Proposition 12.3. 1) *If* $0 \in S$*, then* S *is linearly dependent.*

2) *If* S *is linearly dependent and* $S \subset T \subset V$*, then* T *is linearly dependent.*

3) *If* S *is linearly independent and* $U \subset S \subset V$*, then* U *is linearly independent.*

4) $S \subset V$ *is a basis for* V *iff each element of* V *is a unique linear combination of elements of* S*.*

5) *If* $A = \{v_1, \cdots, v_m\}$ *spans* V*, then some subset of* A *is a basis for* V*.*

6) *Let* W *be a subspace of the finite-dimensional vector space* V*. Then any basis for* W *can be extended to a basis for* V*.*

Theorem 12.1. The Rank Theorem. *For a linear map* $\phi : V \to W$*, we have*

$$\textit{dim}\ V = \textit{dim}\ (\textit{Ker}\ \phi) + \textit{dim}\ (\textit{Im}\ \phi)$$

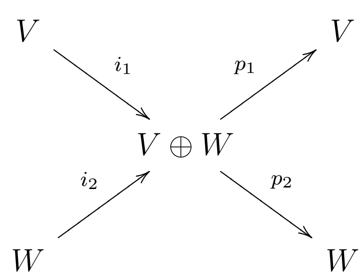
Corollary 12.1. *Let* V *and* W *be finite-dimensional vector spaces. Then we have*

1) $\phi \in \textit{End}(V)$ *is an isomorphism iff* ϕ *is injective iff* ϕ *is surjective.*

2) $V \cong W$ *iff* V *and* W *have the same dimension.*

13. Direct sums, Exact sequence, and Quotients

Given two vector spaces V and W, we can form the *(external) direct sum* of V and W, denoted $V \oplus W$: $V \oplus W$ is a vector space with canonical structure induced from those of V and W, along with natural injection and projection mappings



$$i_1(v) = (v, 0), \qquad p_1(x, y) = x$$

$$i_2(w) = (0, w), \qquad p_2(x, y) = y$$

$$p_1 \circ i_1 = id_V, \qquad p_2 \circ i_2 = id_W$$

Proposition 13.1. 1) *If* $\{v_1, \cdots, v_n\}$ *is a basis for* V *and* $\{w_1, \cdots, w_m\}$ *is a basis for* W*, then* $\{(v_1, 0), \cdots, (v_n, 0), (0, w_1), \cdots, (0, w_m)\}$ *is a basis for* $V \oplus W$*.*

2) $\textit{dim}(V \oplus W) = \textit{dim}(V) + \textit{dim}(W)$*.*

Now, given two subspaces A and B of a vector space V, we can consider a natural mapping $\eta : A \oplus B \to V$ defined by $\eta(a, b) = a + b$. We have

Proposition 13.2. 1) η *is a linear map.*

2) η *is injective if and only if* $A \cap B = \{0\}$*.*

3) η *is surjective if and only if* $A \cup B$ *spans* V*.*

In case η is an isomorphism, we say that V is an *internal direct sum* of A and B. One useful concept is the *exact sequence*: Given vector spaces U, V, W and linear maps $\rho : U \to V$ and $\sigma : V \to W$, we say that the sequence $U \xrightarrow{\rho} V \xrightarrow{\sigma} W$ is *exact* if im(ρ)=ker(σ). Moreover the exact sequence $0 \longrightarrow U \xrightarrow{\rho} V \xrightarrow{\sigma} W \longrightarrow 0$ is said to *split* if there is a linear mapping $\gamma : W \to V$ such that $\sigma \circ \gamma$ is the identity mapping on W.

Proposition 13.3. 1) *For finite-dimensional vector spaces, the above exact sequence splits.*

2) *In the above situation, V is represented as an internal direct sum*

$$V = \rho(U) \oplus \gamma(W)$$

.

Given a subspace W of a vector space V, we can form the quotient space V/W using the equivalence relation. For finite-dimensional vector spaces, we have

Proposition 13.4. 1) *V/W is a vector space.*

2) *The natural projection mapping $\eta : V \to V/W$ is a surjective linear map.*

3) *For a linear map $\phi : U \to V$, we have $U/ker\phi \cong \phi(U)$*

14. Eigenvectors and Eigenvalues

Definition 14.1. 1) Let $\phi : V \to V$ be linear. A subspace W of V is said to be *ϕ-stable* if $\phi(W) \subset W$. If W is ϕ-stable, then $\phi|_W \in \mathrm{End}(W)$.

2) A nonzero vector $v \in V$ is an *eigenvector* for $\phi : V \to V$ if Span(v) is ϕ-stable. The unique λ such that $\phi(v) = \lambda v$ is called an *eigenvalue* for ϕ corresponding to v.

3) Let $V(\lambda) = \{v \in V \mid \phi(v) = \lambda v\}$ and call it the *eigenspace belonging to* λ. The dimension of $V(\lambda)$ is the *geometric multiplicity* of λ.

For example, the set E of elementary functions in the real vector space D of C^∞-functions from $[0, 1]$ to $\mathbb{R}$ is stable under differentiation, but is not stable under integration.

Proposition 14.1. 1) *If v is an eigenvector for ϕ, then rv for any scalar $r \in k$ is an eigenvector for ϕ with the same eigenvalue as v.*

2) *If $\lambda, \mu \in k$ and $\lambda \neq \mu$, then $V(\lambda) \cap V(\mu) = \{0\}$.*

3) *There exists a linear map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $V(\lambda) = \{0\}$ for each $\lambda \in \mathbb{R}$.*

4) *If v is an eigenvector for ϕ with eigenvalue λ, then $\psi(v)$ is an eigenvector for $\psi\phi\psi^{-1}$ with eigenvalue λ.*