the appendix, I list miscellaneous proofs and numerical data.

2. Preliminaries

2.1. **The Gromov-Witten invariants.** Let X be a smooth projective variety and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli stack of n-pointed stable maps of genus g and degree β, i.e. it consists of maps

$$f : (C; x_1, \cdots, x_n) \longrightarrow X$$

such that

- C is a Riemann surface of arithmetic genus $g = h^1(C, \mathcal{O}_C)$ and n marked points $x_1, \cdots, x_n$ with only nodal singularities.
- An algebraic map $f : C \longrightarrow X$ such that $f_*(C) = \beta \in H_2(X, \mathbb{C})$.
- It has no infinitesimal automorphisms fixing the marked points.

For each marked point x_i, consider the line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber over $[C; x_1, \cdots, x_n] \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point x_i. Then define *the ψ-classes* as its first Chern-class, i.e. $\psi_i = c_1(\mathbb{L}_i)$. For each i, let $ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X$ be the evaluation map which sends x_i to its image $f(x_i) \in X$. The construction of *virtual fundamental class*, denoted by $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, allows us to do the intersection theory on $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The *Gromov-Witten invariants* are defined as the intersection numbers

$$\langle \tau_{k_1}(x_1) \cdots \tau_{k_n}(x_n) \rangle^X_{g,d} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \psi_1^{k_1} \cdots \psi_n^{k_n} \, \mathrm{ev}_1^*(Z_1) \cdots \mathrm{ev}_n^*(Z_n)$$

where $Z_1, \cdots, Z_n$ are cohomology classes of X.

2.2. **Localization technique.** In this section, I will briefly summarize various versions of localization formulas [1, 14, 15]. We start with the equivariant cohomology.

2.2.1. *Equivariant Cohomology.* Let G be a compact Lie group acting on M. The equivariant cohomology of M is defined as the ordinary cohomology of the space M_G obtained from a fixed universal G-bundle EG, by the mixing construction

$$M_G = EG \times_G M$$

Here, G acts on the right of EG and on the left of M, and the notation means that we identify $(pg, q) \sim (p, gq)$ for $p \in EG$, $q \in M$, $g \in G$. Hence M_G is the bundle with fibre M over the classifying space BG associated to the universal bundle $EG \longrightarrow BG$. We have natural projection map $\pi : M_G \longrightarrow BG$ and $\sigma : M_G \longrightarrow M/G$, which fits into the mixing diagram of Cartan and Borel:

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BG & \xleftarrow{\ \pi\ } & E \times_G M & \xrightarrow{\ \sigma\ } & M/G \end{array}$$

If G acts smoothly on M, then we have $M_G \cong M/G$. This is not true in general but it turns out that M_G is a better functorial construction and the proper homotopy theoretic quotient of M by G. In any case, the equivariant cohomology, denoted by $H^*_G(M)$, is defined by

$$H^*_G(M) = H^*(M_G)$$

and constitutes a contravariant functor from G-spaces to modules over the base ring $H^*_G := H^*_G(pt) = H^*(BG)$. The map σ defines a natural map $\sigma^* : H^*(M/G) \longrightarrow$

$H^*_G(M)$ which is an isomorphism if G acts freely. The inclusion $i : M \longrightarrow M_G$ induces a natural map $i^* : H^*_G(M) \longrightarrow H^*(M)$.

2.2.2. *Atiyah-Bott Localization Formula.* Let $i : V \hookrightarrow M$ be a map of compact manifolds. The tubular neighborhood of V inside M can be identified with the normal bundle of V. On the total space of the normal bundle, there is the Thom form Φ_V which has compact support in the fibres and integrates to one in each fiber. Extending this form by zero gives a form in M, and multiplying by Φ_V provides a map $H^*(V) \cong H^{*+k}(M, M\backslash V) \longrightarrow H^*(M)$. In particular, the cohomology class $1 \in H^0(V)$ is sent to the Thom class and this class restricts to be the Euler class of the normal bundle of V in M, $\mathcal{N}_{V/M}$. Hence, we see that

$$i^* i_* 1 = e(\mathcal{N}_{V/M})$$

This also holds in equivariant cohomology by same argument applied to V_G, M_G. The theorem of Atiyah and Bott says that an inverse of the Euler class of the normal bundle always exists along the fixed locus of a group action. Precisely, $i^*/e(\mathcal{N}_{V/M})$ is the inverse of i_* in equivariant cohomology, i.e. for any equivariant class ϕ,

$$\phi = \sum_F \frac{i_* i^* \phi}{e(\mathcal{N}_{F/M})}$$

holds where F runs over the fixed locus of the group action. In the integrated form, we have

$$\int_M \phi = \sum_F \int_F \frac{i^* \phi}{e(\mathcal{N}_{F/M})}$$

2.2.3. *Functorial Localization Formula.* Let X and Y be T-manifolds. Assume $f : X \longrightarrow Y$ is a T-equivariant map, $j_E : E \hookrightarrow Y$ is a fixed component in Y, and $i_F : F \hookrightarrow f^{-1}(E)$ is a fixed component in X. For any equivariant class $\omega \in H^*_T(X)$, we have the diagrams;

$$\begin{array}{ccc} F & \xrightarrow{\;i_F\;} & X \\ \downarrow{\scriptstyle g=f|_F} & & \downarrow{\scriptstyle f} \\ E & \xrightarrow{\;j_E\;} & Y \end{array} \qquad\qquad \begin{array}{ccc} \frac{i_F^*(\omega)}{e_T(F/X)} & \xleftarrow{\;i_F^*\;} & \omega \\ \downarrow{\scriptstyle g_!} & & \downarrow{\scriptstyle f_!} \\ g_!\left[\frac{i_F^*(\omega)}{e_T(F/X)}\right] & \xleftarrow{\;j_E^*\;} & f_!(\omega) \end{array}$$

Applying the Atiyah-Bott Localizatio Formula with the naturality relation $f_!(\omega \cdot f^*\alpha) = f_!\omega \cdot \alpha$, we obtain the Functorial Localization Formula:

$$g_!\left[\frac{i_F^*(\omega)}{e_T(F/X)}\right] = \frac{j_E^* f_!(\omega)}{e_T(E/Y)}$$

2.2.4. *Virtual Functorial Localization Formula.* The above Functorial Localization Formula is also valid in the case where X and F are virtual fundamental classes. In this paper, I will use $\left[\overline{\mathcal{M}}_g(X\times\mathbb{P}^1; X\times\{\infty\}\,|\,\beta;\mu)\right]^{vir}$ for X, and $\left[F_\Gamma\right]^{vir}$ for F. Hence for any equivariant class ω, we have:

$$\int_{\left[\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\,|\,\beta;\mu)\right]^{vir}} \omega = \sum_{F_\Gamma}\int_{\left[F_\Gamma\right]^{vir}} \frac{i_\Gamma^*(\omega)}{e_T(F_\Gamma/\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\,|\,\beta;\mu))} \tag{1}$$

2.3. **The moduli space of relative stable morphisms.** In this section, I briefly summarize the definitions and results from [23, 24, 25, 29] with minor modifications. Let X be a smooth projective variety and $D^1,\cdots,D^k$ be disjoint smooth divisors. For $\alpha=1,\cdots,k$, define

$$\Delta(\mathcal{D}^\alpha)(m)=\Delta(\mathcal{D}^\alpha)_1\cup\cdots\cup\Delta(\mathcal{D}^\alpha)_m$$

where $\Delta(\mathcal{D}^\alpha)_i\cong\mathbb{P}(\mathcal{O}_{\mathcal{D}^\alpha}\oplus\mathcal{N}_{\mathcal{D}^\alpha/X})\to\mathcal{D}^\alpha$ for each i,α, and $\mathcal{N}_{\mathcal{D}^\alpha/X}$ denotes the normal sheaf of a subvariety $\mathcal{D}^\alpha$ in X. The projective line bundle $\Delta(\mathcal{D}^\alpha)\to\mathcal{D}^\alpha$ has two distinct sections

$$\mathcal{D}_0^\alpha=\mathbb{P}(\mathcal{O}_{\mathcal{D}^\alpha}\oplus 0),\qquad \mathcal{D}_\infty^\alpha=\mathbb{P}(0\oplus\mathcal{N}_{\mathcal{D}^\alpha/X}).$$

We have $\mathcal{N}_{\mathcal{D}_0^\alpha/\Delta(\mathcal{D}^\alpha)}\cong\mathcal{N}^{-1}_{\mathcal{D}^\alpha/X}$ and $\mathcal{N}_{\mathcal{D}_\infty^\alpha/\Delta(\mathcal{D}^\alpha)}\cong\mathcal{N}_{\mathcal{D}^\alpha/X}$. Then $\Delta(\mathcal{D}^\alpha)(m)$ is constructed by gluing along the two distinct sections of $\Delta(\mathcal{D}^\alpha)_i$'s that correspond to two distinct sections $\mathcal{D}_0^\alpha$ and $\mathcal{D}_\infty^\alpha$. The $\mathbb{C}^*$-action on $\mathcal{O}_{\mathcal{D}^\alpha}$ induces a $\mathbb{C}^*$-action on $\Delta(\mathcal{D}^\alpha)$ such that $\Delta(\mathcal{D}^\alpha)\to\mathcal{D}^\alpha$ is $\mathbb{C}^*$-equivariant, where $\mathbb{C}^*$ acts on $\mathcal{D}^\alpha$ trivially. The two distinct sections $\mathcal{D}_0^\alpha$, $\mathcal{D}_\infty^\alpha$ are fixed under this $\mathbb{C}^*$-action. So there is a $(\mathbb{C}^*)^m$-action on $\Delta(\mathcal{D}^\alpha)(m)$ fixing $\mathcal{D}_0^\alpha,\cdots,\mathcal{D}_m^\alpha$, such that $\Delta(\mathcal{D}^\alpha)(m)\to\mathcal{D}^\alpha$ is $(\mathbb{C}^*)^m$-equivariant, where $(\mathbb{C}^*)^m$ acts on $\mathcal{D}^\alpha$ trivially. The variety

$$X[m^1,\cdots,m^k]=X\cup\bigcup_{\alpha=1}^k\Delta(\mathcal{D}^\alpha)(m^\alpha)$$

with normal crossing singularities is obtained by identifying $\mathcal{D}^\alpha\subset X$ with $\mathcal{D}_0^\alpha\subset\Delta(\mathcal{D}^\alpha)$ under the canonical isomorphism. There is a morphism

$$\pi[m^1,\cdots,m^k]\,:\,X[m^1,\cdots,m^k]\longrightarrow X$$

which contracts $\Delta(\mathcal{D}^\alpha)(m^\alpha)$ to $\mathcal{D}^\alpha$. The $(\mathbb{C}^*)^{m^\alpha}$-action on $\Delta(\mathcal{D}^\alpha)(m^\alpha)$ gives a $(\mathbb{C}^*)^{\sum m^\alpha}$ action on $X[m^1,\cdots,m^k]$ such that $\pi[m^1,\cdots,m^k]$ is $(\mathbb{C}^*)^{m^1+\cdots+m^k}$-equivariant with respect to the trivial action on X. Let $\beta\in H_2(X,\mathbb{Z})$ be a nonzero homology class and μ^α be a partition of d^α, for $\alpha=1,\cdots,k$, where d^α's are defined to be

$$d^\alpha=\int_\beta c_1(\mathcal{O}(\mathcal{D}^\alpha))\geq 0.$$

Define the *relative stable moduli*

$$\overline{\mathcal{M}}_g(X;D^1,\cdots,D^k\mid\beta;\mu^1,\cdots,\mu^k)$$

to be the moduli space of *relative stable morphisms*

$$f:(C;\{x_i^1\}_{i=1}^{l(\mu^1)},\cdots,\{x_i^k\}_{i=1}^{l(\mu^k)})\longrightarrow X[m^1,\cdots,m^k]$$

such that

1. $(C;\{x_i^1\}_{i=1}^{l(\mu^1)},\cdots,\{x_i^k\}_{i=1}^{l(\mu^k)})$ is a connected prestable curve of arithmetic genus g with $\sum_{\alpha=1}^k l(\mu^\alpha)$ marked points.
2. $(\pi[m^1,\cdots,m^k]\circ f)_*[C]=\beta\in H_2(X,\mathbb{Z})$.
3. As Cartier divisors, we have $f^{-1}(\mathcal{D}^\alpha_{(m^\alpha)})=\sum_{i=1}^{l(\mu^\alpha)}\mu_i^\alpha x_i^\alpha$. In particular, if $d^\alpha=0$, then $f^{-1}(\mathcal{D}^\alpha_{(m^\alpha)})$ is empty.
4. The preimage of $\mathcal{D}_l^\alpha$ consists of nodes of C for $l=0,\cdots,m^\alpha-1$. If $f(y)\in\mathcal{D}_l^\alpha$ and C_1,C_2 are two irreducible components of C which intersect at y, then $f|_{C_1}$ and $f|_{C_2}$ have the same contact order to $D^\alpha_{(l)}$ at y.
5. The automorphism group of f is finite.

The construction and proofs in show that $\overline{\mathcal{M}}_g(X;D^1,\cdots,D^k\mid\beta;\mu^1,\cdots,\mu^k)$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$\int_\beta c_1(TX)+(1-g)(\dim X-3)+\sum_{\alpha=1}^k\big(l(\mu^\alpha)-|\mu^\alpha|\big)$$

In order to perform the virtual localization computation on the relative stable moduli $\overline{\mathcal{M}}_g(X;D^1,\cdots,D^k\mid\beta;\mu^1,\cdots,\mu^k)$, we need to compute the Euler classes of the tangent space T^1 and the obstruction space T^2 of $\overline{\mathcal{M}}_g(X;D^1,\cdots,D^k\mid\beta;\mu^1,\cdots,\mu^k)$ at the moduli point

$$f:(C;\{x_i^1\}_{i=1}^{l(\mu^1)},\cdots,\{x_i^k\}_{i=1}^{l(\mu^k)})\longrightarrow X[m^1,\cdots,m^k].$$

This can be done by the following two exact sequences:

$$\begin{aligned}0\longrightarrow&\mathrm{Ext}^0(\Omega_C(R),\mathcal{O}_C)\longrightarrow H^0(\mathbf{D}^\bullet)\longrightarrow T^1\\ \longrightarrow&\mathrm{Ext}^1(\Omega_C(R),\mathcal{O}_C)\longrightarrow H^1(\mathbf{D}^\bullet)\longrightarrow T^2\longrightarrow 0\end{aligned}$$

$$\begin{aligned}0\to H^0\Big(C,f^*\big(\Omega_{X[m^1,\cdots,m^k]}(\sum_{\alpha=1}^k\log\mathcal{D}^\alpha_{m^\alpha})\big)^\vee\Big)&\to H^0(\mathbf{D}^\bullet)\to\bigoplus_{\alpha=1}^k\bigoplus_{l=0}^{m^\alpha-1}H^0_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\\ \to H^1\Big(C,f^*\big(\Omega_{X[m^1,\cdots,m^k]}(\sum_{\alpha=1}^k\log\mathcal{D}^\alpha_{m^\alpha})\big)^\vee\Big)&\to H^1(\mathbf{D}^\bullet)\to\bigoplus_{\alpha=1}^k\bigoplus_{l=0}^{m^\alpha-1}H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\to 0\end{aligned}$$

where

$$R=\sum_{\alpha=1}^k\sum_{i=1}^{l(\mu^\alpha)}x_i^\alpha,\qquad H^1_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\cong H^0(\mathcal{D}_l^\alpha,L_l^\alpha)^{\oplus n_l^\alpha}\big/H^0(\mathcal{D}_l^\alpha,L_l^\alpha),$$

$$H^0_{\mathrm{et}}(\mathbf{R}_l^{\alpha\bullet})\cong\bigoplus_{q\in f^{-1}(\mathcal{D}_l^\alpha)}T_q\big(f^{-1}(\Delta(\mathcal{D}^\alpha)_l)\big)\otimes T_q^*\big(f^{-1}(\Delta(\mathcal{D}^\alpha)_l)\big)\cong\mathbb{C}^{\oplus n_l^\alpha},$$

and n_l^α is the number of nodes over $\mathcal{D}_l^\alpha$. Please refer to [29] for detailed notations.

