

Witten's Conjecture through Cut-and-Join Equation

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USC Geometry & Topology Seminar
December 5, 2005

In this talk, I will present a new proof of Witten's conjecture using localization method. Precisely, I will show that the *Cut-and-Join equation* implies ELSV-formula, the KdV-hierarchy, and the Virasoro constraint.

- Witten's original conjecture.
- Equivalent formulations.
- Previous approaches.
- Localization method.
- Cut-and-Join equation.
- Proof of Witten's conjecture.

Preliminaries

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves, i.e. it consists of Riemann surfaces of genus g and n -marked points with certain stability condition. We can consider two natural cohomology classes on it as follows:

- **ψ -classes** : For each marked point x_i where $1 \leq i \leq n$, consider the line bundle \mathbb{L}_i over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[C, x_1, \dots, x_n] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T_{x_i}^*C$ at the i -th marked point x_i . Then define $\psi_i = c_1(\mathbb{L}_i)$.
- **λ -classes** : Let $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The *Hodge bundle* $\mathbb{E} = \pi_*\omega_\pi$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$. Denote the Chern polynomial by

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g$$

and its j -th Chern class by $\lambda_j = c_j(\mathbb{E})$.

Hodge integral

The *Hodge integral* is defined as the integration of these classes over the moduli space $\overline{\mathcal{M}}_{g,n}$, i.e.

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i} \prod \lambda_j^{l_j}$$

where $\sum k_i + \sum j \cdot l_j = 3g - 3 + n = \dim \overline{\mathcal{M}}_{g,n}$.

The Witten's conjecture, in mathematical formulation, is about the governing relation of Hodge integrals which involve ψ -classes only: Given a quantum field theory Lagrangian \mathcal{L}_0 and operators τ_i , its intersection numbers $\langle \tau_0^{n_0} \tau_1^{n_1} \dots \rangle$ amounts to the Hodge integral as

$$\langle \tau_0^{n_0} \tau_1^{n_1} \dots \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i}$$

where g , n , and k_i are determined by conditions

$$\begin{aligned} n &= \sum n_p \\ k_i &= \#\{ a \mid n_p = i \} \\ \sum k_i &= 3g - 3 + n \end{aligned}$$

Witten's Original Conjecture

Motivated by the Feynman integral and its “total free energy” , E.Witten considered the generating function of the stable intersection theory on moduli space:

$$F(t_0, t_1, \dots) = \sum_{\{n_i\}} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \dots \rangle.$$

and formulated the *Witten's Conjecture* as follows:

2a. The conjecture. Our basic conjecture is that $F(t_0, t_1, \dots)$ is determined by the following two conditions:

(1) The object $U = \partial^2 F / \partial t_0^2$ obeys the KdV equations.

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \dot{U}, \ddot{U}, \dots),$$

where $\dot{U} = \partial U / \partial t_0$, $\ddot{U} = \partial^2 U / \partial t_0^2$, etc., are the derivatives of U w.r.t t_0 , and $R_{n+1}(U, \dot{U}, \ddot{U}, \dots)$ are certain polynomials in U and its t_0 derivatives that are well known in the theory of the KdV equations (and can be defined by a recursion relation that is given below).

(2) In addition, F obeys the “string equation,”

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.$$

The two statements can be summarized by saying that stable intersection theory on moduli space is equivalent to the “hermitian matrix model” of two-dimensional gravity. That formulation was the original context for the conjecture.

[excerpted from “Two-Dimensional Gravity and Intersection Theory on Moduli Space” , *Surveys in Differential Geometry* **1**(1991), p248-250]

Equivalent Formulations

1] The Virasoro constraint formulation.

The Witten conjecture states that the intersection theory on the moduli space of stable curves is governed by the KdV-hierarchy. It is also shown to be equivalent to the following Virasoro constraint:

$$L_n \cdot F = 0, \quad n \geq -1$$

where L_n denote the Virasoro operators.

$$\begin{aligned} L_{-1} &= -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_0} + \sum_{k=1}^{\infty} \left(k + \frac{1}{2}\right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4} \tilde{t}_0^2 \\ L_0 &= -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_1} + \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16} \\ L_n &= -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_{n-1}} + \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k+n}} \\ &\quad + \frac{1}{4} \sum_{i=1}^n \frac{\partial^2}{\partial \tilde{t}_{i-1} \partial \tilde{t}_{n-i}}, \quad \tilde{t}_n = t_{2n+1} \end{aligned}$$

[E.Witten, *On the Kontsevich model and other models of two-dimensional gravity*, Proceeding of the XXth International Conference on Differential Geometric Methods in Theoretical Physics]

2] The Recursion relation formulation.

