18. Cayley-Hamilton Theorem and Spectral Theorem

Let R be a commutative ring with identity. We first observe the following fact

$$M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$$

i.e. the matrix algebra where its entries are from the commutative polynomial ring $R[x]$ is isomorphic the polynomial ring where its coefficients are from the matrix algebra over the ring R. The algebra homomorphism $\phi : M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$ is defined by

$$\phi(A) = \sum_{k=0}^{\infty} A_k x^k \quad \text{where } (A_k)_{ij} = r_k(A_{ij})$$

Here $r_k : R[x] \to R$ denotes the map $r_k(a_0 + \cdots + a_k x^k + \cdots + a_m x^m) = a_k$. It is clear that ϕ is a well-defined map. We only need to show that ϕ is an algebra homomorphism, i.e.

$$\phi(AB) = \phi(A)\phi(B), \quad \text{and} \quad \phi(\lambda A) = \lambda\phi(A).$$

Given an $n \times n$-matrix A over a commutative ring R with unity, consider the element $C = xI - A \in M_{n\times n}(R[x])$. We can apply previous arguments to prove the following formula:

$$(Adj(C))\, C = \det(C) I.$$

The RHS is precisely the definition of the characteristic polynomial of A, i.e.

$$(Adj(C))\,(xI - A) = \sigma_A(x) I$$

Viewed as an identity in the polynomial ring $(M_{n\times n}(R))[x]$, we can evaluate x at A to obtain

$$\sigma_A(A) = 0 \in M_{n\times n}(R)$$

We just proved the Cayley-Hamilton theorem:

Theorem 18.1. *For $A \in M_{n\times n}(R)$, we have*

$$\sigma_A(A) = a_0 I + \cdots + a_n A^n = 0$$

where $\sigma_A(x) = a_0 + \cdots + a_n x^n$ is the characteristic polynomial of A.

As an application, we can prove the following theorem:

Theorem 18.2. **(Spectral Theorem)** *Let A be an $n \times n$-matrix over a field k. Suppose the characteristic polynomial $\sigma_A(x)$ of A has the form*

$$\sigma_A(x) = (x - r_1)^{m_1} \cdots (x - r_l)^{m_l}$$