3. A new approach to the Gromov-Witten theory

In this section, I will illustrate a new approach to derive a recursion relation for any natural cohomology class on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Precisely, let ω be a cohomology class on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ that can be lifted to a equivariant class ω_T on $\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\mid\beta;\mu)$. We obtain a recursion relation for the Gromov-Witten invariants involving ω by the following two steps:

1) **Localization on the relative stable moduli:** We have a natural projection map

$$p:\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\mid\beta;\mu)\longrightarrow\overline{\mathcal{M}}_g(\mathbb{P}^1;\{\infty\}\mid|\mu|;\mu)$$

along with the branching morphism [8],

$$\mathrm{Br}:\overline{\mathcal{M}}_g(\mathbb{P}^1;\{\infty\}\mid|\mu|;\mu)\longrightarrow\mathrm{Sym}^r(\mathbb{P}^1)\cong\mathbb{P}^r$$

Combining these morphisms, we obtain information for the Gromov-Witten invariants involving ω_T through:

$$F(u)=\int_{[\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\mid\beta;\mu)]^{vir}}\omega_T\,\Big(\mathrm{Br}\circ p\Big)^*\Big(\prod_{k\in B}(H-k)\Big)$$

Here, B is any subset of $\{0,1,2,\cdots,r\}$ and H is the hyperplane class of $H^*(\mathbb{P}^r)$ such that $H|_{p_k}=k$ for each fixed point p_k of $\mathbb{P}^r$ under the natural S^1-action. $F(u)$ is, in general, a polynomial of the equivariant parameter u. On the other hand, the virual functorial localization formula (1) applied on the relative stable moduli $\overline{\mathcal{M}}_g(X\times\mathbb{P}^1;X\times\{\infty\}\mid\beta;\mu)$ expresses $F(u)$ as the sum over all fixed locus of S^1-action. Precisely, we have the following expression:

$$F(u)=\sum_{l\in B^c}\Big(\prod_{k\in B}(l-k)\Big)\cdot\Gamma_l(u) \tag{2}$$

where $\Gamma_l(u)$ is the contribution from the fixed locus that are mapped to the fixed point p_l under $\mathrm{Br}\circ p$. These contributions have the following form:

$$\Gamma_l(u)=\sum\Big[\int_{\overline{\mathcal{M}}_{g,m}(X,\tilde{\beta})}i^*(\omega)\;\psi_1^{k_1}\cdots\psi_m^{k_m}\Big]\cdot\Big[\text{previously known data }C(\Gamma_l^k)\Big] \tag{3}$$

where $i^*(\omega)$ is the restriction of ω to the components of fixed locus. The coefficients $C(\Gamma_l^k)$ depends on the partition μ and the splitting-type of the fixed locus which is governed by the following Cut-and-Join operation [42]:

- **Cut-operation :** Geometrically this coreesponds to the pinching of the domain curve along a non-trivial cycle. In terms of localization computation, this corresponds to the Cut-operation on the partition μ:

$$\mu=(\cdots,\mu_i,\cdots)\longrightarrow\nu=(\cdots,p,q,\cdots)\quad,p+q=\mu_i$$

- **Join-operation :** Geometrically this corresponds to the bubbling of the domain curve by pinching a cycle enscribing more than one marked points. In terms of localization computation, this corresponds to the Join-operation on the partition μ:

$$\mu = (\cdots, \mu_i, \mu_j, \cdots) \longrightarrow \eta = (\cdots, \mu_i + \mu_j, \cdots)$$

Moreover the fixed locus that are mapped to a fixed point p_l is precisely those curves that are obtained by performing the Cut-and-Join operation on the fixed locus that are mapped to the fixed point p_{l+1}. For example, there is a unique curve $\mathcal{C}_r$ of genus g and n-marked points that is mapped to p_r. And the fixed locus that are mapped to the branching point p_{r-1} are precisely

- † (Cut-of-type-I) A curve that is obtained by pinching a meridian of $\mathcal{C}_r$. This curve will have arithmetic genus $g-1$ and one more special point coming from the pinching.
- † (Cut-of-type-II) A curve that is obtained by pinching a longitude of $\mathcal{C}_r$. This curve will consist of two smooth components with genus g_1,g_2 such that $g_1 + g_2 = g$.
- † (Join) A curve that is obtained by pinching a cycle that enscribes two marked points, say x_1 and x_2. This curve will consist of two smooth components, one of which has genus g with one less marked points.

As the result, RHS of the relation (2) can be explicitly computed and consists of the Gromov-Witten invariants involving ω. This relation contains enough information to compute all Gromov-Witten invariants. However, we can extract more precise relations from (2) by using the asymptotic analysis as described below.

2) **Asymptotic Analysis:** The relation (2) holds for any given partition μ of any size. Hence it is natural to expect that, if we choose arbitraty μ, we should be able to extract relations on the Gromov-Witten invariants that are independent of the partition μ, i.e. relations between absolute Gromov-Witten invariants. This idea is realized by letting the size of μ to be arbitrarily large $|\mu| \to \infty$. More precisely, we consider the following scaling limit of the partition μ:

$$\text{Write } \mu_i = N \cdot x_i \quad \text{where } N \in \mathbb{Z}, x_i \in \mathbb{Q} \quad \text{ and let } N \to \infty$$

In the localization computation, we encounter the following type of combinatorial number that depends on the partition μ:

$$\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i + k_i}}{\mu_i!}$$

Under the Cut-operation, this number will be replaced by the corresponding combinatorial number for ν. Especially the difference is given by

$$\frac{\mu_i^{\mu_i+k_i}}{\mu_i!} \qquad \longrightarrow \qquad \sum_{p+q=\mu_i} \frac{p^{p+a}q^{q+b}}{p!q!} \tag{4}$$

where a and b depend on k_i and the splitting-type of fixed locus. The asymptotic behaviour of this combinatorial number is given by the following asymptotic formulas [21].

• **Asymptotic Formula :** As $M \longrightarrow \infty$, we have for $a, b \geq 1$ and $k \geq 0$

$$\begin{aligned} e^{-M} \sum_{p+q=M} \frac{p^{p+a}q^{q+b}}{p!q!} &\longrightarrow \frac{1}{2}\Big[\frac{(2a-1)!!(2b-1)!!}{2^{a+b}(a+b)!}\Big]M^{a+b} + o(M^{a+b}) \\ e^{-M} \sum_{p+q=M} \frac{p^{p+k+1}q^{q-1}}{p!q!} &\longrightarrow \frac{n^{k+\frac{1}{2}}}{\sqrt{2\pi}} - \Big[\frac{(2k+1)!!}{2^{k+1}k!}\Big]M^{k} + o(M^{k}) \end{aligned} \tag{5}$$

These asymptotic formulas are obtained by application of the integration by parts and the Stirling's formula

$$n! \sim \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}(1 + \frac{1}{12n} + \cdots).$$

Please see Appendix A for the complete proof. This allows us to derive the limiting equation of the recursion relation under the scaling limit $N \longrightarrow \infty$. Moreover, the asymptotic behaviour does not depend on the specific partition-type of μ. Hence this allows us to extract relations between absolute Gromov-Witten invariants. Precisely, under the scaling limit $N \rightarrow \infty$, we obtain a stratification of the relation (2) with respect to the degree of N. This stratification gives us a system of relations between absolute Gromov-Witten invariants.

4. Localization proof of the Witten conjecture

As an application of the new approach described in the previous section, I summarize the new proof of Witten's conjecture using localization method. Please refer to [21] for details. The famous Witten conjecture [38] claims that stable intersection theory on moduli space is equivalent to the "hermitian matrix model" of two-dimensional gravity. Precisely, E. Witten considered the generating function of the stable intersection theory on moduli space:

$$F(t_0, t_1, \cdots) = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle.$$

and formulated the *Witten's conjecture* as follows: The generating function $F(t_0, t_1, \cdots)$ is determined by the following two conditions

(1) The object $U = \partial^2 F/\partial t_0^2$ obeys the KdV equations.

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \dot{U}, \ddot{U}, \cdots),$$

where $\dot{U} = \partial U/\partial t_0$, $\ddot{U} = \partial^2 U/\partial t_0^2$, etc., are the derivatives of U with respect to t_0, and $R_{n+1}(U, \dot{U}, \ddot{U}, \cdots)$ are certain polynomials in U and its t_0 derivatives that are well known in the theory of the KdV equations (and can be defined by a recursion relation that is given below).

(2) In addition, F obeys the "string equation,"

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.$$

Now there exist several different approaches to this conjecture:

1. M. Kontsevich [22] gave the first proof by constructing the main identity which relates the stable intersection theory on $\overline{\mathcal{M}}_{g,n}$ to its proper combinatorial model. The string partition function $\tau(t)$:

$$\tau(t) = \exp \sum_{g=0}^{\infty} \langle \exp \sum_n t_n \mathcal{O}_n \rangle_g$$

admits an integral representation which involves the following integral over $N \times N$ Hermitian matrix Y of the form [2]

$$\tau(Z) = \rho(Z)^{-1} \int dY \cdot \exp \mathrm{Tr}\big[-\frac{1}{2} Z Y^2 + \frac{i}{6} Y^3 \big]$$

where Z is a second $N \times N$ Hermitian matrix, and $\rho(Z)$ is the one-loop integral

$$\rho(Z) = \int dY \cdot \exp \big[-\frac{1}{2} \mathrm{Tr}\, Z Y^2 \big]$$

2. A. Okounkov-R. Pandharipande [33] gave another approach through the enumeration of branched covering of $\mathbb{P}^1$ using the ELSV-formula [6]:

$$\frac{H_{g,\mu} \cdot |\mathrm{Aut}\ \mu|}{(2g - 2 + |\mu| + l(\mu))!} = \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod(1 - \mu_i \psi_i)}$$

3. M. Mirzhakani [32] derived the Virasoro constraints by connecting the stable intersection theory on $\overline{\mathcal{M}}_{g,n}$ to the Weil-Petersen volume and by using the McShane identity on the Weil-Petersen volume.
4. M. Kazarian-S. Lando [19] obtained an algebro-geometric proof by using the ELSV-formula and the PDEs which govern the generating series of Hurwitz numbers to derive the KdV-equation.

There are a couple of equivalent formulations for the Witten conjecture, namely the Virasoro constraints and the recursion relation for the correlation functions of topological gravity.

- **The Virasoro constraints:** The KdV-hierarchy can be expressed as linear, homogeneous differential equations for the τ-function [2]

$$L_n \cdot \tau = 0, \qquad (n \geq -1)$$

where L_n denote the differential operators

$$\begin{aligned} L_{-1} &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_0} + \sum_{k=1}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4}\tilde{t}_0^2 \\ L_0 &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_1} + \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16} \\ L_n &= -\frac{1}{2}\frac{\partial}{\partial \tilde{t}_{n-1}} + \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{1}{4}\sum_{i=1}^{n}\frac{\partial^2}{\partial \tilde{t}_{i-1}\partial \tilde{t}_{n-i}} \end{aligned}$$

- **The recursion relation for the correlation functions of topological gravity:** R. Dijkgraaf, E. Verlinde, and H. Verlinde derived [2, 3, 37], through physical arguments, the following recursion relation for the correlation functions of topological gravity and showed that it is equivalent to the Virasoro constraints.

$$\begin{aligned} \langle \tilde{\sigma}_n \prod_{k\in S}\tilde{\sigma}_k\rangle_g =& \sum_{k\in S}(2k+1)\langle \tilde{\sigma}_{n+k-1}\prod_{l\neq k}\tilde{\sigma}_l\rangle_g + \frac{1}{2}\sum_{a+b=n-2}\langle \tilde{\sigma}_a\tilde{\sigma}_b\prod_{l\in S}\tilde{\sigma}_l\rangle_{g-1} \\ &+ \frac{1}{2}\sum_{S=X\cup Y, a+b=n-2, g_1+g_2=g}\langle \tilde{\sigma}_a\prod_{k\in X}\tilde{\sigma}_k\rangle_{g_1}\langle \tilde{\sigma}_b\prod_{l\in Y}\tilde{\sigma}_l\rangle_{g_2} \end{aligned} \tag{6}$$

where $\tilde{\sigma}_n = [(2n+1)!!]\sigma_n = [(2n+1)!!]\psi^n$ and

$$\langle \tilde{\sigma}_{k_1}\cdots\tilde{\sigma}_{k_l}\rangle_g = \Big[\prod_{i=1}^{l}(2k_i+1)!!\Big]\int_{\overline{\mathcal{M}}_{g,l}}\psi_1^{k_1}\cdots\psi_l^{k_l}$$

The above recursion relation (6) has the same degeneration type as that of the Cut-and-Join relation. It is proved in [21] to be the limiting equation of the Cut-and-Join relation obtained by applying localization technique on the relative stable moduli $\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$. We summarize the proof below:

Let ω be the trivial class and $B=\{0,1,2,\cdots,r-2\}$. Then the class $\omega\cdot\mathrm{Br}^*\prod_{k\in B}\big(H-k\big)$ has strictly less dimension than the virtual dimension of the relative stable moduli $\overline{\mathcal{M}}_g(\mathbb{P}^1;\{\infty\}\mid|\mu|,\mu)$. Hence the relation (2) becomes

$$0 = r!\Gamma_r + (r-1)!\Gamma_{r-1}$$

As was explained in the previous section, the fixed curves that are mapped to p_{r-1} are precisely the curves obtained by performing the Cut-and-Join operation to the unique curve $\mathcal{C}_r$. This gives the following Cut-and-Join relation [28, 20]:

$$r\Gamma_r = \sum_{i=1}^{n}\Big[\sum_{j\neq i}\frac{\mu_i+\mu_j}{1+\delta_{\mu_j}^{\mu_i}}\Gamma_J^{ij} + \sum_{p=1}^{\mu_i-1}\frac{p(\mu_i-p)}{1+\delta_{\mu_i-p}^{p}}\Big(\Gamma_{C1}^{i,p} + \sum_{g_1+g_2=g,\nu_1\cup\nu_2=\nu}\Gamma_{C2}^{i,p}\Big)\Big] \tag{7}$$

where $\Gamma_J, \Gamma_{C1}, \Gamma_{C2}$ denote the contributions from Join-curve, Cut-of-type-I, and Cut-of-type-II, respectively. Precisely they are defined as follows:

