9. Euclidean Domains

Definition 9.1. A *Euclidean norm* on an integral domain D is a function ν mapping the nonzero elements of D into the nonnegative integers such that

1) For all $a, b \in D$ with $b \neq 0$, there exists q and r in D such that $a = bq + r$, where either $r = 0$ or $\nu(r) < \nu(b)$.
2) For all nonzero $a, b \in D$, we have $\nu(a) \leq \nu(ab)$.

An integral domain D is a *Euclidean domain* if there exists a Euclidean norm on D.

Theorem 9.1. 1) *Every Euclidean domain is a PID. Hence, every Euclidean domain is a UFD.*

2) $\nu(1)$ *is minimal.* $u \in D$ *is a unit if and only if* $\nu(u) = \nu(1)$.
3) *Euclidean Algorithm. Let* a *and* b *be nonzero elements of* D. *We have a system of equations and* r_i*'s:*

$$
\begin{aligned}
a &= bq_1 + r_1, && r_1 = 0 \text{ or } \nu(r_1) < \nu(b)\\
b &= r_1 q_1 + r_2, && r_2 = 0 \text{ or } \nu(r_2) < \nu(r_1)\\
&\cdots\\
r_{i-1} &= r_i q_{i+1} + r_{i+1}, && r_{i+1} = 0 \text{ or } \nu(r_{i+1}) < \nu(r_i)\\
&\cdots
\end{aligned}
$$

The sequence $r_1, r_2, \cdots$ *must terminate with some* $r_s = 0$. *The gcd of* a *and* b *is* r_{s-1}. *Furthermore, if* d *is a gcd of* a *and* b, *then there exists* λ *and* μ *in* D *such that* $d = \lambda a + \mu b$.