

Given an extension field E of F and an element $\alpha \in E$. We can consider an extension field $F(\alpha)$ of F given by adding α to F . Precisely, we have two cases:

When α is algebraic over F : The kernel of the evaluation homomorphism ϕ_α is the maximal ideal $\langle \text{irr}(\alpha, F) \rangle$ generated by the irreducible polynomial for α over F . Hence the image $\phi_\alpha(F[x])$ is a subfield of E isomorphic to $F[x]/\langle \text{irr}(\alpha, F) \rangle$, and it is the smallest subfield of E containing F and α . We denote this field by $F(\alpha)$.

When α is transcendental over F : The kernel of ϕ_α is trivial in this case and ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E . Hence $\phi_\alpha[F[x]]$ is an integral domain which we denote by $F[\alpha]$. The quotient field of $F[\alpha]$ is contained in E and we denote this field by $F(\alpha)$.

Definition 6.3. An extension field E of a field F is a *simple extension of F* if $E = F(\alpha)$ for some $\alpha \in E$.

Theorem 6.4. Let E be a simple extension $F(\alpha)$ of a field F where α is algebraic over F . Let the degree of α over F be $n \geq 1$. Then every element β of $F(\alpha)$ can be uniquely written in the form

$$\beta = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$$

where the $b_i \in F$, i.e. $F(\alpha)$ is a vector space of dimension n over F with a basis $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$.