DEFINITION OF TRANSFER FACTORS IN STANDARD ENDOSCOPY: A SUMMARY

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CONTENTS

1. Introduction 1
2. Review on L-groups 1
3. Abstract Cohomological Formulations 2
4. Story on G-side: a-data 3
5. Story on G-side: χ-data 6
6. Endoscopic Groups and Transfer Factors 8
7. Loose Ends 12
References 12

1. INTRODUCTION

In this short notes, I try to summarize the definition of transfer factors in standard endoscopy theory in a way more clear for myself to read. The materials are mostly extracted from Langlands and Shelstad [2], and some are from Kottwitz [1]. There are some other results, well-known or folklore, that I didn’t include a citation since they are more or less standard now (and I’m lazy).

I do try to write the definitions of a-data and χ-data in a more “symmetric” way and to emphasize the duality between them. For some technical parts, I also try to give an even more brief conceptual outline (see § 4.5 for example).

For anyone who is semi-newcomer to this subject like me, a clear and precise understanding of the functorial formulation of L-groups is necessary. This notes gives an account on this matter, assuming familiarity with the absolute theory regarding the category of reductive groups.

2. REVIEW ON L-GROUPS

2.1. Let F be a local or global field, and G a connected reductive group over F. For convenience, let \( \overline{F} \) be the separable closure of F, and \( \Gamma_F = \text{Gal}(\overline{F}/F) \), and suppose G is split over \( \overline{F} \). If F is local, let \( | \cdot |_F \) be a fixed absolute value of F. For example, we may choose the “usual” absolute value if F = \( \mathbb{R} \) or \( \mathbb{C} \), and \( |\pi|_F = q^{-1} \) for any uniformizer \( \pi \) of F and q is the order of the residue field of F.

For any two Borel pairs \((T_1, B_1)\) and \((T_2, B_2)\) of G over F, any inner automorphism of G carrying \((T_1, B_1)\) to \((T_2, B_2)\) induces the same isomorphism \(T_1 \rightarrow T_2\). So we have canonical isomorphisms on the associated based root datum

\[
(X(T_1), \Delta(T_1, B_1), \hat{X}(T_1), \hat{\Delta}(T_1, B_1)) \rightarrow (X(T_2), \Delta(T_2, B_2), \hat{X}(T_2), \hat{\Delta}(T_2, B_2)).
\]

Thus we obtain a diagram of based root data indexed by Borel pairs of G, on which \( \Gamma_F \) acts as automorphisms. One can thus form the limit of this diagram, represented by an abstract based root datum \((X, \Delta, \hat{X}, \hat{\Delta})\), on which \( \Gamma_F \)-acts. We shall call this the canonical based root datum of G, denoted

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by $\Psi_0(G)$. The Weyl group together with its simple reflections attached to $\Psi_0(G)$ is denoted by $\Omega_0(G)$. Note that both $\Psi_0(G)$ and $\Omega_0(G)$ encodes the $\Gamma_T$-action.

More generally, if $\Gamma$ is any group acting on $G$ over $F$, then $\Gamma$ acts on $\Psi_0(G)$ and $\Omega_0(G)$ as well.

2.2. Take the $\Gamma_T$-dual of $\Psi_0(G)$, because $C$ is algebraically closed, we can obtain a $C$-reductive group $\tilde{G}$ with a splitting $\langle \tilde{T}, \tilde{B}, \{X_\alpha\}\rangle$, on which $\Gamma_T$ acts. The Weyl group $\Omega_T$ of $\tilde{T}$ in $\tilde{G}$ may be identified with $\Omega_0(G)$, compatible with $\Gamma_T$-action.

