A lower bound for disconnection by simple random walk

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What we talk about when we talk about disconnection

- Consider (continuous-time) simple random walk \((X_t)_{t \geq 0}\) on \(\mathbb{Z}^d, d \geq 3\), started from 0. Denote its law by \(P_0\).
- Trace: \(\mathcal{I} = X_{[0, \infty)}\). Vacant set: \(\mathcal{V} = \mathbb{Z}^d \setminus \mathcal{I}\).
- For compact \(K \subset \mathbb{R}^d\) and \(N \in \mathbb{N}\), look at its discrete blow-up
  \[ K_N \triangleq \{x \in \mathbb{Z}^d; d_\infty(NK, x) \leq 1\} \]
- Event of interest \(A_N \triangleq \left\{ K_N \not\leftrightarrow \infty \right\} \): “no path in \(\mathcal{V}\) connects \(K_N\) with infinity”.

\(\mathcal{I}\)
The asymptotic lower bound

- Question is trivial if $d = 1$ or $d = 2$ (recurrence).

**Theorem (L. 14’)**

*One has the following asymptotic lower bound:*

$$\liminf_{N \to \infty} \frac{1}{N^{d-2}} \log P_0[A_N] \geq -\frac{u^{**}}{d} \text{cap}_{\mathbb{R}^d}(K).$$

- $\text{cap}_{\mathbb{R}^d}(K)$: the Brownian capacity of $K$.
  
  $= \inf \mathcal{E}(g, g)$, where infimum runs over all compactly supported non-negative $g \in H^1$ such that $g \geq 1$ on $K$.

- $u^{**} \in (0, \infty)$ is one of the critical thresholds for the percolation on the vacant set of *random interlacements*.

- Is the lower bound tight?
The same lower bound for a slightly different setup

- Let $S_N \triangleq B_\infty(0, N) \cap \mathbb{Z}^d$.
- Fix $M > 1$ and consider $B_N \trianglerighteq \{ S_N \leftrightarrow \partial S_{MN} \}$.

Proof for the lower bound of $P_0[A_N]$ also works for $P_0[B_N]$.

Theorem (L. 14')

$$\liminf_{N \to \infty} \frac{1}{N^{d-2}} \log P_0[B_N] \geq -\frac{u^{**}}{d} \text{cap}_{\mathbb{R}^d}([-1, 1]^d).$$
An accompanying upper bound

Theorem (L. 14’)

\[ \liminf_{N \to \infty} \frac{1}{N^{d-2}} \log P_0[B_N] \geq - \frac{u^{**}}{d} \text{cap}_{\mathbb{R}^d}([-1, 1]^d). \]

Theorem (Sznitman 14’)

\[ \limsup_{N \to \infty} \frac{1}{N^{d-2}} \log P_0[B_N] \leq - \frac{\bar{u}}{d} \text{cap}_{\mathbb{R}^d}([-1, 1]^d). \]

- \( \bar{u} \): another percolation threshold for the vacant set of random interlacements.
- \( 0 < \bar{u} \leq u^{**} < \infty \).
- Conjecture: \( \bar{u} (= u_*) = u^{**} \)?
Change of measure

- A useful classical inequality: for $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $A$ s.t. $\tilde{\mathbb{P}}[A] \neq 0$,

$$\mathbb{P}[A] \geq \tilde{\mathbb{P}}[A] \exp\left(-\left(\mathbb{H}(\tilde{\mathbb{P}}|\mathbb{P}) + c\right)/\tilde{\mathbb{P}}[A]\right).$$

- If $\tilde{\mathbb{P}}[A] \approx 1$, then $\mathbb{P}[A] \gtrapprox \exp(-\mathbb{H}(\tilde{\mathbb{P}}|\mathbb{P})).$

- Hence, we need a family of tilted probability measures $\tilde{\mathbb{P}}_N$, corresponding to (possibly non-homogeneous) Markov chains, such that
  - $\tilde{\mathbb{P}}_N[A_N] \approx 1$;
  - $\mathbb{H}(\tilde{\mathbb{P}}_N|\mathbb{P}_0)$ is minimized.
A la recherche de la Connectivity Decay

- Let $K^\delta \triangleq B(K, \delta)$ and write $K^\delta_N$ for its discrete blow-up.

- Let $\Gamma \triangleq \partial K^\delta_N/2$ be a “strip” between $\partial K_N$ and $\partial K^\delta_N$.