- The unique fixed curve that is mapped to the branching point p_r

$$\Gamma_r = \frac{1}{|\mathrm{Aut}\mu|}\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g^{\vee}(1)}{\prod(1-\mu_i\psi_i)}$$

- Join curve that is obtained by joining i-th and j-th marked points:

$$\Gamma_J^{ij} = \frac{1}{|\mathrm{Aut}\eta|}\prod_{k=1}^{n-1}\frac{\eta_k^{\eta_k}}{\eta_k!}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\Lambda_g^{\vee}(1)}{\prod(1-\eta_k\psi_k)}, \qquad \eta\in J_{ij}(\mu)$$

- Cut curve that is obtained by pinching around the i-th marked point:

$$\Gamma_{C1}^{i} = \frac{1}{|\mathrm{Aut}\nu|}\prod_{k=1}^{n+1}\frac{\nu_k^{\nu_k}}{\nu_k!}\int_{\overline{\mathcal{M}}_{g-1,n+1}}\frac{\Lambda_{g-1}^{\vee}(1)}{\prod(1-\nu_k\psi_k)}, \qquad \nu\in C_i(\mu)$$

- Cut curve that is obtained by splitting around the i-th marked point:

$$\Gamma_{C2}^{i} = \Big[\prod_{k=1}^{n+1}\frac{\nu_k^{\nu_k}}{\nu_k!}\Big]\prod_{s=1,2}\frac{1}{|\mathrm{Aut}\ \nu_s|}\int_{\overline{\mathcal{M}}_{g_s,n_s}}\frac{\Lambda_{g_s}^{\vee}(1)}{\prod(1-\nu_{s,k}\psi_k)}, \qquad \nu\in C_i(\mu)$$

where $\Lambda_g^{\vee}(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g\lambda_g$ is the total Chern-class of the dual Hodge bundle. Applying the scaling limit $N\to\infty$ gives a stratification for $\Gamma_r, \Gamma_J, \Gamma_{C1}$, and Γ_{C2} with respect to N, i.e. we have expansions of the form

$$\Big[\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\Big]\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda_g^{\vee}(1)}{\prod(1-\mu_i\psi_i)} = \sum_{(k_i)}\Big[\prod_{i=1}^{n}\frac{\mu_i^{\mu_i+k_i}}{\mu_i!}\Big]\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i} + \text{lower } N\text{-degree terms}$$

where $(k_i) = (k_1,\cdots,k_n)$ runs over the sequences of non-negative integers such that $\sum_{i=1}^{n}k_i = 3g-3+n = \dim_{\mathbb{C}}\overline{\mathcal{M}}_{g,n}$. Note that the top-degree terms consist of Hodge integrals of only ψ-classes since the total Chern-class of dual Hodge-bundle $\Lambda_g^{\vee}(1) = 1-\lambda_1+\cdots\pm\lambda_g$ do not involve the scaling parameter N. By applyting the asymptotic formulas (.1), we obtain a system of relations between linear Hodge integrals on $\overline{\mathcal{M}}_{g,n}$ from (7). The highest N-degree relation turns out to be trivial:

$$(x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i} - (x_1+\cdots+x_n)\prod\frac{x_i^{k_i-1/2}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_i^{k_i} = 0$$

The second-highest N-degree relation is the following:

$$
\begin{aligned}
0 = \sum_{i=1}^{n} \Big[& \frac{(2k_i+1)!!}{2^{k_i+1}k_i!} x_i^{k_i} \prod_{j\neq i} \frac{x_j^{k_j-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_j^{k_j} \quad - \\
& \sum_{k+l=k_i-2} \frac{(2k+1)!!(2l+1)!!}{2^{k_i+1}k_i!} x_i^{k_i} \prod_{j\neq i} \frac{x_j^{k_j-1/2}}{\sqrt{2\pi}} \Big(\int_{\overline{\mathcal{M}}_{g-1,n+1}} \psi_1^k \psi_2^l \prod \psi_j^{k_j} \\
& + \sum_{g_1+g_2=g, I\cup J=\{2,\cdots,n\}} \int_{\overline{\mathcal{M}}_{g_1,|I|+1}} \psi_1^k \prod_{j\in I} \psi_j^{k_j} \int_{\overline{\mathcal{M}}_{g_2,|J|+1}} \psi_1^l \prod_{j\in J} \psi_j^{k_j} \Big) \\
& - \sum_{j\neq i} \frac{(x_i+x_j)^{k_i+k_j-1/2}}{\sqrt{2\pi}} \prod_{l\neq i,j} \frac{x_l^{k_l-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n-1}} \psi^{k_i+k_j-1} \prod \psi_l^{k_l} \Big]
\end{aligned} \tag{8}
$$

This relation is identical to the recursion relation for the correlation functions of topological gravity (6) which can be seen as follows: Introduce formal variables $s_i \in \mathbb{R}_{>0}$ and recall the Laplace Transformation:

$$
\int_0^\infty \frac{x^{k-1/2}}{\sqrt{2\pi}} e^{-x/2s} dx = (2k-1)!!\ s^{k+1/2}, \qquad \int_0^\infty x^k e^{-x/2s} dx = k!\ (2s)^{k+1}
$$

After taking the Laplace transformation to the above relation (8), we recover the recursion relation for the correlation functions of topological gravity (6)

$$
\begin{aligned}
\langle \tilde{\sigma}_n \prod_{k\in S} \tilde{\sigma}_k \rangle_g \quad = \quad & \sum_{k\in S} (2k+1) \langle \tilde{\sigma}_{n+k-1} \prod_{l\neq k} \tilde{\sigma}_l \rangle_g + \frac{1}{2} \sum_{a+b=n-2} \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l\in S} \tilde{\sigma}_l \rangle_{g-1} \\
& + \frac{1}{2} \sum_{S=X\cup Y, a+b=n-2, g_1+g_2=g} \langle \tilde{\sigma}_a \prod_{k\in X} \tilde{\sigma}_k \rangle_{g_1} \langle \tilde{\sigma}_b \prod_{l\in Y} \tilde{\sigma}_l \rangle_{g_2}
\end{aligned}
$$

Since this recursion relation is equivalent to the Virasoro constraints and the Witten's conjecture, this finishes the proof of Witten's conjecture through localization technique. As a remark, the system of relations given by the stratification of (7) may give more identities between linear Hodge-integrals.

5. Proof of Marinõ-Vafa formula

In this section, I summarize the survey note in [30] about the recent proof of Marinõ-Vafa formula in [28]. Based on the duality between open topological string theory on the deformed conifold T^*S^3 and the closed topological string theory on the resolved conifold, M. Marinõ and C. Vafa [31] conjectured a closed formula about the generating series of the triple Hodge integrals for all genera and any number of marked points in terms of the Chern-Simons invariants, or equivalently in terms of the representations and combinatorics of symmetric groups. The precise statement is as follows:

The Marinõ-Vafa conjecture is an identity between the geometry of the moduli spaces of stable curves and Chern-Simons knot invariants, or the combinatorics of the representation theory of symmetric groups. Let us first introduce the geometric side. For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 1)$, we define the triple Hodge integral to be,

$$\mathcal{G}_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

where the coefficient

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|Aut(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

These expressions arise naturally from localization computations on the moduli spaces of relative stable maps into $\mathbf{P}^1$ with ramification type μ at ∞.We now introduce the generating series

$$\mathcal{G}_\mu(\lambda;\tau) = \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}G_{g,\mu}(\tau).$$

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for any partition μ. These p_{μ_j} correspond to $\mathrm{Tr}\, V^{\mu_j}$ in the notations of string theorists. The generating series for all genera and all possible marked points are defined to be

$$\mathcal{G}(\lambda;\tau;p)^\bullet = \exp\Big(\sum_{|\mu|\geq 1} G_\mu(\lambda;\tau)p_\mu\Big),$$

which encode complete information of the triple Hodge integrals we are interested in.

Next I introduce the representation theoretical side. Let χ_μ denote the character of the irreducible representation of the symmetric group $S_{|\mu|}$, indexed by μ with $|\mu| = \sum_j \mu_j$. Let $C(\mu)$ denote the conjugacy class of $S_{|\mu|}$ indexed by μ. Introduce

$$\mathcal{W}_\mu(\lambda) = \prod_{1\leq a<b\leq l(\mu)}\frac{\sin\left[(\mu_a-\mu_b+b-a)\lambda/2\right]}{\sin\left[(b-a)\lambda/2\right]}\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i}2\sin\left[(v-i+l(\mu))\lambda/2\right]}.$$

This has an interpretation in terms of quantum dimension in Chern-Simons knot theory. We define the following generating series

$$\mathcal{R}(\lambda;\tau;p)^\bullet = \sum_{|\mu|\geq 0}\left(\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda)\right)p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and

$$\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i)$$

for a partition μ which is also standard for representation theory of symmetric groups. There is the relation $z_\mu = |Aut(\mu)|\mu_1 \cdots \mu_{l(\mu)}$. Finally we can give the precise statement of *the Mariño-Vafa conjecture:*

$$\textbf{Marinõ-Vafa conjecture:} \qquad \mathcal{G}(\lambda;\tau;p)^\bullet = \mathcal{R}(\lambda;\tau;p)^\bullet$$

This conjecture was first proved by C.C. Liu, K. Liu, and J. Zhou by showing that both sides have the same initial data, i.e.;

$$\mathcal{G}(\lambda,0,p)^\bullet = \exp\Big(\sum_{d=1}^{\infty} \frac{p_d}{2d\,\sin(\frac{\lambda d}{2})}\Big) = \mathcal{R}(\lambda,0,p)^\bullet$$

and satisfy the following Cut-and-Join relation: for $\Omega = \mathcal{G}^\bullet$ and $\Omega = \mathcal{R}^\bullet$, we have

$$\frac{\partial \Omega}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left(ijp_{i+j}\frac{\partial^2 \Omega}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial \Omega}{\partial p_{i+j}} \right).$$

Since this Cut-and-Join relation completely determines Ω for any given initial condition, we conclude the identity of $\mathcal{G}^\bullet$ and $\mathcal{R}^\bullet$ which is the Marinõ-Vafa conjecture.

Now let us explain how the new approach I illustrated in this paper applies to this case. Let $\pi : \mathcal{U}_{g,\mu} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ and $P : \mathcal{T}_{g,\mu} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ be the universal domain curve and the universal target, respectively. There is an evaluation map $F : \mathcal{U}_{g,\mu} \to \mathcal{T}_{g,\mu}$ and a contraction map $\tilde{\pi} : \mathcal{T}_{g,\mu} \to \mathbb{P}^1$. Let $\mathcal{D}_{g,\mu} \subset \mathcal{U}_{g,\mu}$ be the divisor corresponding to the $l(\mu)$ marked points. Define

$$V_D = R^1\pi_*(\mathcal{O}_{\mathcal{U}_{g,\mu}}(-\mathcal{D}_{g,\mu})) \qquad \text{and} \qquad V_{D_d} = R^1\pi_*\tilde{F}^*\mathcal{O}_{\mathbb{P}^1}(-1),$$

where $\tilde{F} = \tilde{\pi}\circ F : \mathcal{U}_{g,\mu} \to \mathbb{P}^1$. The fibers of V_D and V_{D_d} at

$$\big[f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m]\ \big] \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$$

are $H^1(C,\mathcal{O}_C(-D))$ and $H^1(C,\tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))$, respectively, where $D = x_1 + \ldots + x_{l(\mu)}$, and $\tilde{f} = \pi[m]\circ f$. Note that $H^0(C,\mathcal{O}_C(-D)) = H^0(C,\tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, so V_D and V_{D_d} are vector bundles of ranks $l(\mu)+g-1$ and $d+g-1$, respectively. The obstruction bundle

$$V = V_D \oplus V_{D_d}$$

is a vector bundle of rank $r = 2g-2+d+l(\mu) = \mathrm{vdim}\,\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$. We integrate the equivariant Euler class of V over the relative stable moduli $\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)$ to obtain

$$K^\bullet_\mu(\lambda) = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1,\mu)]^{vir}} e_T(V)$$

where $K^\bullet_\mu(\lambda)$ is of zero u-degree and depends on μ and λ. On the other hand, the localization computation gives a relation of the form (3). In this case, the 'previously known data' in (3) turns out to be double Hurwitz numbers. Precisely, we reach the following convolution formula of Hodge integrals with double Hurwitz numbers

$$K^\bullet_\mu(\lambda) = \sum_{|\nu|=|\mu|} \mathcal{G}^\bullet_\nu(\lambda,\tau,p) z_\nu \Phi^\bullet_{\nu,\mu}(-i\tau\lambda)$$

where $\Phi^\bullet(\lambda)$ is a generating series of double Hurwitz numbers. This convolution formula can be inverted to give the convolution expression of Hodge integrals [27]

$$\mathcal{G}^\bullet(\lambda,\tau,p) = \sum_{|\mu|\geq 0} z_\mu K^\bullet_\mu(\lambda)\Phi^\bullet_\mu(i\tau\lambda,p,1)$$

It is a direct consequence, from this expression, that $\mathcal{G}^\bullet$ satisfies the Cut-and-Join relation since the generating series of double Hurwitz numbers $\Phi^\bullet$ also satisfies it, hence finishing the proof of the Marinõ-Vafa formula. Let us end this section with several consequences [42] of the Marinõ-Vafa formula obtained by comparing the coefficients in τ in the Taylor expansions of the two expressions $\mathcal{G}^\bullet$ and $\mathcal{R}^\bullet$: we have a simple proof of the λ_g-conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\cdots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\cdot\frac{|B_{2g}|}{(2g)!}, \tag{9}$$

and the following identities for Hodge integrals

$$\int_{\overline{\mathcal{M}}_g}\lambda_{g-1}^3 = \int_{\overline{\mathcal{M}}_g}\lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}, \tag{10}$$

$$\int_{\overline{\mathcal{M}}_{g,1}}\frac{\lambda_{g-1}}{1-\psi_1} = b_g\sum_{i=1}^{2g-1}\frac{1}{i} - \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1,g_2>0}}\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}b_{g_1}b_{g_2}, \tag{11}$$

where B_{2g} are Bernoulli numbers, $b_g = 1$ if $g=0$, and $b_g = \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}$ if $g>0$.

6. Various recursion relations

In this section, I present a general method to obtain recursion relations for the Hodge integrals with up-to three λ-classes involved in it. This method gives a useful algorithm when one computes the Gromov-Witten invariants for the point target.

• **Recursion relation for linear Hodge integrals.** We have the following recursion relation that computes linear Hodge integrals, i.e. any Hodge integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_j$$

where $j+\sum k_i = 3g-3+n$ is explicitly expressed as a combination of the Hodge integrals of the same form that is strictly lower in the ordering with respect to $2g+n$. Here is the explicit formula [20]: Let $e=(k_1,\cdots,k_n)$ be a partition where k_i's are allowed to be zero.

Theorem 6.1. *For any partition μ and e with $|e| < |\mu| + l(\mu) - \chi$, we have*

$$\big[\lambda^{l(\mu)-\chi}\big] \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\lambda) z_\nu \mathcal{D}^\bullet_{\nu,e}(\lambda) = 0 \tag{12}$$

where the sum is taken over all partitions ν of the same size as μ.