The Weil group $W_F$ acts on these objects via projection $W_F \to \Gamma_T$, and we can form $L$-group

$$L = \tilde{G} \times W_F.$$

2.3. If $G$ is quasi-split over $F$, fix an $F$-splitting $\langle T, B, \{X_\alpha\}\rangle$ of $G$. The resulting based root system is $\langle \mathcal{X}(T), \Delta(T, B), \tilde{\Delta}(T, B)\rangle$ on which $\Gamma_T$ acts, and in this case it may be identified with $\Psi_0(G)$. Let $\Omega_T$ be the Weyl group of $(G, T)$, then $\Gamma_T$ acts on $\Omega_T$ as well. We may also identify $\Omega_0(G)$ with $\Omega_T$ that is compatible with $\Gamma_T$-action.

2.4. If a group $\Gamma$ acts on a reductive group $H$ over an algebraically closed field, say $C$, the resulting $\Gamma$-action on $\Psi_0(H)$ induces a $\Gamma$-action on some splitting of $H$, hence also an (other) action of $\Gamma$ on $H$. However, the two actions may not coincide, and the original $\Gamma$-action may not fix a splitting of $H$ at all. Therefore we have the following definition.

**Definition 2.1.** Let $H$ be a reductive group over $C$, and $\Gamma$ a group acting on $H$. Such action is called an $L$-action if it stabilizes a splitting.

The action of $W_F$ on $\tilde{G}$ in forming $L$ is thus an $L$-action by definition.

Suppose we have a split extension

$$1 \longrightarrow \tilde{G} \longrightarrow S \longrightarrow W_F \longrightarrow 1,$$

then $S$ is not necessarily an $L$-group since $W_F$-action induced by a splitting of this extension may not be an $L$-action. Nonetheless, for a fixed $\Gamma_T$-splitting of $\tilde{G}$, we may attach to this extension an $L$-action. Let $c : W_F \to S$ be any splitting of this extension, then it induces map $W_F \to \text{Aut}(\tilde{G})$, hence $W_F \to \text{Out}(\tilde{G})$, the latter depending only on the extension but not $c$. Using a fixed $\Gamma_T$-splitting $S_{\text{spl}}$, we may identify $\text{Out}(\tilde{G})$ with the subgroup of $\text{Aut}(\tilde{G})$ that fixes $S_{\text{spl}}$. Therefore we obtain an $L$-action of $W_F$ on $\tilde{G}$. We will call this the $L$-action associated with $S$ and splitting $S_{\text{spl}}$. We shall use the earlier fixed splitting $\langle \tilde{T}, \tilde{B}, \{X_\alpha\}\rangle$ for $S_{\text{spl}}$.

3. **Abstract Cohomological Formulations**

3.1. Let $X$ be a $Z$-lattice, and $R \subset X$ a finite subset such that $-R = R$. A gauge $p$ of $R$ is a map $R \to \{-1, 1\}$ such that $p(-\alpha) = -p(\alpha)$ for all $\alpha \in R$.

For example, if $R \subset X = X(T)$ be the root system for maximal torus $T \subset G$, and $B$ a Borel containing $T$, then $B$ determines a gauge $p_B$ on $R$ such that $p_B(\alpha) = 1$ if and only if $\alpha$ is a root of $T$ in $B$.

If $O \subset R$ be a subset such that $R = O \bigcup -O$ is a disjoint union, then one can define gauge $p_O(\alpha) = 1$ if and only if $\alpha \in O$.

3.2. Let $\Sigma = \Gamma \times \langle \epsilon \rangle$ be a group acting on $X$ and $R$ such that $\epsilon$ acts as $-1$. In our applications we will have $\epsilon^2 = 1$ so we will assume this as well, even though it’s not required in many of the results.
below. Then we define a product notation for any $r$-tuple $\alpha = (a_1, \ldots, a_r) \in \Sigma^r$
\[
\prod_{\alpha: \alpha} = \prod_{\alpha: \alpha_1, \ldots, \alpha_r} := \prod_{\alpha \in R} p((a_1 \cdots \alpha_s^{-1} \alpha) = (-1)^{s+1}
\]

Let $k^x$ be a field and let $\Sigma$ acts on $k$ trivially, then it acts on $k^x \otimes \mathbb{Z} X$, and we denote $c \otimes \lambda$ by $c^\lambda$.

