

## 15. DUAL SPACES

**Definition 15.1.** Let  $V$  be a vector space over a field  $k$ . Consider the set  $V^*$  of linear maps

$$\phi : V \rightarrow k$$

We can endow  $V^*$  the vector space structure by defining, for  $r \in k$  and  $\phi, \psi \in V^*$ ,

$$(r\phi)(v) = r\phi(v)$$

$$(\phi + \psi)(v) = \phi(v) + \psi(v)$$

This vector space  $V^*$  is called the *dual space* of  $V$ , and its elements are called *functionals*.

**Proposition 15.1.** *If  $V$  is finite-dimensional, then  $V \cong V^*$  and  $\dim(V) = \dim(V^*)$ .*

*Proof.* We construct the *dual basis*  $\alpha^* = \{v_1^*, \dots, v_n^*\}$  of  $V^*$  for a given basis  $\alpha = \{v_1, \dots, v_n\}$  of  $V$ . This gives isomorphism between  $V$  and  $V^*$  by sending  $v_i \leftrightarrow v_i^*$ .  $\square$

Since  $V^*$  is the dual of  $V$ , we can think of the dual space of  $V^*$ , i.e.  $(V^*)^*$ : Define the map

$$\rho : V \rightarrow (V^*)^*$$

by  $\rho(v)(\phi) = \phi(v)$ . This  $\rho$  is a linear map. We would like to show that  $\rho$  is an isomorphism, but this is not true in general. In fact we can show that

$\rho$  is injective. In case  $V$  is finite-dimensional,  $\rho$  is an isomorphism.

Key idea in showing the injectivity of  $\rho$  is the following observation:

For any nonzero  $v \in V$ , there is a linear map  $\phi \in V^*$  such that  $\phi(v) = 1$ .

Intuitive idea is to observe that, for any given nonzero vector  $v \in V$ , we can find a basis containing  $v$ . For finite-dimensional vector spaces, this is obvious since the process of adding linearly independent vectors must stop. In general, we rely on the *Zorn's lemma*.

**Zorn's lemma:** Let  $S$  be a partially ordered set. Suppose that every chain in  $S$  has an upper bound in  $S$ . Then  $S$  contains a maximal element.