Here $\big[\lambda^a\big]$ means taking the coefficient of λ^a. Let me first introduce some notations;

$$\begin{aligned}
\mathcal{D}_{g,\nu,e} = \quad & \frac{\nu_1^{\nu_1-2}}{\nu_1!} && \text{, if } \big(g, l(\nu)+l(e)\big) = \big(0,1\big) \\
& \frac{1}{|\mathrm{Aut}\ \nu|} \frac{\nu_1^{\nu_1}\nu_2^{\nu_2}}{\nu_1!\nu_2!} \frac{1}{\nu_1+\nu_2} && \text{, if } \big(g, l(\nu), l(e)\big) = \big(0,2,0\big) \\
& \frac{\nu_1^{\nu_1}}{\nu_1!} \sum_{k=0}^{e_1} \frac{1}{\nu_1^{1+k}} \binom{e_1}{k} && \text{, if } \big(g, l(\nu), l(e)\big) = \big(0,1,1\big) \\
& \frac{1}{l(e)!\ |\ \mathrm{Aut}\ \nu\ |} \Big[\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}\Big] \int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}} \frac{\Lambda_g^\vee(1) \prod_{j=1}^{l(e)} (1-\psi_j)^{e_j}}{\prod_{i=1}^{l(\nu)} \big(1-\nu_i\psi_i\big)} && \text{, otherwise} \\
\mathcal{D}(\lambda,p,q) = \quad & \sum_{|\nu|\geq 1} \sum_{g\geq 0} \lambda^{2g-2+l(\nu)} p_\nu q_e \mathcal{D}_{g,\nu} \\
\mathcal{D}^\bullet(\lambda,p,q) = \quad & \exp\big(\mathcal{D}(\lambda,p,q)\big) =: \sum_{|\nu|\geq 0} \lambda^{-\chi+l(\nu)} p_\nu q_e \mathcal{D}^\bullet_{\chi,\nu,e} = \sum_{|\nu|\geq 0} p_\nu q_e \mathcal{D}^\bullet_\nu(\lambda)
\end{aligned}$$

where p_i, q_j's are formal variables with $p_\nu = p_{\nu_1} \times \cdots \times p_{\nu_{l(\nu)}}$ and $q_e = q_{e_1} \times \cdots \times q_{e_{l(e)}}$.

Proof. For any given μ and χ such that $|\mu| + l(\mu) > \chi$, applying the localization formula to the class $\prod_{j=1}^n \psi_j^{k_j} \mathrm{ev}_j^* H$ where H is the hyperplane class of $\mathbb{P}^1$ yields;

$$\begin{aligned}
0 &= \int_{\overline{\mathcal{M}}_{\chi,n}(\mathbb{P}^1,\mu)} \prod_{j=1}^n \psi_j^{k_j} \mathrm{ev}_j^* H \qquad \text{since} \quad \deg \prod_{j=1}^n \psi_j^{k_j} \mathrm{ev}_j^* H < \dim \overline{\mathcal{M}}_{\chi,n}(\mathbb{P}^1,\mu) \\
&= \sum_{\Gamma_0 \in G^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|} \int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n (u-\psi_j)^{k_j} \mathrm{ev}_j^* H_T}{e_T(\mathcal{N}^{vir}_{\Gamma_0})} \\
&\quad + \sum_{\Gamma \in G^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_\Gamma|} \int_{\overline{\mathcal{M}}_\Gamma} \frac{\prod_{j=1}^n (u-\psi_j)^{k_j} \mathrm{ev}_j^* H_T}{e_T(\mathcal{N}^{vir}_\Gamma)}
\end{aligned}$$

Here H_T is the lift of H to the equivariant hyperplane class. Choose H in such a way that $H(0) = 0$ and $H(\infty) = u$, then we have $H_T(0) = u$ and $H_T(\infty) = 0$. Also let $u = 1$ for simplicity, then the formula reduces to:

$$\sum_{\Gamma_0 \in \overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|} \int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma_0})} + \sum_{\Gamma \in \overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_\Gamma|} \int_{\overline{\mathcal{M}}_\Gamma} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_\Gamma)} = 0$$

where $\overline{G}^0_{\chi,n}$ and $\overline{G}_{\chi,n}$ are the set of graphs corresponding to fixed locus with $m = 0$ and $m > 0$ with all the marked points $z_1, \cdots, z_n$ concentrated on the vertices in $V(\Gamma_0)^{(0)}$ and $V(\Gamma)^{(0)}$, respectively. We can compute the summand for Γ_0 as follows: (please refer to Appendix A for details.)

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma_0})} &= \Big[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}\Big] \times \Big[\prod_I \int_{\{pt\}} \frac{1}{\mu_{v,1}}\Big] \times \Big[\prod_{II} \int_{\{pt\}} \frac{1}{\frac{1}{\mu_{v,1}} + \frac{1}{\mu_{v,2}}}\Big] \\
&\times \Big[\prod_S \int_{\overline{\mathcal{M}}_{g(v),l(\mu(v))+l(e(v))}} \frac{\Lambda^\vee_{g(v)}(1) \prod_{j=1}^{j(v)} (1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\mu(v))} \big(\frac{1}{\mu_{v,i}} - \psi_{v,i}\big)}\Big] \\
=& \Big[\prod_I \mu_{v,1} \frac{\mu_{v,1}^{\mu_{v,1}-2}}{\mu_{v,1}!}\Big] \times \Big[\prod_{II} \mu_{v,1}\mu_{v,2} \frac{\mu_{v,1}^{\mu_{v,1}} \mu_{v,2}^{\mu_{v,2}}}{\mu_{v,1}!\mu_{v,2}!} \frac{1}{\mu_{v,1}+\mu_{v,2}}\Big] \\
&\times \Big[\prod_S \big(\prod_{i=1}^{l(\mu(v))} \mu_{v,i}\big)\Big(\prod_{i=1}^{l(\mu(v))} \frac{\mu_{v,i}^{\mu_{v,i}}}{\mu_{v,i}!}\Big) \int_{\overline{\mathcal{M}}_{g(v),l(\mu(v))+l(e(v))}} \frac{\Lambda^\vee_{g(v)}(1) \prod_{j=1}^{j(v)} (1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\mu(v))} \big(1 - \mu_{v,i}\psi_{v,i}\big)}\Big] \\
=& \prod_{V(\Gamma_0)} z_{\mu(v)} j(v)! \mathcal{D}_{g(v),\mu(v)}
\end{aligned}$$

Similarly, we can compute the summand for Γ as follows:

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{\Gamma}} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma})} &= \Big[\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}\Big] \times \Big[\prod_I \int_{\{pt\}} \frac{1}{\nu_{v,1}}\Big] \times \Big[\prod_{II} \int_{\{pt\}} \frac{1}{\frac{1}{\nu_{n,1}} + \frac{1}{\nu_{v,2}}}\Big] \\
&\times \Big[\prod_S \int_{\overline{\mathcal{M}}_{g(v),l(\nu(v))+l(e(v))}} \frac{\Lambda^\vee_{g(v)}(1) \prod_{j=1}^{j(v)} (1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\nu(v))} \big(\frac{1}{\nu_{v,i}} - \psi_{v,i}\big)}\Big] \times \Big[\int_{\overline{\mathcal{M}}^{(1)}_\Gamma} \frac{-\prod \nu_i}{1+\psi^t}\Big] \\
=& \Big[\prod_I \nu_{v,1} \frac{\nu_{v,1}^{\nu_{v,1}-2}}{\nu_{v,1}!}\Big] \times \Big[\prod_{II} \nu_{v,1}\nu_{v,2} \frac{\nu_{v,1}^{\nu_{v,1}} \nu_{v,2}^{\nu_{v,2}}}{\nu_{v,1}!\nu_{v,2}!} \frac{1}{\nu_{v,1}+\nu_{v,2}}\Big] \\
&\times \Big[\prod_S \big(\prod_{i=1}^{l(\nu(v))} \nu_{v,i}\big)\Big(\prod_{i=1}^{l(\nu(v))} \frac{\nu_{v,i}^{\nu_{v,i}}}{\nu_{v,i}!}\Big) \int_{\overline{\mathcal{M}}_{g(v),l(\nu(v))+l(e(v))}} \frac{\Lambda^\vee_{g(v)}(1) \prod_{j=1}^{j(v)} (1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\nu(v))} \big(1 - \nu_{v,i}\psi_{v,i}\big)}\Big] \\
&\times \Big[(-1)^{-\chi+l(\mu)+l(\nu)} \prod_{i=1}^{l(\nu)} \nu_i\Big] \int_{\overline{\mathcal{M}}^{(1)}_\Gamma} \big(\psi^t\big)^{-\chi_\infty+l(\mu)+l(\nu)-1} \\
=& \Big[\prod_{V(\Gamma_0)} z_{\nu(v)}\, j(v)!\, \mathcal{D}_{g(v),\nu(v),e(v)}\Big] \times \Big[(-1)^{-\chi+l(\mu)+l(\nu)} \prod_{i=1}^{l(\nu)} \nu_i\Big] \int_{\big[\overline{\mathcal{M}}^{(1)}_\Gamma\big]^{vir}} \big(\psi^t\big)^{-\chi_\infty+l(\mu)+l(\nu)-1}
\end{aligned}$$

And the integration over $\overline{\mathcal{M}}_\Gamma^{(1)}$ can be related to Double Hurwitz Numbers as follows:

$$H^\bullet_{\chi_\infty}(\mu,\nu) = \frac{\big(-\chi_\infty + l(\mu) + l(\nu)\big)!}{|\mathrm{Aut}\ \mu|\,|\mathrm{Aut}\nu|} \int_{[\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\mu,\nu)//\mathbb{C}^*]^{vir}} (\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1} =$$
$$\frac{\big(-\chi_\infty + l(\mu) + l(\nu)\big)!}{|\mathrm{Aut}\ \mu|\,|\mathrm{Aut}\nu|\Big(\prod n_k!\Big)\Big(\prod_{V(\Gamma)^{(1)}} |\mathrm{Aut}\ \mu(v)|\ |\mathrm{Aut}\ \nu(v)|\Big)} \int_{\big[\overline{\mathcal{M}}_\Gamma^{(1)}\big]^{vir}} \big(\psi^t\big)^{-\chi_\infty+l(\mu)+l(\nu)-1}$$

since marked points in $\overline{\mathcal{M}}_\Gamma^{(1)}$ are ordered. Recall that $|A_{\Gamma_0}|$ and $|A_\Gamma|$ are given by:

$$\begin{aligned} |A_{\Gamma_0}| =& \Big(\prod_{i=1}^{l(\mu)} \mu_i\Big) \prod_k m_k!\ \big(j(v_k)!\,|\,\mathrm{Aut}\ \mu(v_k)\,|\big)^{m_k} \\ |A_\Gamma| =& \Big(\prod_{i=1}^{l(\nu)} \nu_i\Big)\Big(\prod_k m_k!\ \big(j(v_k)!\,|\,\mathrm{Aut}\ \nu(v_k)\,|\big)^{m_k}\Big)|\mathrm{Aut}\ \mu\,| \\ &\times \Big(\prod n_k!\Big)\Big(\prod_{V(\Gamma)^{(1)}} j(v)!\,|\,\mathrm{Aut}\ \mu(v)|\ |\mathrm{Aut}\ \nu(v)|\Big) \end{aligned}$$

Also observe that

$$\frac{\prod_{V(\Gamma_0)} z_{\mu(v)}\, j(v)!\,\mathcal{D}_{g(v),\mu(v),e(v)}}{\Big(\prod_{i=1}^{l(\mu)} \mu_i\Big) \prod_k m_k!\,\big(j(v)!\,|\,\mathrm{Aut}\ \mu(v_k)\,|\big)^{m_k}} = \frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\cdots,m_l}\prod_k \mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}$$

which is the coefficient of $\lambda^{-\chi+l(\mu)}p_\mu q_e$ in the expansion of $\mathcal{D}^\bullet(\lambda,p,q)$. Now the original equation can be simplified as follows;

$$\begin{aligned}
0 =& \sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|}\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma_0})} + \sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_\Gamma|}\int_{\overline{\mathcal{M}}_\Gamma} \frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma})}\\
=& \sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\cdots,m_l}\prod_k \mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}\\
&+ \sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \Big[\frac{1}{|V(\Gamma)^{(0)}|!}\binom{|V(\Gamma)^{(0)}|}{m_1,\cdots,m_l}\prod_k \mathcal{D}_{g(v_k),\nu(v_k),e(v_k)}\\
&\times \frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}\prod_{i=1}^{l(\nu)}\nu_i}{|\mathrm{Aut}\ \mu|\Big(\prod n_k!\Big)\Big(\prod_{V(\Gamma)^{(1)}}|\mathrm{Aut}\ \mu(v)|\ |\mathrm{Aut}\ \nu(v)|\Big)}\int_{\left[\overline{\mathcal{M}}^{(1)}_\Gamma\right]^{vir}}\left(\psi^t\right)^{-\chi_\infty+l(\mu)+l(\nu)-1}\Big]\\
=& \sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\cdots,m_l}\prod_k \mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}\\
&+ \sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \Big[\frac{1}{|V(\Gamma)^{(0)}|!}\binom{|V(\Gamma)^{(0)}|}{m_1,\cdots,m_l}\prod_k \mathcal{D}_{g(v_k),\nu(v_k),e(v_k)}\\
&\times \frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}z_\nu}{|\mathrm{Aut}\ \mu|\ |\mathrm{Aut}\ \nu|}\int_{[\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\mu,\nu)//\mathbb{C}^*]^{vir}}(\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1}\Big]\\
=&\mathcal{D}^\bullet_{\chi,\mu,e} + \sum_\nu \sum_{-\chi_\infty+l(\mu)+l(\nu)\neq 0} \frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e}\\
=&\sum_\nu\sum_{\chi_0,\chi_\infty}\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e}
\end{aligned}$$

where $\chi=\chi_0+\chi_\infty-2\,l(\nu)$ and the initial value formula for Double Hurwitz numbers is used at the last equality. Summing over χ yields that we have for all $|e|+\chi<|\mu|+l(\mu)$

$$\left[\lambda^{l(\mu)-\chi}\right]\sum_{|\nu|=|\mu|}\Phi^\bullet_{\mu,\nu}(-\lambda)z_\nu\mathcal{D}^\bullet_{\nu,e}(\lambda)=0$$

□

It is enough to consider the case of $\mu=(d)$ for some positive integers d to compute all Hodge integrals with at-most one λ-class. And in this case, we have a closed formula for the Double Hurwitz Numbers as follows;

Theorem 6.2. ([13], Theorem 3.1.) *Let $r = r^g_{(d),\beta}$. For $g \geq 0$, and $\beta \vdash d$ with n parts,*

$$H^g_{(d),\beta} = r!d^{r-1}\big[t^{2g}\big]\prod_{k\geq 1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} = \frac{r!d^{r-1}}{2^{2g}}\sum_{\lambda\vdash g}\frac{\xi_{2\lambda}S_{2\lambda}}{|Aut\ \lambda|}$$

where $r = 2g-1+l(\beta)$ and $c_1 = (\text{number of 1's in } \beta) - 1$, $c_k = (\text{number of } k\text{'s in } \beta)$ for $k > 1$. big$[t^{2g}\big]$ means taking the coefficient of t^{2g}.

