

7. VECTOR SPACES

Definition 7.1. Let F be a field. A *vector space over F* (or F -vector space) consists of an abelian group V under addition together with an operation of scalar multiplication of each element of V by each element of F on the left, i.e. equipped operations:

$$\begin{aligned} + : V \times V &\longrightarrow V, & (\alpha, \beta) &\mapsto \alpha + \beta && \text{addition operation} \\ \cdot : F \times V &\longrightarrow V, & (a, \alpha) &\mapsto a \cdot \alpha && \text{scalar multiplication} \end{aligned}$$

satisfying compatibility relations

$$a \cdot (b \cdot \alpha) = (ab)\alpha, \quad (a + b)\alpha = a \cdot \alpha + b \cdot \alpha, \quad a \cdot (\alpha + \beta) = a \cdot \alpha + a \cdot \beta, \quad 1 \cdot \alpha = \alpha$$

One can show that, from compatibility conditions, we have for a vector space V over F :

$$0 \cdot \alpha = 0, \quad a \cdot 0 = 0, \quad (-a) \cdot \alpha = a \cdot (-\alpha) = -(a \cdot \alpha)$$

The most basic idea about vector spaces is that of linear independence:

Definition 7.2. Let V be a vector space over F . Consider a collection of vectors $S = \{\alpha_i \mid i \in I\}$ where I is an index set (possibly infinite).

- 1) S *span* V if for every $\beta \in V$, we have a finite subset $J \subset I$ and scalars $a_j \in F$ for each $j \in J$ such that

$$\beta = \sum_{j \in J} a_j \alpha_j$$

Expressions of this form is called a *linear combination of the α_j 's*.

- 2) Vectors in S are *linearly independent over F* if, for any finite subset $J \subset I$ and scalars a_j for each $j \in J$, we have $\sum_{j \in J} a_j \alpha_j = 0$ in V if and only if $a_j = 0$ for all $j \in J$. If the vectors are not linearly independent over F , they are *linearly dependent over F* .
- 3) A vector space V over a field F is *finite dimensional* if there is a finite subset of V whose vectors span V .
- 4) S is called a *basis for V over F* if the vectors in S span V and are linearly independent.

Theorem 7.1. *In a finite-dimensional vector space, every spanning set of vectors contains a subset which is a basis. Consequently, any finite-dimensional vector space has a finite basis.*

Theorem 7.2. *Let $S = \{\alpha_1, \dots, \alpha_r\}$ be a finite set of linearly independent vectors of a finite-dimensional vector space V over a field F . Then S can be enlarged to a basis for V over F . Furthermore, if $B = \{\beta_1, \dots, \beta_n\}$ is any basis for V over F , then $r \leq n$. Consequently, any two bases of a finite-dimensional vector space V over F have the same number of elements.*

Definition 7.3. If V is a finite-dimensional vector space over a field F , the number of elements in a basis is the *dimension of V over F* .

Theorem 7.3. Let E be an extension field of F , and let $\alpha \in E$ be algebraic over F . If $\deg(\alpha, F) = n$, then $F(\alpha)$ is an n -dimensional vector space over F with a basis $\{1, \alpha, \dots, \alpha^{n-1}\}$. Furthermore, every element β of $F(\alpha)$ is algebraic over F , and $\deg(\beta, F) \leq \deg(\alpha, F)$.