

Now we can find explicit form of the inverse matrix for any given nonsingular matrix:

Definition 16.7. For an $n \times n$ -matrix A , define the *classical adjoint* of A by

$$\text{Adj}(A) = (c_{ij}) \quad \text{where } c_{ij} = (-1)^{i+j} \det(A_{ji})$$

where A_{ji} is the minor matrix of A obtained by removing j -th row and i -th column of A .

Theorem 16.2. For a square matrix A , we have

$$A(\text{Adj}(A)) = (\text{Adj}(A))A = (\det(A))I$$

If, in particular, $\det(A) \neq 0$, then $A^{-1} = (\det(A))^{-1} \text{Adj}(A)$.

17. CHARACTERISTIC POLYNOMIAL

Definition 17.1. The *characteristic polynomial* of a square matrix $A \in \text{Mat}_{n \times n}(k)$ is

$$\sigma_A(x) = \det(xI - A)$$

The determinant in the definition can be formally understood through Laplace expansion as long as the matrix has its entries in a commutative ring. In our case, the ring we're using is the polynomial ring $k[x]$ over k . The Cayley-Hamilton theorem asserts the following

$$\sigma_A(A) = 0 \in \text{Mat}_{n \times n}(k)$$

Proposition 17.1. $\lambda \in k$ is an eigenvalue of A if and only if $x - \lambda$ is a factor of $\sigma_A(x)$

Definition 17.2. The *algebraic multiplicity* of an eigenvalue λ of an endomorphism $\phi : V \rightarrow V$ is the maximum dimension of the subspaces $0 \subset \ker \psi_\lambda \subset \ker \psi_\lambda^2 \subset \dots$. Denote the subspaces $\ker \psi_\lambda$ as $V_i(\lambda) = \{ v \in V \mid v \in \ker(\psi_\lambda^i) \}$.

Lemma 17.1. $V_i(\lambda) \cap V_j(\mu) = \{0\}$ for $\lambda \neq \mu$.

Definition 17.3. The monic polynomial $m(x) \in k[x]$ of minimal degree such that $m(A) = 0$ is called the *minimal polynomial* of A .

Proposition 17.2. *Minimal polynomials have the following properties:*

- 1) $\deg m(x) \leq \deg \sigma_A(x)$.
- 2) $m(x)$ is unique.
- 3) $m(x)$ divides any polynomial which vanishes at A . In particular, $m(x)$ divides $\sigma(x)$.
- 4) $m(x)$ and $\sigma(x)$ have the same irreducible factors.
- 5) Conjugate matrices have the same minimal polynomials.