Here the Double Hurwitz number is counted with multiplicity, hence in our notation it will read as follows

$$\frac{H^\bullet_{\chi_\infty}((d),\nu)}{(-\chi_\infty+1+l(\nu))!} = \frac{d^{-\chi_\infty+l(\nu)}}{|\text{Aut }\nu|}\big[t^{2g}\big]\prod_{k\geq 1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} \tag{13}$$

And in this case, (12) can be written as

$$\sum_{|\nu|=d}\sum_{\chi_0,\chi_\infty}\mathcal{D}^\bullet_{\chi_0,\nu,e}z_\nu(-1)^{-\chi_\infty+1+l(\nu)}\frac{d^{-\chi_\infty+l(\nu)}}{|\text{Aut }\nu|}\big[t^{2g}\big]\prod_{k\geq 1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} = 0 \tag{14}$$

Now fix ν and consider the case where there are m vertices in $V(\Gamma)^{(0)}$. Then we have splitting of χ_0, ν, and e into $\{g_1,\cdots,g_m\}$, $\{\nu(v_1),\cdots,\nu(v_m)\}$, and $\{e(v_1),\cdots,e(v_m)\}$ such that $e(v_i)$'s are allowed to be empty and

$$\sum_{i=1}^m(2-2g_i) = \chi_0, \qquad \bigcup_{i=1,\cdots,m}\nu(v_i) = \nu, \qquad \bigcup_{i=1,\cdots,m}e(v_i) = e$$

Each vertex will correspond to a certain Hodge integral on $\overline{\mathcal{M}}_{g(v),l(\nu(v))+l(e(v))}$ with dimension $3g(v)-3+l(\nu(v))+l(e(v))$. There are conditions on m, $\chi, \chi_0, \chi_\infty, l(e(v))$, and $l(\nu(v))$: let $\overline{g}(v)$ denote $\sum_{w\neq v} g(w)$,

$$m \leq l(\nu), \quad l(\nu(v)) \leq l(\nu)-m+1, \quad \chi = \chi_0+\chi_\infty-2l(\nu), \quad l(e(v)) \leq l(e)$$
$$\chi_\infty \leq 2\min\{\ l((d)), l(\nu)\ \} = 2, \qquad \chi_0 = \sum_{i=1}^m(2-2g(v_i)) = 2m-2g(v)-2\overline{g}(v)$$

From these conditions, we can deduce that

$$\begin{aligned}3g(v) = \quad & 3m - 3\overline{g}(v) - \frac{3}{2}\chi_0 = 3m-3\overline{g}(v) - \frac{3}{2}\chi + \frac{3}{2}\chi_\infty - 3l(\nu)\\ \leq & -\frac{3}{2}\chi + 3m - \frac{3}{2}\big(2l(\nu)-\chi_\infty\big)\end{aligned}$$

where equality holds if and only if $\overline{g}(v) = \sum_{w\neq v} g(w) = 0$. Now we can find the upper bound for the dimension of Hodge integral as follows.

$$\begin{aligned}3g(v)-3+l(\nu(v))+l(e(v)) \leq\ & -\frac{3}{2}\chi+3m-3-\frac{3}{2}\big(2l(\nu)-\chi_\infty\big)+l(\nu(v))+l(e)\\ \leq\ & \Big[3g-3+l(e)+1\Big]+\Big[3m-3-3l(\nu)+\frac{3}{2}\chi_\infty+l(\nu)-m\Big]\\ \leq\ & \Big[3g-3+l(e)+1\Big]+\Big[2\big(m-l(\nu)\big)+\frac{3}{2}\big(\chi_\infty-2\big)\Big]\\ \leq\ & 3g-2+l(e)\end{aligned}$$

and the equality holds if and only if

$$m=l(\nu),\quad \chi_\infty=2,\quad e(w)=\emptyset \text{ for all } w\neq v,\quad g(w)=0 \text{ for all } w\neq v$$

,i.e. when each part of ν is splitted into separate vertices, and all the marked points other than the ramification divisor as well as all genuses are concentrated on one vertex on the 0-th side. Now we can compute any Hodge integral with at-most one λ-class as follows: Say we want to compute Hodge integrals of the form

$$\int_{\overline{M}_{g,n+1}}\psi_0^{k_0}\cdots\psi_n^{k_n}\lambda_j$$

where $j+\sum_{i=0}^n k_i=3g-2+n$. Assume $0\leq k_0\leq\cdots\leq k_n$ and let $e=(k_1,\cdots,k_n)$ and $\chi=2-2g$. Then for any positive integer d such that $d>\chi+|e|-1$, the recursion formula (14) expresses the top-dimensional Hodge integrals in terms of lower-dimensional Hodge integrals as follows:

$$\sum_{|\nu|=d}\Big[\Big(\prod_{i=1}^{l(\nu)}\frac{\nu_i^{\nu_i-1}}{\nu_i!}\Big)\frac{(-1)^{l(\nu)-1}d^{l(\nu)-2}}{n!}\sum_{i=1}^{l(\nu)}\frac{\nu_i^2}{|\mathrm{Aut}\ \hat{\nu}_i|}\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\Lambda_g^\vee(1)\prod_{j=1}^n(1-\psi_j)^{k_j}}{1-\nu_i\psi_0}\Big]$$

$$=\text{terms consisting of lower dimensional Hodge integrals only...}$$

where Aut $\hat{\nu}_i$ is the automorphism group of the partition $\hat{\nu}_i=(\nu_1,\cdots,\hat{\nu}_i,\cdots,\nu_{l(\nu)})$. In this expression, the Hodge integral term expands to:

$$\begin{aligned}&\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\Lambda_g^\vee(1)\prod_{j=1}^n(1-\psi_j)^{k_j}}{1-\nu_i\psi_0}\\ &=\int_{\overline{\mathcal{M}}_{g,n+1}}\big(1-\lambda_1+\lambda_2+\cdots+(-1)^g\lambda_g\big)\Big(\sum_{a_0=0}^{\infty}\nu_i^{a_0}\psi_0^{a_0}\Big)\prod_{j=1}^n\Big[\sum_{a_j=0}^{k_j}(-1)^{a_j}\binom{k_j}{a_j}\psi_j^{a_j}\Big]\\ &=\sum_{k+\sum a_j=3g-2+n}(-1)^{3g-2+n}\Big[\nu_i^{a_0}\prod_{j=1}^n\binom{k_j}{a_j}\Big]\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_0^{a_0}\times\cdots\times\psi_n^{a_n}\lambda_k\end{aligned}$$

Hence the previous expression can be written as

$$\text{(15)}\qquad \sum_{k+\sum a_j=3g-2+n}C_d(k,(a_j))\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_0^{a_0}\times\cdots\times\psi_n^{a_n}\lambda_k=\text{lower dimensional terms...}$$

where $C_d(k,(a_j))$ are constants defined as follows:

$$C_d\big(k,(a_j)\big)=\sum_{|\nu|=d}\Big[\Big(\prod_{i=1}^{l(\nu)}\frac{\nu_i^{\nu_i-1}}{\nu_i!}\Big)\frac{(-1)^{l(\nu)-1}d^{l(\nu)-2}}{n!}\sum_{i=1}^{l(\nu)}\frac{\nu_i^2}{|\mathrm{Aut}\ \hat{\nu_i}|}\Big(\nu_i^{a_0}\prod_{j=1}^{n}\binom{k_j}{a_j}\Big)\Big]$$

Now we have infinitely many linear relations of finitely many Hodge integrals of fixed dimension $3g-2+n$ with at-most one λ-class since the equation (15) holds for all positive integers d such that $d>1-2g+\sum k_j$. Moreover the coefficients $C_d(k,(a_j))$ form Vandermonde-type matrices and it can be proved that one can always find a set of positive integers $\{d_1,\cdots,d_l\}$ which will give linearly independent relations to solve for all the Hodge integrals of given dimension $3g-2+n$ with at-most one λ-class in terms of the values of lower-dim'l Hodge integrals with at-most one λ-class. So we just proved the following theorem;

Theorem 6.3. *Any given Hodge integral with one λ-class:*

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_j \tag{16}$$

where $k_1,\cdots,k_n\in\mathbb{N}\cup\{0\}$, $j\in\{0,1,2,\cdots,g\}$, is explicitly expressed as a polynomial in terms of lower-dimensional Hodge integrals with one λ-class. Therefore it computes all Hodge integrals with one λ-class.

It is clear that Theorem 6.3 can be implemented. We can use the formula (13) to compute Double Hurwitz numbers. And since the values of Hodge integrals will be too big to fit in the usual 4-byte integer data type, we will need a library for multi-precision computing. There are several free libraries on the web and GNU-MP is one of them which provides well-organized C++ class interfaces as well as documentations.

For any given Hodge integral with one λ-class

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_j$$

Let $e=(k_1,\cdots,k_{n-1})$ and $\chi=2-2g$. Start with $d=\chi-1+|e|,\cdots$ and find linear relations (12) as follows: Run over partitions ν of size d. For a fixed ν, run over pairs (χ_0,χ_∞) which satisfies $\chi=\chi_0+\chi_\infty-2\,l(\nu)$, $\chi_\infty\le 2$, $\chi_0,\chi_\infty\in 2\,\mathbb{Z}$. Now for a fixed pair (χ_0,χ_∞), we can compute Double Hurwitz Number $\mathcal{D}^\bullet_{\chi_\infty}(\,(d)\,,\nu)$ as follows: When $c_1\ge 0$, i.e. when there are one or more 1's in ν, we have

$$[t^{2g}]\prod_{k\ge1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k}=\sum_{(b_k)}\prod_{b_k\ne0}\sum_{(a_j^k)}\frac{k^{2b_k}}{2^{2b_k}\prod_j(2a_j^k+1)!}$$

where $b_k=0$ if $c_k=0$ and $\chi_\infty=2-2g_\infty$, $\sum_k b_k=g_\infty$, $\sum_{j=1,\cdots,c_k}a_j^k=b_k$ with $b_k,a_j^k\in\mathbb{N}\cup\{0\}$. When $c_1=-1$, i.e. when there is no 1 in ν, let h be the minimum numbered part of ν. Then we have $c_h \geq 1$ and

$$\Big(\frac{\sinh(t/2)}{t/2}\Big)^{-1}\Big(\frac{\sinh(ht/2)}{ht/2}\Big) = \frac{1}{h}\Big(e^{(h-1)t/2} + e^{(h-3)t/2} + \cdots + e^{(-h+1)t/2}\Big)$$

$$= \begin{cases} \sum_{m=0}^{\infty} \Big[\frac{1}{h\,2^{2m-1}(2m)!}\Big(\sum_l l^{2m}\Big)\Big]t^{2m}, & \text{when } h \text{ is even, } l = 1, 3, \cdots, h-1 \\ \frac{1}{h} + \sum_{m=0}^{\infty} \Big[\frac{1}{h\,2^{2m-1}(2m)!}\Big(\sum_l l^{2m}\Big)\Big]t^{2m}, & \text{when } h \text{ is odd, } l = 2, 4, \cdots, h-1 \end{cases}$$

Hence we have

$$\prod_{k\geq 1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} = \Big(\frac{\sinh(t/2)}{t/2}\Big)^{-1}\Big(\frac{\sinh(ht/2)}{ht/2}\Big)\Big(\frac{\sinh(ht/2)}{ht/2}\Big)^{c_h-1}\prod_{k>h}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k}$$

$$=\Big(\frac{1}{h}\delta_{h,\mathrm{odd}} + \sum_{m=0}^{\infty}\Big[\frac{1}{h\,2^{2m-1}(2m)!}\Big(\sum_l l^{2m}\Big)\Big]t^{2m}\Big)\Big(\frac{\sinh(ht/2)}{ht/2}\Big)^{c_h-1}\prod_{k>h}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k}$$

and in this case, for $g > 0$;

$$[t^{2g}]\prod_{k\geq 1}\Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} = \sum_{(b_k)}\Big[\frac{1}{h\,2^{2b_1-1}(2b_1)!}\Big(\sum_l l^{2b_1}\Big)\Big]\prod_{b_k\neq 0,\, k\geq h}\sum_{(a_j^k)}\frac{k^{2b_k}}{2^{2b_k}\prod_j(2a_j^k+1)!}$$

where $b_h = 0$ if $c_h = 1$ and $b_k = 0$ if $c_k = 0$ for $k \neq h$. Using these formulas and (13), we can compute Double Hurwitz numbers. In order to compute $\big[\lambda^{l(\nu)-\chi_0}\big]\mathcal{D}^{\bullet}_{\nu,e}(\lambda)$, first run over the number of vertices $m = 1, 2, \cdots, l(\nu)$. For each m, find all possible groupings of ν, e, and all possible splittings of χ_0. Note that e admits groupings with empty components. Now find all triples $(\nu(v), e(v), g(v))$ according to the equivalence condition of vertices discussed in Section 3. Then the contribution of $\big[\lambda^{l(\nu)-\chi_0}\big]\mathcal{D}^{\bullet}_{\nu,e}(\lambda)$ will be the product of the combination factor $1/\prod m_i!$ and the expansions of $\mathcal{D}_{g(v),\nu(v),e(v)}$ which can be obtained by

$$\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}\frac{\Lambda_g^{\vee}(1)\prod_{j=1}^{l(e)}\big(1-\psi_j\big)^{e_j}}{\prod_{i=1}^{l(\nu)}\big(1-\nu_i\psi_i\big)}$$

$$= \int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}[1-\lambda_1+\cdots+(-1)^g\lambda_g]\prod_{j=1}^{l(e)}[\sum_{(\tilde{l}_j)}(-1)^{\tilde{l}_j}\binom{e_j}{\tilde{l}_j}\psi_j^{\tilde{l}_j}]\prod_{i=1}^{l(\nu)}[\sum_{l_i}\nu_i^{l_i}\psi_i^{l_i}]$$

$$= \sum_{k,(l_i),(\tilde{l}_j)}(-1)^{k+\sum\tilde{l}_j}[\prod_{j=1}^{l(e)}\binom{e_j}{\tilde{l}_j}][\prod_{i=1}^{l(\nu)}\nu_i^{l_i}]\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}\lambda_k\prod_{i=1}^{l(\nu)}\psi_i^{l_i}\prod_{j=1}^{l(e)}\psi_j^{\tilde{l}_j}$$

where $l_i \geq 0$, $0 \leq \tilde{l}_j \leq e_j$, $0 \leq k \leq g$, and $k+\sum_i l_i+\sum_j \tilde{l}_j = 3g-3+l(\nu)+l(e)$. Some of them will have maximum dimension $3g-3+n$ for the situations described above. Those are treated as unknowns and all others are lower-dimensional Hodge integrals or initial values which are already computed. Summing over all pairs (χ_0, χ_∞) and ν will give a linear relation between Hodge integrals of the dimension $3g-3+n$. Now we can follow same step as above for other values of d and obtain more linear relations. Observe that the number of unknowns are independent of d and actually bounded by the number of partitions of $3g-3+n$, and hence we will have a system of linear relations which can be solved by simple Gaussian Elimination method. Thus in each dimension, it amounts to solve a matrix equation of size $N \times N$ where N is at-worst-case the number of partitions of dimension. Here is an example:

$$d=1, g=2, e=\emptyset, \quad d=2, g=2, e=\emptyset, \quad \text{and } d=3, g=2, e=\emptyset;$$

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = 0,\\
&7\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 15\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 31\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 - \frac{1}{240} = 0,\\
&25\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 90\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 301\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 - \frac{5}{48} = 0\\
\Longrightarrow &\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 = \frac{7}{5760}, \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 = \frac{1}{480}, \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{1}{1152}
\end{aligned}$$

The first value matches with λ_g-formula (9), since we have

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k = 0, \quad \text{for } m>0 \implies B_4 = -\frac{1}{30}$$

$$\binom{2*2+1-3}{2} \frac{2^{2*2-1}-1}{2^{2*2-1}} \frac{|B_{2*2}|}{(2*2)!} = \frac{7}{5760}$$

The second value matches with (11), since we have

$$b_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, \quad \text{for } g>0 \implies b_1 = \frac{1}{24}, \quad b_2 = \frac{7}{5760}$$

$$b_g \sum_{i=1}^{2g-1} \frac{1}{i} - \frac{1}{2} \sum_{g_1+g_2=g} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2} = \frac{7}{5760}\Big(1+\frac{1}{2}+\frac{1}{3}\Big) - \frac{1}{2}\frac{1}{3!}\Big(\frac{1}{24}\Big)^2 = \frac{1}{480}$$

And the last value matches with the result in [43], p36.

