

## 16. MATRIX, LINEAR TRANSFORMATION, AND BASE CHANGE

In this section, we will discuss how to relate a linear mapping with a matrix. As was discussed before, we know that

A linear mapping  $\phi : V \rightarrow W$  corresponds to an  $n \times m$ -matrix  $M(\phi)$  under a choice of bases for  $V$  and  $W$  where  $\dim(V) = m$  and  $\dim(W) = n$ .

One key thing to remember in this correspondence is the following fact:

A linear mapping  $\phi : V \rightarrow W$  is independent of a choice of bases for  $V$  and  $W$ .

Hence, depending on our choice of bases, the corresponding matrices will look different and we would like to choose certain bases that will make matrix computation easier and intuitive. Let us first fix a notation for matrix corresponding to a linear mapping:

**Definition 16.1.** Let  $\phi : V \rightarrow W$  be a linear mapping between two vector spaces  $V$  and  $W$  over a fixed field  $k$  with dimension  $m$  and  $n$ , respectively. Fix bases  $A = \{v_1, \dots, v_m\}$  for  $V$  and  $B = \{w_1, \dots, w_n\}$  for  $W$ . The matrix associated to  $\phi$  with respect to bases  $A$  and  $B$  is the  $n \times m$ -matrix over  $k$  given by

$$M_A^B(\phi) = (c_{ij}) \quad \text{where } \phi(v_j) = \sum_i c_{ij} w_i$$

**Example 16.1.** Rotation in  $\mathbb{R}^2$  and in  $\mathbb{R}^3$ , Projection, Embedding...

The natural question to ask at this point would be "What happens if we change our choice of bases?" The following observation answers to this question:

**Proposition 16.1.** Let  $A' = \{v'_1, \dots, v'_m\}$  and  $B' = \{w'_1, \dots, w'_n\}$  be different bases for  $V$  and  $W$ , respectively. Then we have the following formula between matrices corresponding to  $\phi$  with respect to these bases

$$M_{A'}^{B'}(\phi) = M_B^{B'}(1) M_A^B(\phi) M_{A'}^A(1)$$

where  $M_{A'}^A(1) = (a_{ii'})$  for  $v_i = \sum_{i'} a_{ii'} v_{i'}$  and  $M_B^{B'}(1) = (b_{j'j})$  for  $w_{j'} = \sum_j b_{j'j} w_j$ .

In this sense, we may call  $M_{A'}^A(1)$  the *base-change-matrix*. The composition of linear mappings corresponds to the multiplication of the associated matrices

**Proposition 16.2.** Let  $\psi : U \rightarrow V$  and  $\phi : V \rightarrow W$  be linear mappings. Then we have

$$M_A^C(\phi \circ \psi) = M_B^C(\phi) M_A^B(\psi)$$

where  $A$ ,  $B$ , and  $C$  are bases for  $U$ ,  $V$ , and  $W$ , respectively.