**Lemma 3.1.** The 2-cochain
\[
t_p(\sigma, \tau) = \prod_{\alpha: \alpha, \sigma, \tau} (-1)^{\alpha}
\]
is a 2-cocycle $\Sigma^2 \rightarrow k^x \otimes \mathbb{Z} X$. Moreover, if $q$ is another gauge, $t_p / t_q$ is a coboundary.

3.3. If $p$ is a gauge, then so is $-p$, and we define for a pair of gauge $(p, q)$ and an $r$-tuple $\alpha \in \Sigma^r$ another product
\[
p,q = \prod_{\alpha: \alpha} := \prod_{\alpha \in R} p((a_1 \cdots \alpha_s^{-1} \alpha) = (-1)^{s+1}
\]

Then if we define 1-cochain of $\Gamma$
\[
s_{p/q}(\sigma) = \prod_{\alpha: \alpha, \sigma} (-1)^{\alpha} = (-1)^{\alpha}
\]
then one can show that
\[
\partial s_{p/q} = t_p / t_q,
\]
as cochains of $\Gamma$ (not $\Sigma$).

4. **Story on $G$-side: $\alpha$-data**

4.1. Let $G$ be quasi-split and fix a splitting as before. Let $U_\alpha$ be the root groups of $G$ such that $X_\alpha = U_\alpha(1)$. We can define
\[
n: \Omega_T \times \Gamma_F \longrightarrow N_G(T) \times \Gamma_F
\]
\[
w \times \sigma \longmapsto n(w) \times \sigma,
\]
where $n(w)$ is such that if $w = s_{\alpha_1} \cdots s_{\alpha_r}$ is a reduced expression, then
\[
n(w) = n(s_{\alpha_1}) \cdots n(s_{\alpha_r}),
\]
\[
n(s_{\alpha}) := U_\alpha(1)U_{{-\alpha}}(-1)U_\alpha(1),
\]
\[
n(1) := 1.
\]

One can show that $n(w)$ is independent of the reduced expression hence is well defined, and that $n: \Omega_T \rightarrow N_G(T)$ is $\Gamma_F$-equivariant. Therefore for $\theta \in \Omega_T \times \Gamma_F$, $n(\theta)$ acts on $T$ as $\theta$, and
\[
t(\theta_1, \theta_2) := n(\theta_1)n(\theta_2)n(\theta_1 \theta_2)^{-1}
\]
is a 2-cocycle of $\Omega_T \times \Gamma_F$ in $T(\overline{F})$. 
Lemma 4.1. We have that 
\[ t(\theta_1, \theta_2) = t_{p^B}(\theta_1, \theta_2) = \prod_{\alpha: 1, \theta_1, \theta_2} (-1)^{\hat{\alpha}}, \]
where \( p_B \) is the gauge determined by \( B \) on \( \hat{R}(G, T) \).

Note that the definition of \( t \) depends on root vectors \( X_\alpha \), while the right-hand side of the Lemma above doesn’t.

4.2. The 2-cocycle \( t \) is in fact a coboundary when restricted to certain subgroup of \( \Omega_T \times \Gamma_F \), and a splitting can be found with the help of so-called \( a \)-data. The abstract formulation is as follows. Retain notations in the subsection about \( \Sigma \) acting on \( X \) and \( R \). Suppose \( \Sigma \) acts on \( \bar{k}/k \) such that \( e \) still acts trivially (but not necessarily for \( \Gamma \)).

Definition 4.2. An \( a \)-datum is a \( \Gamma \)-equivariant map 
\[ a: R \to \bar{k}^\times \]
\[ \alpha \mapsto a_\alpha \]
such that \( a_{-\alpha} = -a_\alpha \) (i.e. \( \alpha \) is “\( e \)-antivariant”). A \( b \)-datum is a \( \Gamma \)-equivariant map \( b: R \to \bar{k}^\times \) that is also \( e \)-equivariant (hence \( \Sigma \)-equivariant).