- Note that $A^c_N = \{K_N \leftrightarrow \infty\} \implies \{K_N \leftrightarrow \partial K^\delta_N\} \implies \bigcup_{x \in \Gamma} \{x \leftrightarrow \partial B(x, N^{1/3})\}.$

- $|\Gamma| = O(N^{d-1}).$

- Hence, $\tilde{P}_N[\{x \leftrightarrow \partial M\}] \leq e^{-N^c} \implies \tilde{P}_N[A^c_N] \leq e^{-N^{c'}}.$
Deus ex machina: Random Interlacements

- How “much” tilting to ensure $\tilde{P}_N[A_N] \geq 1 - \exp(-N^{c'})$?
- Random interlacements: The natural model for the study of traces left by random walk.
- Random subset $\mathcal{I}^u \subset \mathbb{Z}^d, d \geq 3$: the traces of a Poissonian random collection of infinite trajectories, with intensity parameter $u > 0$.
- The connectivity decay we need: for $d \geq 3$, $\exists u_{**}(d) \in (0, \infty)$ such that
  - for all $u > u_{**}$, the connectivity of the vacant set $\mathcal{V}^u \triangleq \mathbb{Z}^d \backslash \mathcal{I}^u$ decays stretched-exponentially fast, i.e., $\exists c > 0$, s.t.
    \[ \mathbb{P}^u[0 \leftrightarrow \partial B(0, N)] \leq e^{-N^c} ; \]
  - and for all $u \in (0, u_{**})$ the connectivity decays slower than any stretched exponential function, i.e., such $c$ does not exist.
Only local couplings needed

- Only need to find couplings of $\tilde{P}_N$ and \( P^{u**+\epsilon} \) on all 
  \( M = B_\infty(x, N^{1/3}) \) for \( x \in \Gamma \), i.e., on \( M \), with high probability
  - \( \mathcal{I} \cap M \supset \mathcal{I}^{u**+\epsilon} \cap M \), or
  - \( \mathcal{V} \cap M \subset \mathcal{V}^{u**+\epsilon} \cap M \), equivalently.
The tilted random walk

- Generator of a (possibly non-homogeneous) MC

\[ \overline{L}h(x) = \frac{1}{2d} \sum_{|e|=1} f(x+e)/f(x) \left( h(x+e) - h(x) \right), \]

where \( f \) is to be chosen to our advantage.
- If \( f \equiv C \), \( \overline{L} \) is the generator of simple random walk.
- Transition rate from \( x \) to \( x+e \): \( f(x+e)/f(x) \).
- MC lands more often on the vertices where \( f \) is larger.
- Reversibility measure: \( \pi(x) = f^2(x) \).

- Construction of the tilted random walk \( \tilde{P}_N \):
  - Let MC run with the generator \( \overline{L} \) up to some \( T_N \) deterministic;
  - After \( T_N \) it is “released” and runs as SRW happily ever after.
- \( H(\tilde{P}_N|P_0) \) is proportional to \( T_N \) and the discrete Dirichlet form of \( f \).
- Need to minimize both \( T_N \) and the Dirichlet form!
Mount Roraima: NOT the best choice of $f$
The story of $f$

- Let $U \triangleq B(0, R)$ for large $R$ and $U_N$ be its discrete blow-up.

- Let $f = 1$ on $K_N^\delta$, $f = 0$ outside $U_N$.

- In $U_N \setminus K_N^\delta$, choose $f$ as discretized Brownian potential of $K_N^\delta$ w.r.t $U$: best choice to minimize Dirichlet form!
Mount Fuji, the optimal shape of $f$
The story of $f$ reloaded; denouement

- Let $f = 1$ on $K_N^\delta$, $f = 0$ outside $U_N$.
- In $U_N \setminus K_N^\delta$, choose $f$ as discretized Brownian potential of $K^\delta$ w.r.t $U$.

Choose $T_N$ such that: on $K_N^\delta$, the expected occupation time of the tilted walk equals that of random interlacements with intensity $u^{**} + \epsilon$.

- This ensures the feasibility of the couplings!
- In the asymptotic lower bound $\exp(-u^{**} \text{cap}_{\mathbb{R}^d}(K) N^{d-2}/d)$,
  - $u^{**}$ will appear from the choice of $T_N$, taking $\epsilon \to 0$;
  - $\text{cap}_{\mathbb{R}^d}(K)$ and $d$ will appear from the limit of the Dirichlet form of $f$, taking $R \to \infty$ and $\delta \to 0$.
  - $N^{d-2}$ comes from both terms.
Comparison with results for Random Interlacements

Theorem (L.-Sznitman 13’, Sznitman 14’)

Consider \( u \in (0, u_{**}) \). Similar to the definition of \( A_N \) and \( B_N \), let \( A_u^N \triangleq \{ K_N \not
\infty \} \) and \( B_u^N \triangleq \{ S_N \not
\partial S_{MN} \} \). Then,

\[
\lim \inf_{N \to \infty} \frac{1}{N^{d-2}} \log \mathbb{P}^u[A_u^N] \geq -\frac{1}{d}(\sqrt{u_{**}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K),
\]

and

\[
\lim \sup_{N \to \infty} \frac{1}{N^{d-2}} \log \mathbb{P}^u[B_u^N] \leq -\frac{1}{d}(\sqrt{u} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}([-1, 1]^d).
\]

▶ If \( u > u_{**} \), \( \mathbb{P}^u[A_N] \to 1 \) as \( N \to \infty \). Hence the problem is non-trivial only when \( u < u_{**} \).

▶ Intuitively, the case for SRW is a limiting case (\( u \to 0 \)) of the disconnection problem for random interlacements.

▶ Proof of lower bound includes similar tiltings, but with different scheme.
Thanks for your attention!