•**Recursion relation for quadratic Hodge integrals.** Consider the non-relative moduli space of stable curves of degree d, i.e. $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$. This space has dimension $2g-2+2d$. Introduce $\mathbb{C}^*$-action on $\mathbb{P}^1$ by $t \cdot [z, w] = [tz, w]$. Then we can apply localization technique on this moduli space with induced $\mathbb{C}^*$-action. The normal bundle contribution at a fixed locus can be computed to be

$$\frac{1}{e_T(\mathcal{N}_\Gamma^{vir})} = \Big[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i-1}}{\mu_i!}(-u^2)^{-\mu_i}\Big] \prod_{V_0} \Big[\frac{\Lambda_{g_0}^\vee(u)/u}{\prod\big(\frac{1}{\mu_i} - \psi_i/u\big)}\Big] \prod_{V_\infty} \Big[\frac{\Lambda_{g_\infty}^\vee(-u)/(-u)}{\prod\big(\frac{1}{\mu_i} + \psi_i/u\big)}\Big]$$

where μ is a partition determined by the splitting type of rational tails over the projective line $\mathbb{P}^1$. Let $V \to \overline{\mathcal{M}}_g(\mathbb{P}^1, d)$ be the vector bundle of deformation of domain curve. For a point $f : C \to \mathbb{P}^1$, we have $V|_f = H^1(C, \mathcal{O}_C)$. Then at a fixed locus parametrized by the graph Γ,

$$i_\Gamma^*\big(e_T(V)\big) = \Big[\prod_{V_0} \Lambda_{g_0}^\vee(\tau\, u)\Big] \cdot \Big[\prod_{V_\infty} \Lambda_{g_\infty}^\vee(\tau\, u)\Big]$$

where V_0 and V_∞ are the sets vertices corresponding to the components of the domain curve C that are contracted over 0 and ∞, respectively. Hence, the total contribution of a fixed locus parametrized by Γ is given by

$$\frac{i_\Gamma^*\big(e_T(V)\big)}{e_T(\mathcal{N}_\Gamma^{vir})} = a_\mu \Big[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}(-u^2)^{-\mu_i}\Big] \prod_{V_0} \Big[\frac{\Lambda_{g_0}^\vee(u)\Lambda_{g_0}^\vee(\tau\, u)/u}{\prod\big(1-\mu_i\psi_i/u\big)}\Big] \prod_{V_\infty} \Big[\frac{\Lambda_{g_\infty}^\vee(-u)\Lambda_{g_\infty}^\vee(\tau\, u)/(-u)}{\prod\big(1+\mu_i\psi_i/u\big)}\Big]$$

By letting $\tau = 1$ and using the Mumford's relation

$$\Lambda_g^\vee(u)\Lambda_g^\vee(-u) = (-1)^g u^{2g},$$

I obtained the following recursion relation for the quadratic Hodge integrals:

$$(17) \quad 0 = \sum_{n=1}^{d} \sum_{\Gamma_{l,n}} \frac{(-1)^{C_\Gamma}}{|\mathrm{Aut}\ \Gamma|} \Big[\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\Big] \prod_{V_0} \int_{\overline{\mathcal{M}}_{g_0,l'}} \frac{\Lambda_{g_0}^\vee(1)^2}{\prod(1-\mu_i\psi_i)} \prod_{V_\infty} \int_{\overline{\mathcal{M}}_{g_\infty,l''}} \frac{1}{\prod(1-\mu_i\psi_i)}$$

where $0 \le l \le g$ denotes the arithmetic genus of the fixed locus. Hence this recursion relation expresses the quadratic Hodge integrals in terms of Hodge integrals with ψ-classes only. As a remark, we have more precise relations by isolating the coefficients of homogeneous u-terms. This stratification in terms of u has a geometric explanation as the type of the corresponding graph Γ for the fixed locus. By the same argument as I did for the linear Hodge integral recursion relation, the relation (17) explicitly expresses any given quadratic Hodge integral in terms of lower-dimensional ones. Here's a very simple example for the case of $d = 1$ and $g = 2$:

$$\text{Case } g_0 = 2,\ g_\infty = 0: \quad \int_{\overline{\mathcal{M}}_{2,1}} \frac{(1-\lambda_1+\lambda_2)^2}{1-\psi} = -2\int_{\overline{\mathcal{M}}_{2,1}} \lambda_2\lambda_1\psi + 4\int_{\overline{\mathcal{M}}_{2,1}} \lambda_2\psi^2 - 2\int_{\overline{\mathcal{M}}_{2,1}} \lambda_1\psi^3 + \int_{\overline{\mathcal{M}}_{2,1}} \psi^4$$

$$\text{Case } g_0 = 0,\ g_\infty = 2: \quad \int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-\psi} \int_{\overline{\mathcal{M}}_{2,1}} \frac{1}{1+\psi} = \int_{\overline{\mathcal{M}}_{2,1}} \psi^4$$

$$\text{Case } g_0 = 1,\ g_\infty = 1: \quad \int_{\overline{\mathcal{M}}_{1,1}} \frac{(1-\lambda_1)^2}{1-\psi} \int_{\overline{\mathcal{M}}_{1,1}} \frac{1}{1+\psi} = \frac{-1}{24}\Big[\int_{\overline{\mathcal{M}}_{1,1}} \psi - 2\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1\Big]$$

Plugging in the values for the lower-dimensional Hodge integrals, we obtain

$$\int_{\overline{\mathcal{M}}_{2,1}} \lambda_2\lambda_1\psi = \frac{1}{5}\Big(\frac{1}{24}\Big)^2$$

which matches with the equation (10) and also to the prediction of $\lambda_g\lambda_{g-1}$-conjecture (Please look at the last section for the statement of $\lambda_g\lambda_{g-1}$-conjecture).

•**Recursion relation for triple Hodge integrals.** The Marinõ-Vafa formula can be used to compute triple Hodge integrals. In the original proof of the Marinõ-Vafa formula in [28], it is shown that

$$J^1_{g,\mu}(\tau) = -\frac{d}{d\tau} J^0_{g,\mu}(\tau) \tag{18}$$

which is the Cut-and-Join equation between the contributions of the fixed locus that are mapped to the branching points p_r and p_{r-1}. These fixed locus are classified in the proof of Witten conjecture. Hence we have a system of relations which are polynomial in τ. It can be shown that the above relation obeys the induction hypothesis for recursion algorithm. Combining with the linear- and quadratic- Hodge integrals as shown above, we can compute all triple Hodge integrals through the system of relations obtained from (18).

7. Future Research problems

In this section, I list the following open problems to which our new approach can be applied. Each of them has an equivalent formulation in the form of recursion relations which has the same structure as that of the Cut-and-Join relation.

7.1. **Generalized Witten conjecture.** Consider a series of integrable hierarchies KdV_r, where $r = 2, 3, \cdots$, called the generalized KdV, or Gelfand-Dickey hierarchies. E.Witten generalized his original conjecture [40, 41], suggesting that for each r there should exist moduli spaces and cohomology classes on them whose intersection numbers assemble into the formal τ-function of the KdV_r-hierarchy. The corresponding moduli spaces of higher spin curves are constructed and the zero-genus case of the conjecture has been proved in [16, 17]. A brief idea of the construction is as follows [36]: Let $a_1, \cdots, a_n \in \{0, \cdots, r-1\}$ be integers assigned to the marked points $x_1, \cdots, x_n$ such that $2g - 2 - \sum a_i$ is divisible by r. On a smooth curve C, there are r^{2g} different line bundles $\mathcal{T}$ with an identification

$$\mathcal{T}^{\otimes r} \simeq K(-\sum a_i x_i).$$

The space of smooth curves endowed with such a line bundle $\mathcal{T}$ is denoted by $\mathcal{M}^{1/r}_{g;a_1,\cdots,a_n}$. The compactified space, denoted by $\overline{\mathcal{M}}^{1/r}_{g;a_1,\cdots,a_n}$, is constructed in [16] and is called the *moduli space of stable r-spin curves.* The construction uses the *Jarvis-Vistoli twisted curves*, i.e. curves that are themselves endowed with an orbifold structure. It is a smooth stack with a finite projection embedding

$$p : \overline{\mathcal{M}}^{1/r}_{g;a_1,\cdots,a_n} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Its analogue of the Gromov-Witten classes is constructed in [16] and is called a *virtual class* $c_{g,n}^{1/r}(\mathbf{a})$ in $H^\bullet(\overline{\mathcal{M}}_{g;a_1,\cdots,a_n}^{1/r})$. In the physics notation, we write

$$\langle \sigma_{m_1,a_1}\cdots\sigma_{m_n,a_n}\rangle_g = \int_{\overline{\mathcal{M}}_{g;a_1,\cdots,a_n}^{1/r}} c_{g,n}^{1/r}(\mathbf{a})\psi_1^{m_1}\cdots\psi_n^{m_n}.$$

There is a conjectural recursion relation [26] for these intersection numbers:

$$\frac{h+1}{h}\langle\sigma_{m,1}\prod_{l=1}^{s}\sigma_{n_l,\alpha_l}\rangle_g = \sum_l \Big(n_l+\frac{\alpha_l}{h}\Big)\langle\sigma_{m+n_l-1,\alpha_l}\prod_{j\neq l}\sigma_{n_j,\alpha_j}\rangle_g + \frac{1}{2}\sum_{n=2}^{m}\sum_{\alpha\in I}$$
$$\Big[\langle\sigma_{n-2,\alpha}\sigma_{m-n,h-\alpha}\prod_{l=1}^{s}\sigma_{n_l,\alpha_l}\rangle_{g-1} + \sum_{\substack{S=X\cup Y\\ g=g_1+g_2}}\langle\sigma_{n-2,\alpha}\prod_{l\in X}\sigma_{n_l,\alpha_l}\rangle_{g_1}\langle\sigma_{m-n,h-\alpha}\prod_{l\in Y}\sigma_{n_l,\alpha_l}\rangle_{g_2}\Big]$$

where h is the dual Coxeter number such that $\alpha\in I$ if and only if $h-\alpha\in I$. It has the same structure as that of the Cut-and-Join relation. The approach illustrated in this paper can be applied by using localization technique on the moduli space of relative stable maps with higher spin structure and using the factorization rules [34, 35] for the virtual class $c_{g,n}^{1/r}(\mathbf{a})$.

7.2. **General Virasoro conjecture.** Let V be a non-singular projective variety. The *general Virasoro conjecture for* V asserts vanishing relations on the total Gromov-Witten potential

$$Z(V) = \exp\Big(\sum_{g\geq 0}\hbar^{g-1}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1\cdots k_n a_1\cdots a_n} t_{k_n}^{a_n}\cdots t_{k_1}^{a_1}\langle\tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle_g^V\Big).$$

Precisely, there are differential operators L_k [4, 5] which annihilates $Z(V)$, i.e.

$$L_k Z(V) = 0 \quad \text{for all } k\geq -1. \tag{19}$$

It is known that (19) is equivalent to the following recursion relation [9]:

$$0=\sum_{i=0}^{k+1}\Big(-\big[\frac{3-r}{2}\big]_i^k (R^i)_0^b\langle\langle\tau_{k-i+1,b}\rangle\rangle_g^V + \sum_{m=0}^{\infty}\big[\mu_a+m+\frac{1}{2}\big]_i^k (R^i)_a^b t_m^a\langle\langle\tau_{m+k-i,b}\rangle\rangle_g^V$$
$$+\frac{1}{2}\sum_{m=i-k}^{-1}(-1)^m\big[\mu_a+m+\frac{1}{2}\big]_i^k(R^i)^{ab}\big(\langle\langle\tau_{-m-1,a}\tau_{m+k-i,b}\rangle\rangle_{g-1}^V$$
$$+\sum_{g_1+g_2=g}\langle\langle\tau_{-m-1,a}\rangle\rangle_{g_1}^V\langle\langle\tau_{m+k-i,b}\rangle\rangle_{g_2}^V\big)\Big)+\frac{\delta_{g,0}}{2}(R^{k+1})_{ab}t_0^a t_0^b+\delta_{k,0}\delta_{g,1}\rho(V)$$

where R_a^b is the matrix associated to multiplication on the affine superspace $H(V)$ by the first Chern class $c_1(V)$ of V, defined by

$$R_a^b\gamma_b = c_1(V)\cup\gamma_a$$

This conjectural relation has the same structure as that of the Cut-and-Join relation except that the first Chern class of X is involved in it. The approach illustrated in this paper can be applied to this conjecture by considering a general relative stable moduli $\overline{\mathcal{M}}_g(d\times\mu, X\times\mathbb{P}^1)$ relative to the divisor $X\times\{\infty\}$ and its natural projection map $\overline{\mathcal{M}}_g(d\times\mu, X\times\mathbb{P}^1)\longrightarrow\overline{\mathcal{M}}_g(\mu,\mathbb{P}^1)$.

7.3. **Faber's conjecture on Hodge integrals.** In [7], C. Faber obtained a set of conjectures concerning the tautological Chow ring $R^\bullet(\mathcal{M}_g)$. The following identity on Hodge integrals is one of them:

$$\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!}\cdot\frac{1}{\prod_{j=1}^k(2e_j-1)!!}=\langle\tau_{e_1}\cdots\tau_{e_k}\tau_{2g}\rangle-\sum_{j=1}^{k}\langle\tau_{e_1}\cdots\tau_{e_{j-1}}\tau_{e_j+2g-1}\tau_{e_j+1}\cdots\tau_{e_k}\rangle$$
$$+\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\tau_{e_1}\cdots\tau_{e_k}\rangle+\frac{1}{2}\sum_{\underline{k}=I\amalg J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{e_i}\rangle\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{e_i}\rangle$$

The fourth-highest N-degree relation given by the recursion relation obtained in the proof of Witten's conjecture strongly suggests this conjectural identity. As a remark, this conjectural identity implies [10], through an application of the degree 0 Virasoro conjecture for $\mathbb{P}^2$, the following $\lambda_g\lambda_{g-1}$-conjecture

$$\Big[\prod_{i=1}^n(2k_i-1)!!\Big]\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\lambda_{g-1}\psi_1^{k_1}\cdots\psi_n^{k_n}=(2g-3+n)!\frac{|B_{2g}|}{2^{2g-1}(2g)!}$$

where B_{2g} denotes the Bernoulli number.