E.Verlinde, H.Verlinde, and R.Dijkgraaf derived the following recursion relation for the correlation functions of the topological gravity

$$\begin{aligned}
 \langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g &= \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_{n+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g \\
 &+ \frac{1}{2} \sum_{a,b} \left[\langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g-1} \right. \\
 &\left. + \sum_{\tilde{g}, X} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{\tilde{g}} \langle \tilde{\sigma}_b \prod_{l \in X^c} \tilde{\sigma}_l \rangle_{g-\tilde{g}} \right]
 \end{aligned}$$

where $\tilde{\sigma}_n = (2n + 1)!! \psi^n$ and

$$\langle \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \left[\prod_{k \in S} (2k + 1)!! \right] \int_{\overline{\mathcal{M}}_{g,|S|}} \prod_{k \in S} \psi^k$$

Physically, they considered the contact terms which occur when two operators collide in the form

$$\mathcal{O}_i(z) \cdot \mathcal{O}_j(0) \sim \mathcal{O}_k(0) \delta^2(z)$$

,and showed that the above recursion relation is equivalent to the Virasoro constraint.

Previous Approaches

1] Matrix model and Kontsevich's proof

Witten conjecture was first proved by M.Kontsevich using combinatorial model of the moduli space and matrix model.

The string partition function $\tau(t)$:

$$\tau(t) = \exp \sum_{g=0}^{\infty} \langle \exp \sum_n t_n \mathcal{O}_n \rangle_g$$

admits an integral representation which involves the following integral over a $N \times N$ Hermitian matrix Y of the form

$$\tau(Z) = \rho(Z)^{-1} \int dY \cdot \exp \operatorname{Tr} \left[-\frac{1}{2} ZY^2 + \frac{i}{6} Y^3 \right]$$

where Z is a second $N \times N$ Hermitian matrix, and $\rho(Z)$ is the one-loop integral

$$\rho(Z) = \int dY \cdot \exp \left[-\frac{1}{2} \operatorname{Tr} ZY^2 \right]$$

This requires a map from the matrix Z to the coupling constants t_n , which is given by

$$t_n = -\frac{1}{n} \operatorname{Tr} Z^{-n}$$

2] Approaches using ELSV formula

- A. Okounkov and R. Pandharipande gave a different approach using the enumeration of branched covering of \mathbb{P}^1 through the ELSV-formula:

$$\frac{H_{g,\mu} \cdot |\text{Aut } \mu|}{(2g - 2 + |\mu| + l(\mu))!} = \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod(1 - \mu_i \psi_i)}$$

and using Kontsevich's matrix model. Here $H_{g,\mu}$ is the single Hurwitz number with ramification type at ∞ given by the partition μ .

- Recently, M.E. Kazarian and S.E. Lando obtained an algebro-geometric proof by using the ELSV-formula and the PDEs which govern the generating series of Hurwitz numbers to derive the KdV-equation.

3] Approach using Weil-Petersen volumes

M. Mirzakhani derived the Virasoro constraint using Weil-Petersen volumes on moduli spaces of bordered Riemann surfaces.

Localization method

Let $\overline{\mathcal{M}}_g(\mu, \mathbb{P}^1)$ be the moduli space of relative stable morphisms, i.e. it consists of mapping

$$f : [C ; x_1, \dots, x_{l(\mu)}] \longrightarrow (\mathbb{P}^1, \infty)$$

with prescribed ramification μ at the divisor ∞

$$f^{-1}(\infty) = \sum \mu_i x_i$$

We can induce S^1 -action on this space from the natural S^1 -action on \mathbb{P}^1 given by $t \cdot [u, v] = [tu, v]$. Then localization method can be applied and we can express an integration of any natural class ω as the summation over fixed locus of S^1 -action:

$$\int_{\overline{\mathcal{M}}_g(\mu, \mathbb{P}^1)} \omega = \sum_{F : \text{fixed locus}} \int_F \frac{i_F^*(\omega)}{e(\mathcal{N}_F)}$$

Also the moduli space of relative stable morphisms admits the branching morphism:

$$\text{Br} : \overline{\mathcal{M}}_g(\mu, \mathbb{P}^1) \longrightarrow \text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$$

where $r = 2g - 2 + |\mu| + l(\mu) = \text{v.dim } \overline{\mathcal{M}}_g(\mu, \mathbb{P}^1)$.

Cut-and-Join equation

Consider the hyperplane class $H \in H^*(\mathbb{P}^r)$ with prescribed values $H(p_k) = k$ for $k = 0, \dots, r$ and p_k is the k -th fixed point of \mathbb{P}^r . Let $\omega = \text{Br}^* \prod_{k=0}^{r-2} (H - k)$. The localization method applied to ω yields:

$$r\Gamma_r = \Gamma_{r-1}$$

where Γ_r and Γ_{r-1} denote the contributions from the fixed locus that are mapped under the branching morphism to p_r and p_{r-1} , respectively.

