

Now we move on to Curtis' book "Abstract Linear Algebra".

11. VECTOR SPACES

Vector spaces over a field is defined as before. We'll mainly look at some more examples of vector spaces:

- 1) \mathbb{R}^n .
- 2) Let V consist of all functions $f \in C^2(\mathbb{R})$ which satisfy

$$\frac{d^2 f}{dx^2} + a \frac{df}{dx} + bf = 0$$

Then V is a real vector space. For example, in the special case $a = 0$ and $b > 0$, one can check that $\cos(\sqrt{b}x)$ and $\sin(\sqrt{b}x)$ are elements in V . If, moreover, we only look at analytic functions, i.e. those that can be written as a convergent power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

, then $\{\cos(\sqrt{b}x), \sin(\sqrt{b}x)\}$ is a basis for V .

- 3) Let V be the set of all continuous functions from $[0, 1]$ to \mathbb{R} . V is a real vector space which is infinite-dimensional.

Proposition 11.1. *A subset W of a vector space V is a subspace iff W is closed under linear combinations. Arbitrary intersection of subspaces in a vector space is again a subspace.*

Definition 11.1. Let A be a subset of a vector space V . The $\text{Span}(A)$ is the intersection of all subspaces of V which contain A . Equivalently, it is a subspace of V consisting of all finite linear combinations of vectors in A .

Example 11.1. 1) $\text{Span}(\{(t, t^2, t^3) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}) = \mathbb{R}^3$. We only need three vectors to span whole \mathbb{R}^3 . This is called the twisted cubic. It's also related to the Vandermonde determinant.

- 2) Let A be the following set of 2×2 -matrices with real coefficients:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then one can see that $\text{Span}(A) = \text{Mat}_{2 \times 2}(\mathbb{R})$, i.e. the set of all 2×2 -matrices over \mathbb{R} .

12. LINEAR MAPS

Definition 12.1. A mapping $\phi : V \rightarrow W$ between two vector spaces is *linear* if it preserves linear combinations, i.e. for any $a, b \in k$ and $x, y \in V$,

$$\phi(ax + by) = a\phi(x) + b\phi(y)$$

The importance of a basis for vector spaces is reflected in the following observation

Let $A = \{v_1, \dots, v_n\}$ and $B = \{w_1, \dots, w_m\}$ be bases for vector spaces V and W , respectively. Any linear map $\phi : V \rightarrow W$ is completely determined by its effect on the bases, i.e. the coefficients c_{ij} where

$$\phi(v_i) = \sum_{j=1}^m c_{ij} w_j$$

completely determines the linear map ϕ . Hence we can identify the linear map ϕ with its associated matrix under the choice of bases for V and W .

$$\begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nm} \end{pmatrix}$$

Proposition 12.1. *If $\phi : V \rightarrow W$ is a linear isomorphism between vector spaces, then $\phi^{-1} : W \rightarrow V$ is also a linear isomorphism.*

Now we can think of an algebra structure on the vector space of linear mappings, i.e.

$$\text{End}(V) = \{ \phi : V \rightarrow V \mid \phi \text{ is linear} \}$$

forms an algebra, called the endomorphism space of V .

Proposition 12.2. 1) *Let A be an associative algebra with unit element 1. Then the set U of all units in A is a group under multiplication.*

2) *$\phi \in \text{End}(V)$ is a unit if and only if ϕ is an isomorphism.*