Appendix A. Proof of Asymptotic Formulas In this section, I prove the following asymptotic formula:

Proposition .1. *As $n\longrightarrow\infty$, we have for $k,l\geq 0$*

$$e^{-n}\sum_{p+q=n}\frac{p^{p+k+1}q^{q+l+1}}{p!q!}\longrightarrow\frac{1}{2}\Big[\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}\Big]n^{k+l+2}+o(n^{k+l+2})$$
$$e^{-n}\sum_{p+q=n}\frac{p^{p+k+1}q^{q-1}}{p!q!}\longrightarrow\frac{n^{k+\frac{1}{2}}}{\sqrt{2\pi}}-\Big[\frac{(2k+1)!!}{2^{k+1}k!}\Big]n^k+o(n^k)$$

Proof. Let m be an integer such that $1 < m < n$ and consider three ranges of p, q as follows:

$$R_l = \{ (p,q) \mid p > n-m \text{ and } q < m\}$$
$$R_c = \{ (p,q) \mid m \leq p, q \leq n-m \}$$
$$R_r = \{ (p,q) \mid p < m \text{ and } q > n-m\}$$

Recall the Stirling's formula;

$$n! = \frac{\sqrt{2\pi}n^{n+1/2}}{e^n}\Big(1+\frac{1}{12n}+\cdots\Big)$$

For the summation over R_c, let $m = n\epsilon$ and $p = nx$ for some $\epsilon, x \in \mathbb{R}_{>0}$ so that $m, p \in \mathbb{N}$, then we have

$$\begin{aligned}
e^{-n}\sum_{p=m}^{n-m}\frac{p^{p+k+1}}{p!}\frac{q^{q+l+1}}{q!} &= \sum_{p=m}^{n-m}\frac{1}{2\pi}p^{k+\frac{1}{2}}q^{l+\frac{1}{2}}\big[1+o(1)\big] \\
&= \frac{n^{k+l+2}}{2\pi}\sum_{p=m}^{n-m}x^{k+\frac{1}{2}}(1-x)^{l+\frac{1}{2}}\frac{1}{n}+o(n^{k+l+2}) \\
&\longrightarrow \frac{n^{k+l+2}}{2\pi}\int_{\epsilon}^{1-\epsilon}x^{k+\frac{1}{2}}(1-x)^{l+\frac{1}{2}}dx+o(n^{k+l+2}) \qquad \text{as } n \text{ goes to } \infty \\
&= \frac{n^{k+l+2}}{2\pi}\frac{(2k+1)!!(2l+1)!!}{(2(k+l)+3)!!}\int_{\epsilon}^{1-\epsilon}\frac{(1-x)^{k+l+\frac{3}{2}}}{\sqrt{x}}dx+o(n^{k+l+2})+O(\sqrt{\epsilon}) \\
&= \frac{1}{2}\Big[\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!}\Big]n^{k+l+2}+o(n^{k+l+2})+O(\sqrt{\epsilon})
\end{aligned}$$

As $n \longrightarrow \infty$, we can send $\epsilon \longrightarrow 0$. For the summation over R_l and R_r, the top-degree terms belong to $O(n^{k+1/2})$ and $O(n^{l+1/2})$, respectively. Since we assume $k, l \geq 0$, both cases belong to $o(n^{k+l+2})$, and this proves the first formula. For the second formula, R_l has highest order of $n^{k+1/2}$ and one can show that the leading term in the asymptotic behaviour is $n^{k+1/2}/\sqrt{2\pi}$. After integration by parts, R_c gives the second highest term in the asymptotic behaviour

$$\begin{aligned}
e^{-n}\sum_{p=m}^{n-1}\frac{p^{p+k+1}}{p!}\frac{q^{q-1}}{q!} &= \sum_{p=m}^{n-1}\frac{1}{2\pi}p^{k+\frac{1}{2}}q^{l-\frac{3}{2}}\big[1+o(1)\big] = \frac{n^k}{2\pi}\sum_{p=m}^{n-1}x^{k+\frac{1}{2}}(1-x)^{-3/2}\frac{1}{n}+o(n^k) \\
&\longrightarrow \frac{n^k}{2\pi}\int_{\epsilon}^{1}x^{k+\frac{1}{2}}(1-x)^{-3/2}dx+o(n^k) \qquad \text{as } n \text{ goes to } \infty \\
&= \frac{n^{k+1/2}}{\sqrt{2\pi}}-\frac{n^k}{2\pi}(2k+1)\int_{\epsilon}^{\delta}\frac{x^{k-\frac{1}{2}}}{\sqrt{1-x}}dx+o(n^k) \\
&= \frac{n^{k+1/2}}{\sqrt{2\pi}}-\Big[\frac{(2k+1)!!}{2^{k+1}k!}\Big]n^k+o(n^k)+O(\sqrt{\epsilon})
\end{aligned}$$

This proves the second formula. ∎

I tested these formulas up to $N = 24,000$ through numerical computation. I used multi-precision library in order to keep whole computation error-free. First I list the comparison results for the first few cases of the first asymptotic formula.

$(k,l)=(0,0)$

$n=2000$: ratio= 0.99941, $n=4000$: ratio= 0.999707

$n=6000$: ratio= 0.999805, $n=8000$: ratio= 0.999855

$n=10000$: ratio= 0.999884

$(k,l)=(1,0)$

$n=2000$: ratio= 0.99941, $n=4000$: ratio= 0.999707

$n=6000$: ratio= 0.999805, $n=8000$: ratio= 0.999855

$n=10000$: ratio= 0.999884

$(k,l)=(1,1)$

$n=2000$: ratio= 0.999641, $n=4000$: ratio= 0.999821

$n=6000$: ratio= 0.999881, $n=8000$: ratio= 0.999911

$n=10000$: ratio= 0.999929

$(k,l)=(2,0)$

$n=2000$: ratio= 0.999271, $n=4000$: ratio= 0.999638

$n=6000$: ratio= 0.99976, $n=8000$: ratio= 0.999821

$n=10000$: ratio= 0.999857

$(k,l)=(2,1)$

$n=2000$: ratio= 0.999641, $n=4000$: ratio= 0.999821

$n=6000$: ratio= 0.999881, $n=8000$: ratio= 0.999911

$n=10000$: ratio= 0.999929

$(k,l)=(2,2)$

$n=2000$: ratio= 0.999685, $n=4000$: ratio= 0.999843

$n=6000$: ratio= 0.999896, $n=8000$: ratio= 0.999922

$n=10000$: ratio= 0.999938

$(k,l)=(3,0)$

$n=2000$: ratio= 0.999113, $n=4000$: ratio= 0.99956

$n=6000$: ratio= 0.999708, $n=8000$: ratio= 0.999782

$n=10000$: ratio= 0.999826

$(k,l)=(3,1)$

$n=2000$: ratio= 0.999609, $n=4000$: ratio= 0.999805

$n=6000$: ratio= 0.999871, $n=8000$: ratio= 0.999903

$n=10000$: ratio= 0.999923

$(k,l) = (3,2)$

$n = 2000$: ratio= 0.999685, $n = 4000$: ratio= 0.999843
$n = 6000$: ratio= 0.999896, $n = 8000$: ratio= 0.999922
$n = 10000$: ratio= 0.999938

$(k,l) = (3,3)$

$n = 2000$: ratio= 0.999704, $n = 4000$: ratio= 0.999853
$n = 6000$: ratio= 0.999902, $n = 8000$: ratio= 0.999927
$n = 10000$: ratio= 0.999942

$(k,l) = (4,0)$

$n = 2000$: ratio= 0.998948, $n = 4000$: ratio= 0.999479
$n = 6000$: ratio= 0.999654, $n = 8000$: ratio= 0.999741
$n = 10000$: ratio= 0.999794

$(k,l) = (4,1)$

$n = 2000$: ratio= 0.999567, $n = 4000$: ratio= 0.999784
$n = 6000$: ratio= 0.999857, $n = 8000$: ratio= 0.999893
$n = 10000$: ratio= 0.999915

$(k,l) = (4,2)$

$n = 2000$: ratio= 0.99967, $n = 4000$: ratio= 0.999836
$n = 6000$: ratio= 0.999891, $n = 8000$: ratio= 0.999919
$n = 10000$: ratio= 0.999935

$(k,l) = (4,3)$

$n = 2000$: ratio= 0.999704, $n = 4000$: ratio= 0.999853
$n = 6000$: ratio= 0.999902, $n = 8000$: ratio= 0.999927
$n = 10000$: ratio= 0.999942

$(k,l) = (4,4)$

$n = 2000$: ratio= 0.999715, $n = 4000$: ratio= 0.999858
$n = 6000$: ratio= 0.999906, $n = 8000$: ratio= 0.99993
$n = 10000$: ratio= 0.999944

$(k,l) = (5,0)$

$n = 2000$: ratio= 0.998779, $n = 4000$: ratio= 0.999395
$n = 6000$: ratio= 0.999599, $n = 8000$: ratio= 0.9997
$n = 10000$: ratio= 0.999761

$(k, l) = (5, 1)$

$n = 2000$: ratio= 0.999519, $n = 4000$: ratio= 0.99976
$n = 6000$: ratio= 0.999841, $n = 8000$: ratio= 0.999881
$n = 10000$: ratio= 0.999905

$(k, l) = (5, 2)$

$n = 2000$: ratio= 0.999649, $n = 4000$: ratio= 0.999825
$n = 6000$: ratio= 0.999884, $n = 8000$: ratio= 0.999913
$n = 10000$: ratio= 0.999931

$(k, l) = (5, 3)$

$n = 2000$: ratio= 0.999695, $n = 4000$: ratio= 0.999848
$n = 6000$: ratio= 0.999899, $n = 8000$: ratio= 0.999925
$n = 10000$: ratio= 0.99994

$(k, l) = (5, 4)$

$n = 2000$: ratio= 0.999715, $n = 4000$: ratio= 0.999858
$n = 6000$: ratio= 0.999906, $n = 8000$: ratio= 0.99993
$n = 10000$: ratio= 0.999944

$(k, l) = (5, 5)$

$n = 2000$: ratio= 0.999721, $n = 4000$: ratio= 0.999861
$n = 6000$: ratio= 0.999908, $n = 8000$: ratio= 0.999931
$n = 10000$: ratio= 0.999946

$(k, l) = (6, 0)$

$n = 2000$: ratio= 0.998609, $n = 4000$: ratio= 0.999311
$n = 6000$: ratio= 0.999543, $n = 8000$: ratio= 0.999658
$n = 10000$: ratio= 0.999727

$(k, l) = (6, 1)$

$n = 2000$: ratio= 0.99947, $n = 4000$: ratio= 0.999736
$n = 6000$: ratio= 0.999824, $n = 8000$: ratio= 0.999869
$n = 10000$: ratio= 0.999895

$(k, l) = (6, 2)$

$n = 2000$: ratio= 0.999623, $n = 4000$: ratio= 0.999812
$n = 6000$: ratio= 0.999876, $n = 8000$: ratio= 0.999907
$n = 10000$: ratio= 0.999926

$(k,l) = (6,3)$

$n = 2000$: ratio= 0.999682, $n = 4000$: ratio= 0.999842
$n = 6000$: ratio= 0.999895, $n = 8000$: ratio= 0.999922
$n = 10000$: ratio= 0.999938

$(k,l) = (6,4)$

$n = 2000$: ratio= 0.999709, $n = 4000$: ratio= 0.999855
$n = 6000$: ratio= 0.999904, $n = 8000$: ratio= 0.999928
$n = 10000$: ratio= 0.999943

$(k,l) = (6,5)$

$n = 2000$: ratio= 0.999721, $n = 4000$: ratio= 0.999861
$n = 6000$: ratio= 0.999908, $n = 8000$: ratio= 0.999931
$n = 10000$: ratio= 0.999946

$(k,l) = (6,6)$

$n = 2000$: ratio= 0.999726, $n = 4000$: ratio= 0.999864
$n = 6000$: ratio= 0.99991, $n = 8000$: ratio= 0.999933
$n = 10000$: ratio= 0.999946

Next I list the comparison results for the first few cases between actual values for $N \leq 10,000$ and its expected values predicted by the 2nd asymptotic formula. From the data, we can see clearly ratios approach 1 as N grows for each case. (1st and 2nd means the ratio for the coefficients of top N-degree term and 2nd-highest N-degree term, respectively.)