Suppose \( a \)-data exists for \( \Sigma \)-action on \( R \), and \( p \) a gauge of \( R \), then we form 1-cochain of \( \Gamma \)
\[ u_p(\sigma) := \prod_{\alpha: 1, \sigma} a_\alpha^\alpha \in \bar{k}^\times \otimes_Z X. \]

Lemma 4.3. Viewing \( t_p \) as a 2-cocycle of \( \Gamma \) with value in \( \bar{k}^\times \otimes_Z X \supset k^\times \otimes_Z X \), we have that 
\[ \partial u_p = t_p. \]

Similarly, if \( b \)-data exists, we can form 1-cochain that is in fact a cocycle:
\[ v_p(\sigma) := \prod_{\alpha: 1, \sigma} b_\alpha^\alpha \in \bar{k}^\times \otimes_Z X. \]

4.3. To emphasize the duality to \( \chi \)-data later, here we use \( F \) instead of \( k \). Let \( \Gamma = \Gamma_F \). For \( \alpha \in R \), let \( F_\alpha \) be its splitting field, and \( F_{\pm \alpha} \) be the splitting field of \( \pm \alpha \). Let \( \Gamma_\alpha \) and \( \Gamma_{\pm \alpha} \) be their respective absolute Galois group. Then \( [F_\alpha : F_{\pm \alpha}] = 1 \) if \( \Gamma_F \)-orbit of \( \alpha \) doesn’t contain \( -\alpha \) or \( 2 \) otherwise.

Suppose we have \( a \)-data for \( \Gamma \)-action on \( R \). Then we always have \( a_\alpha \in F_\alpha^\times \). If moreover \( [F_\alpha : F_{\pm \alpha}] = 2 \), we must have that \( \sigma(a_\alpha) = -a_\alpha \) for the unique non-trivial element \( \sigma \in \Gamma_{\pm \alpha}/\Gamma_\alpha \). This means \( a_\alpha^2 = -1 \) viewed as elements of group \( F_\alpha^\times / Nm_{F_\alpha/F_{\pm \alpha}}(F_\alpha^\times) \).

4.4. Let \( T \subset G \) be a maximal \( F \)-torus. Let \( h \in G(F) \) be a chosen transporter from \( T \) to \( T \), i.e. \( Ad_h(T) = T \). Then \( h^{-1} \sigma(h) \) acts on \( T \) by conjugation, hence \( h^{-1} \sigma(h) \in N_G(T) \), whose image in \( \Omega_T \) is denoted by \( \omega_T(\sigma) \). Thus if we denote by \( \sigma_T \) the action of \( \sigma \) on \( T \) by transporting that on \( T \) to \( T \) using \( h \), then \( \sigma_T = \omega_T(\sigma) \times \sigma \in \Omega_T \times \Gamma_F \). Let \( \Gamma_T \) be the group generated by \( \sigma_T \). Clearly \( \sigma_T \) depends only on the choice of \( B = Ad_h(B) \), not \( h \) itself.

The action of \( \Sigma = \Gamma_T \times \langle e \rangle \) on \( \hat{R}(G, T) \subset \hat{X}(T) \) admits \( a \)-data, which transports to \( \Gamma_T \times \langle e \rangle \)-action on \( R(G, T) \). Let \( \{ a_\alpha \} \) be an \( a \)-datum. We have gauge \( p = p_B \) on \( \hat{R}(G, T) \), so we have 
\[ x_p(\sigma_T) = \prod_{\alpha: 1, \sigma_T} a_\alpha^\alpha, \]
whose coboundary is
\[ \partial_p (\sigma_T, \tau_T) = n(\sigma_T)n(\tau_T)n(\sigma_T\tau_T)^{-1}. \]

Since \( a^{-1}_\chi \) is also an \( a \)-datum, we also have that
\[ \partial \chi_p^{-1} = t_p. \]