This is *the Cut-and-Join equation*, and is of same type as the single Hurwitz number formula:

$$H_{g,\mu} = \sum_{\nu \in J(\mu)} I_1(\nu) H_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) H_{g-1,\nu} + \sum \binom{r-1}{2g_1 - 2 + |\nu^1| + l(\nu^1)} I_3(\nu^1, \nu^2) H_{g_1,\nu^1} H_{g_2,\nu^2}.$$

except that the single Hurwitz numbers are replaced by the Hodge integrals

$$H_{g,\mu} = \frac{r!}{|\text{Aut } \mu|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod (1 - \mu_i \psi_i)}$$

Hence the Cut-and-Join equation recovers the ELSV-formula since both sides satisfy identical recursion relation with same initial values.

Proof of Witten's conjecture

Application of degree analysis to the combinatorial data (i.e. Cut-and-Join equation):

- **Combinatorics:** Cut and join of cycles:

Cut: a $(i + j)$ -cycle is cut into an i -cycle and a j -cycle, denote the set by $C(\mu)$:

Join: an i -cycle and a j -cycle are joined to an $(i + j)$ -cycle, denote the set by $J(\mu)$:

- **Degree Analysis:** Scale the partition μ

$$\mu_i = N \cdot x_i, \quad \text{for some } x_i \in \mathbb{R} \text{ and } N \in \mathbb{N}$$

and let N tend to ∞ . The asymptotic behaviour of Cut-and-Join equation can be obtained by using the asymptotic formula:

$$e^{-n} \sum_{p+q=n} \frac{p^{p+k+1} q^{q+l+1}}{p!q!} \\ \longrightarrow \frac{1}{2} \left[\frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!} \right] n^{k+l+2} + o(n^{k+l+2})$$

$$e^{-n} \sum_{p+q=n} \frac{p^{p+k+1} q^{q-1}}{p!q!} \\ \longrightarrow \frac{n^{k+\frac{1}{2}}}{\sqrt{2\pi}} - \left[\frac{(2k+1)!!}{2^{k+1}k!} \right] n^k + o(n^k)$$

[Proof is an application of integration by parts and Stirling's formula $n! \sim \sqrt{2\pi} e^{-n} n^{n+1/2} (1 + 1/12n + \dots)$]

and the stratification given by

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1 - \mu_i \psi_i)} = \left[\sum_k \prod x_i^{k_i} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i} \right] N^{3g-3+n} \\ + \text{lower } N\text{-degree terms}$$

The resulting relation is a system of vanishing equations between linear Hodge integrals. Its first non-trivial equation is the recursion relation of topological gravity obtained by E.Verlinde, H.Verlinde, and R.Dijkgraaf.

- The highest N -degree gives a trivial identity:

$$0 = (x_1 + \cdots + x_n) \prod \frac{x_i^{k_i-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i} \\ - (x_1 + \cdots + x_n) \prod \frac{x_i^{k_i-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{k_i}$$

- The second highest N -degree gives a recursion relation between Cut-and-Join graphs:

$$0 = \sum_{i=1}^n \left[\frac{(2k_i + 1)!!}{2^{k_i+1} k_i!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_j^{k_j} \right. \\ - \sum_{j \neq i} \frac{(x_i + x_j)^{k_i+k_j-1/2}}{\sqrt{2\pi}} \prod_{l \neq i,j} \frac{x_l^{k_l-1/2}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n-1}} \psi^{k_i+k_j-1} \prod \psi_l^{k_l} \\ - \frac{1}{2} \sum_{k+l=k_i-2} \frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j-1/2}}{\sqrt{2\pi}} \times \\ \left. \left(\int_{\overline{\mathcal{M}}_{g-1,n+1}} \psi_1^k \psi_2^l \prod \psi_j^{k_j} + \sum \int_{\overline{\mathcal{M}}_{g_1,n_1}} \psi_1^k \prod \psi_j^{k_j} \int_{\overline{\mathcal{M}}_{g_2,n_2}} \psi_1^l \prod \psi_j^{k_j} \right) \right]$$

- Lower N -degree strata will give relations for Hodge integrals involving non-trivial λ -class in terms of lower-dimensional ones.

After taking the Laplace transformations,

$$\int_0^\infty \frac{x^{k-1/2}}{\sqrt{2\pi}} e^{-x/2s} dx = (2k-1)!! s^{k+1/2}$$

$$\int_0^\infty x^k e^{-x/2s} dx = k! (2s)^{k+1}$$

we recover the recursion relation for the correlation functions of the topological gravity.

$$\begin{aligned} \langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g &= \sum_{k \in S} (2k+1) \langle \tilde{\sigma}_{n+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g \\ &+ \frac{1}{2} \sum_{a,b} \left[\langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g-1} \right. \\ &+ \left. \sum_{\tilde{g}, X} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{\tilde{g}} \langle \tilde{\sigma}_b \prod_{l \in X^c} \tilde{\sigma}_l \rangle_{g-\tilde{g}} \right] \end{aligned}$$

THANK YOU!