$k=0$

$N=2000:$ 1st: 1.02866, 2nd: 0.988225 $N=4000:$ 1st: 1.02013, 2nd: 0.99165
$N=6000:$ 1st: 1.01639, 2nd: 0.993174 $N=8000:$ 1st: 1.01417, 2nd: 0.994084
$N=10000:$ 1st: 1.01266, 2nd: 0.994706

$k=1$

$N=2000:$ 1st: 1.04334, 2nd: 0.976436 $N=4000:$ 1st: 1.03037, 2nd: 0.983293
$N=6000:$ 1st: 1.0247, 2nd: 0.986342 $N=8000:$ 1st: 1.02134, 2nd: 0.988163
$N=10000:$ 1st: 1.01906, 2nd: 0.989408

$k=2$

$N=2000:$ 1st: 1.05454, 2nd: 0.968661 $N=4000:$ 1st: 1.03814, 2nd: 0.977763
$N=6000:$ 1st: 1.03099, 2nd: 0.981815 $N=8000:$ 1st: 1.02676, 2nd: 0.984237
$N=10000:$ 1st: 1.02389, 2nd: 0.985893

$k=3$

$N=2000:$ 1st: 1.06399, 2nd: 0.962478 $N=4000:$ 1st: 1.04468, 2nd: 0.973357
$N=6000:$ 1st: 1.03628, 2nd: 0.978207 $N=8000:$ 1st: 1.03131, 2nd: 0.981106
$N=10000:$ 1st: 1.02794, 2nd: 0.983088

$k=4$

$N=2000:$ 1st: 1.07236, 2nd: 0.957204 $N=4000:$ 1st: 1.05044, 2nd: 0.969594
$N=6000:$ 1st: 1.04093, 2nd: 0.975122 $N=8000:$ 1st: 1.03531, 2nd: 0.978428
$N=10000:$ 1st: 1.03151, 2nd: 0.980689

$k=5$

$N=2000:$ 1st: 1.07996, 2nd: 0.952534 $N=4000:$ 1st: 1.05567, 2nd: 0.966259
$N=6000:$ 1st: 1.04514, 2nd: 0.972386 $N=8000:$ 1st: 1.03893, 2nd: 0.976052
$N=10000:$ 1st: 1.03473, 2nd: 0.97856

Appendix B. Normal bundle computation

In this section, I will summarize the computations of $e_T(\mathcal{N}_\Gamma^{vir})$ in [28] which will be needed for localization computation. Denote by (ω) the 1-dimensional representation of $\mathbb{C}^*$ given by $\lambda \cdot z = \lambda^\omega z$ for $\lambda \in \mathbb{C}^*$, $z \in \mathbb{C}$. For a given graph $\Gamma \in G_{g,n}(\mathbb{P}^1, \mu)$, let

$$[\ f : (C, x_1, \cdots, x_{l(\mu)}, z_1, \cdots, z_n) \longrightarrow \mathbb{P}^1[m]\ ] \tag{20}$$

be a fixed point of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ associated to Γ. Given a flag $(v, e) \in F(\Gamma)$, denote by $q_{(v,e)} \in C$ the node at which C_v and C_e intersect. Also let $\psi_{(v,e)}$ denote the first Chern class of the cotangent line bundles over $\overline{\mathcal{M}}_\Gamma$, i.e. the fiber at the fixed point (20) is given by $T^*_{q(v,e)}C_v$. The Euler class $e_T(\mathcal{N}^{vir}_\Gamma)$ is given by;

$$\frac{1}{e_T(\mathcal{N}^{vir}_\Gamma)} = \frac{e_T(\hat{T}^2)}{e_T(\hat{T}^1)}$$

where $\hat{V}$ denotes the moving part of any vector bundle V, and T^1, T^2 are the tangent space and the obstruction space of $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$, respectively. Here, T^1 and T^2 can be computed through the following two exact sequences [24]:

$$0 \to \mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C) \to H^0(\mathbf{D}^\bullet) \to T^1 \to \mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C) \to H^1(\mathbf{D}^\bullet) \to T^2 \to 0$$

$$0 \longrightarrow H^0(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee) \longrightarrow H^0(\mathbf{D}^\bullet) \longrightarrow \bigoplus_{l=0}^{m-1} H^0_{et}(\mathbf{R}^\bullet_l)$$

$$\longrightarrow H^1(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee) \longrightarrow H^1(\mathbf{D}^\bullet) \longrightarrow \bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet) \longrightarrow 0$$

where $\omega_{\mathbb{P}^1[m]}$ is the dualizing sheaf of $\mathbb{P}^1[m]$, $D = x_1 + \cdots + x_{l(\mu)}$ is the branch divisor, and for $n_l =$ the number of nodes over $p_1^{(l)}$;

$$H^0_{et}(\mathbf{R}^\bullet_l) \cong \bigoplus_{q\in f^{-1}(p_1^{(l)})} T_q\big(f^{-1}(\mathbb{P}^1_{(l)})\big) \;\cong\; \mathbb{C}^{\oplus n_l}$$

$$H^1_{et}(\mathbf{R}^\bullet_l) \cong \big(T_{p_1^{(l)}}\mathbb{P}^1_{(l)} \otimes T_{p_1^{(l)}}\mathbb{P}^1_{(l+1)}\big)^{\oplus(n_l-1)}$$

Recall the map $\pi[m] : \mathbb{P}^1[m] \longrightarrow \mathbb{P}^1$, and observe that for $\hat{f} = \pi[m]\circ f$ we have

$$f^*\big(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)})\big)^\vee \cong \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)$$

Let F_Γ be the set of fixed points associated to $\Gamma \in G_{\chi,n}(\mathbb{P}^1,\mu)$ and assume that

$$\big[\; f : (C, x_1, \cdots, x_{l(\mu)}, z_1, \cdots, z_n) \longrightarrow \mathbb{P}^1[m] \;\big] \in F_\Gamma \subset \overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$$

The $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$ induces $\mathbb{C}^*$-actions on

$$\mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C) \quad , \qquad H^0(C,\hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \quad , \qquad \bigoplus_{l=0}^{m-1} H^0_{et}(\mathbf{R}^\bullet_l)$$

$$\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C) \quad , \qquad H^1(C,\hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \quad , \qquad \bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet_l)$$

The moving part of each of these groups form vector bundles over $\overline{\mathcal{M}}_\Gamma$. I will use the same notation $\hat{\ }$ to denote the induced vector bundles. In particular,

$$\widehat{\bigoplus_{l=0}^{m-1} H^0_{et}(\mathbf{R}^\bullet_l)} = 0, \qquad \text{and}$$

$$\widehat{\bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet_l)} = \begin{cases} 0, & \text{if } m = 0 \\ H^1_{et}(\mathbf{R}^\bullet_0) = \big(T_{p_1^{(0)}}\mathbb{P}^1_{(0)} \otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)}\big)^{\oplus(n_0-1)}, & \text{if } m > 0 \end{cases}$$

Hence we have;

$$\frac{1}{e_T(\mathcal{N}_\Gamma^{vir})}=\frac{e_T(\hat{T}^2)}{e_T(\hat{T}^1)}=\frac{e_T\big(\widehat{\mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C)}\big)e_T\big(\widehat{H^1(C,\hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)e_T\big(\widehat{\bigoplus_{l=0}^{m-1}H^1_{et}(\mathbf{R}_l^\bullet)}\big)}{e_T\big(\widehat{H^0(C,\hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)e_T\big(\widehat{\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C)}\big)}$$

<u>Case $m=0$:</u>
For each $v\in V(\Gamma_0)$ and $\mu_{v,1},\cdots,\mu_{v,l(\mu(v))}$ the ramification type in the vertex v, we have under the convention to write $\mu_{v,2}=\infty$ when $g(v)=0,l(\mu(v))=l(e(v))=1$;

$$\widehat{\bigoplus_{l=0}^{m-1}H^1_{et}(\mathbf{R}_l^\bullet)_v}=0$$

$$\mathrm{Ext}^0(\widehat{\Omega_C(D)},\mathcal{O}_C)_v=\begin{cases}\left(\frac{1}{\mu_{v,1}}\right), & \text{if } v\in \text{I}\\ 0, & \text{if } v\in \text{II or S}\end{cases}$$

$$\mathrm{Ext}^1(\widehat{\Omega_C(D)},\mathcal{O}_C)_v=\begin{cases}0, & \text{if } v\in \text{I}\\ \left(\frac{1}{\mu_{v,1}}+\frac{1}{\mu_{v,2}}\right), & \text{if } v\in \text{II}\\ \bigoplus_{i=1}^{l(\mu(v))}T_{q_{(v,e_{v,i})}}C_v\otimes T_{q_{(v,e_{v,i})}}C_{e_{v,i}}, & \text{if } v\in \text{S}\end{cases}$$

Hence we can compute their contributions to be;

$$e_T\big(\widehat{\bigoplus_{l=0}^{m-1}H^1_{et}(\mathbf{R}_l^\bullet)_v}\big)=1$$

$$e_T\big(\mathrm{Ext}^0(\widehat{\Omega_C(D)},\mathcal{O}_C)_v\big)=\begin{cases}\frac{u}{\mu_{v,1}}, & \text{if } v\in \text{I}\\ 1, & \text{if } v\in \text{II or S}\end{cases}$$

$$e_T\big(\mathrm{Ext}^1(\widehat{\Omega_C(D)},\mathcal{O}_C)_v\big)=\begin{cases}1, & \text{if } v\in \text{I}\\ \frac{u}{\mu_{v,1}}+\frac{u}{\mu_{v,2}}, & \text{if } v\in \text{II}\\ \prod_{i=1}^{l(\mu(v))}\left(\frac{u}{\mu_{v,i}}-\psi_{v,i}\right), & \text{if } v\in \text{S}\end{cases}$$

For the contributions from the rest, consider the normalization sequence when $v\in$S;

$$0\to\hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\to\big(\hat{f}\mid_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\oplus\bigoplus_{i=1}^{l(\mu(v))}\big(\hat{f}\mid_{C_{e_{v,i}}}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\to\bigoplus_{i=1}^{l(\mu(v))}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\to 0$$

The corresponding long exact sequence becomes

$$
\begin{aligned}
0 \longrightarrow & H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow H^0\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^0\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\
& \longrightarrow \bigoplus_{i=1}^{l(\mu(v))} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\
& \longrightarrow H^1\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^1\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow 0
\end{aligned}
$$

and the representations of $\mathbb{C}^*$ are given by

$$
\begin{aligned}
0 \to & H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to H^0\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \oplus \bigoplus_{i=1}^{l(\mu(v))} \Big(\bigoplus_{a=1}^{\mu_{v,i}} (\frac{a}{\mu_{v,i}})\Big) \to \\
& \bigoplus_{i=1}^{l(\mu(v))} (1) \to H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to H^1\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \to 0
\end{aligned}
$$

Hence their ratio can computed as;

$$
\frac{e_T\big(H^1(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}}(1))\big)}{e_T\big(H^0(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}}(1))\big)} = \prod_v \Big[\Lambda^\vee_{g(v)}(u) u^{l(\mu(v))-1} \prod_{i=1}^{l(\mu(v))} \big(\frac{\mu_{v,i}^{\mu_{v,i}}}{\mu_{v,i}!} u^{-\mu_{v,i}}\big)\Big]
$$

which also works for the case of $v \in$I or $v \in$II.

Case $m > 0$: Let ψ^t be the first Chern class of the line bundle over $\overline{\mathcal{M}}_\Gamma^{(1)}$ whose fiber at $[\ \hat{f} : \hat{C} \longrightarrow \mathbb{P}^1(m)\ ]$ is $T^*_{p_1^{(0)}}\mathbb{P}^1(m)$. So $\psi^t = \nu_{v,i}\psi_{(v,e_{v,i})}$ for $v \in V(\Gamma)^{(1)},\ (v, e_{v,i}) \in F$. By similar observation as in the case of $m = 0$, we can find that

$$
\begin{aligned}
& \widehat{\bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet_l)} = \Big(T_{p_1^{(0)}}\mathbb{P}^1_{(0)} \otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)}\Big)^{|E(\Gamma)|-1} \\
& \mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} \Big(\frac{1}{\nu_{v,1}}\Big), & \text{if } v \in \text{I} \\ 0, & \text{if } v \in \text{II or S} \end{cases} \\
& \mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} 0, & \text{if } v \in \text{I} \\ \Big(\frac{1}{\nu_{v,1}} + \frac{1}{\nu_{v,2}}\Big), & \text{if } v \in \text{II} \\ \bigoplus_{i=1}^{l(\nu(v))} T_{q_{(v,e_{v,i})}} C_v \otimes T_{q_{(v,e_{v,i})}} C_{e_{v,i}}, & \text{if } v \in \text{S or T} \end{cases}
\end{aligned}
$$

Hence we can compute their contributions to be;

$$e_T\big(\widehat{\bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet_l)}\big) = \big(-u-\psi^t\big)^{|E(\Gamma)|-1}$$

$$e_T\big(\mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} \frac{u}{\nu_{v,1}}, & \text{if } v \in \mathrm{I} \\ 1, & \text{if } v \in \mathrm{II} \text{ or } \mathrm{S} \end{cases}$$

$$e_T\big(\mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} 1, & \text{if } v \in \mathrm{I} \\ \frac{u}{\nu_{v,1}} + \frac{u}{\nu_{v,2}}, & \text{if } v \in \mathrm{II} \\ \prod_{i=1}^{l(\nu(v))}\Big(\frac{u}{\nu_{v,i}} - \psi_{v,i}\Big), & \text{if } v \in \mathrm{S} \\ \prod_{i=1}^{l(\nu(v))}\Big(\frac{-u}{\nu_{v,i}} - \psi_{v,i}\Big), & \text{if } v \in \mathrm{T} \end{cases}$$

Also consider the following normalization sequence

$$0 \to \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \to \bigoplus_{S\cup T}\big(\hat{f}\mid_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{e\in E(\Gamma)}\big(\hat{f}\mid_{C_e}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \to$$
$$\bigoplus_{II}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{S}\Big(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\Big) \oplus \bigoplus_{T}\Big(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\Big) \to 0$$

and the corresponding long exact sequence

$$0 \to H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to \bigoplus_{S\cup T} H^0\big(C_v, \big(\hat{f}\mid_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{e\in E(\Gamma)} H^0\big(C_e, \big(\hat{f}\mid_{C_e}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to$$
$$\bigoplus_{II}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{S}\Big(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\Big) \oplus \bigoplus_{T}\Big(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\Big) \to H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to$$
$$\bigoplus_{ST} H^1\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{e\in E(\Gamma)} H^1\big(C_e, (\hat{f}\mid_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to 0$$

The representations of $\mathbb{C}^*$ are given by

$$0 \to H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to \bigoplus_{S} H^0\big(C_v, \mathcal{O}_{C_v}\big)\otimes(1) \oplus \bigoplus_{T} H^0\big(C_v, \mathcal{O}_{C_v}\big)\otimes(0) \oplus \bigoplus_{e\in E(\Gamma)}\Big(\bigoplus_{a=1}^{d(e)}\big(\frac{a}{d(e)}\big)\Big)$$
$$\to \bigoplus_{II}(1) \oplus \bigoplus_{S}\Big(\bigoplus_{(v,e)\in F}(1)\Big) \oplus \bigoplus_{T}\Big(\bigoplus_{(v,e)\in F}(0)\Big) \to H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to$$
$$\bigoplus_{S} H^1\big(C_v, \mathcal{O}_{C_v}\big)\otimes(1) \oplus \bigoplus_{T} H^1\big(C_v, \mathcal{O}_{C_v}\big)\otimes(0) \longrightarrow 0$$

from which we can compute their ratio to be;

$$\frac{e_T\big(H^1(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)}{e_T\big(H^0(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)} = \prod_{V(\Gamma)^{(0)}}\Big[\Lambda^\vee_{g(v)}(u)u^{l(\nu(v))-1}\Big]\prod_{i=1}^{l(\nu)}\Big[\frac{\nu_i^{\nu_i}}{\nu_i!}u^{-\nu_i}\Big]$$

After combining all the contributions, we find the following Feynman rules;

$$\frac{1}{e_T(\mathcal{N}^{vir}_{\Gamma_0})} = \Big[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} u^{-\mu_i}\Big] \times \Big[\prod_I \frac{u}{\mu_{v,1}}\Big] \times \Big[\prod_{II} \frac{u}{\frac{u}{\mu_{v,1}} + \frac{u}{\mu_{v,2}}}\Big] \times \Big[\prod_S \frac{\Lambda^\vee_{g(v)}(u)}{u} \Big(\prod_{i=1}^{l(\mu(v))} \frac{u}{\frac{u}{\mu_{v,i}} - \psi_{v,i}}\Big)\Big]$$

$$\frac{1}{e_T(\mathcal{N}^{vir}_{\Gamma})} = \frac{-\prod \nu_i}{u+\psi^t} \times \Big[\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!} u^{-\nu_i}\Big] \times \Big[\prod_I \frac{u}{\nu_{v,1}}\Big] \times \Big[\prod_{II} \frac{u}{\frac{u}{\nu_{n,1}} + \frac{u}{\nu_{v,2}}}\Big] \times \Big[\prod_S \frac{\Lambda^\vee_{g(v)}(u)}{u} \Big(\prod_{i=1}^{l(\nu(v))} \frac{u}{\frac{u}{\nu_{v,i}} - \psi_{v,i}}\Big)\Big]$$

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