Definition 16.2. A square matrix $M \in \text{Mat}_{n\times n}(k)$ is called *nonsingular* if there is another matrix $N \in \text{Mat}_{n\times n}(k)$ such that $MN = I$ and $NM = I$ where I is the identity matrix.

Proposition 16.3. *A linear mapping $\phi : V \to W$ is an isomorphism if and only if the associated matrix $M_A^B(\phi)$ is nonsingular for any choice of bases A and B.*

A linear mapping is independent of our choice of bases, hence any intrinsic quantity of a linear mapping should be invariant under a change of bases.

Definition 16.3. Two matrices A and B are called *conjugate* to each other if there is a nonsingular matrix C such that $A = CBC^{-1}$.

We will look at two quantities which stay fixed in any conjugacy class, namely the determinant and the trace.

Definition 16.4. The *trace* of a square matrix $A = (a_{ij})$ is given by

$$\textbf{Tr}(A) = \sum_i a_{ii}$$

Proposition 16.4. *For any matrix B conjugate to A, we have* **Tr**(A)=**Tr**(B).

We first define exterior algebra of a vector space:

Definition 16.5. Let V be a finite dimensional vector space and let $B = \{v_1, \cdots, v_n\}$ be a basis. The *exterior algebra* $\Lambda(V)$ is the (graded) vector space

$$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V) = \Lambda^0(V) \oplus \cdots \oplus \Lambda^n(V)$$

where each $\Lambda^k(V)$ is a vector space with a basis given by $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$. The algebra structure is given by $v_i \wedge v_j = -v_j \wedge v_i$ and its unique extension to $\Lambda(V)$.

Proposition 16.5. *dim*$\Lambda^k(V) = \binom{n}{k}$ *and dim*$\Lambda(V) = 2^n$.

Now we can define the determinant: Let A be a $n \times n$-matrix over $\mathbb{R}$. Choose the standard basis $\{e_1, \cdots, e_n\}$. For each $i = 1, 2, \cdots, n$, e_iA is a $1 \times n$-matrix which can be considered as a vector in $\mathbb{R}^n$. Hence

$$e_1A \wedge \cdots \wedge e_nA \in \Lambda^n(V).$$

Since $\Lambda^n(V)$ is 1-dimensional with basis $e_1 \wedge \cdots \wedge e_n$, there exists a constant det(A) such that

$$e_1A \wedge \cdots \wedge e_nA = \textbf{det}(A)e_1 \wedge \cdots \wedge e_n$$

Definition 16.6. The constant det(A) is the *determinant* of A.