Thus the map
\[ \Gamma_T \to N_G(T) \times \Gamma_F \]
\[ \sigma_T \mapsto x_p(\sigma_T)n(\sigma_T) \]
is a homomorphism, hence induces 1-cocycle \( \sigma_T \mapsto x_p(\sigma_T)n(\omega_T(\sigma)) =: m(\sigma_T). \)

Transporting by \( \text{Ad}_{h \times 1} \) inside \( G \times \Gamma_F \), one has map
\[ \Gamma_F \to N_G(T) \times \Gamma_F \]
\[ \sigma \mapsto h x_p(\sigma_T)n(\sigma_T)h^{-1} = h m(\sigma_T)\sigma(h)^{-1} \times \sigma, \]
whose image lies in \( T \times \Gamma_F \). One thus obtains a 1-cocycle of \( \Gamma_F \) in \( T \), whose cohomology class depends only possibly on \( B = \text{Ad}_h(B), \) not \( h \). In fact, a long computation would show it doesn’t depend on \( B \) either. Call this cohomology class in \( H^1(F, T) \) by \( \lambda_T \).

4.5. Here we try to summarize the construction of \( \lambda_T \) using \( a \)-data more conceptually. To begin with, we have extension
\[ 1 \to T \to N_G(T) \times \Gamma_F \to \Omega_T \times \Gamma_F \to 1. \] (4.5.1)
The choice of a \( \Gamma_F \)-equivariant set-theoretic section \( n : \Omega_T \to N_G(T) \) gives a set-theoretic section \( n \times \text{id} \) of this extension, which in turn gives a 2-cocycle of \( \Omega_T \times \Gamma_F \) in \( T \). We still use \( n \) instead of \( n \times \text{id} \) for convenience.

Given \( T \), we choose \( B \) containing \( T \), and \( h \in G \) such that \( \text{Int}_h \) maps \( (T, B) \) to \( (T, B) \). Then we obtain another splitting of \( \Omega_T \times \Gamma_F \) via map \( \Gamma_F \to \Gamma_T \). Restricting the extension (4.5.1) to \( \Gamma_T \), we have
\[ 1 \to T \to N_T \to \Gamma_T \to 1, \]
\[ 1 \to T \to N_G(T) \times \Gamma_F \to \Omega_T \times \Gamma_F \to 1, \]
where the square on the right is Cartesian. This extension of \( \Gamma_T \) is split, and a choice of an \( a \)-datum provides a splitting \( x_p n \), whose composition with (set-theoretic) projection \( N_T \to N_G(T) \) gives 1-cocycle \( m \).

Transporting using \( \text{Int}_h \), one has split extension
\[ 1 \to T \to N_T := hN_T h^{-1} \to \Gamma_T \to 1, \]
where in fact \( N_T = T \times \Gamma_F \subset N_G(T) \times \Gamma_F \). The difference between the natural splitting \( T \times \Gamma_F \) and the splitting obtained using an \( a \)-datum gives a class \( \lambda_T \in H^1(F, T) \). The effect of all the choices in the construction of \( \lambda_T \) as well as its various naturalities can be summarized as follows:

(1) it doesn’t depend on \( h \) or \( B \),
(2) a change in splitting \( (T, B, \{X_\alpha\}) \) modifies \( \lambda_T \) by an element in the image of map
\[ \text{coker}(G(F) \to G_{AD}(F)) \to H^1(F, Z) \to H^1(F, T), \]
(3) a change in \(a\)-datum resulting a \(b\)-datum by taking quotient of two \(a\)-data. Forming \(1\)-cocycle \(v_p\) using the same definition as \(x_p\), but with \(a\)-data replaced with \(b\)-data (so it is indeed a cocycle), then \(\lambda_T\) is modified by \(hv_p h^{-1}\).

(4) The construction of \(\lambda_T\) is compatible with conjugation of triples \((T, B, \{a_\alpha\})\).

(5) Finally, if instead \(F\) is global, one can carry out the same construction for \(F\), and for any place \(v\) of \(F\), \(\lambda_{v, T}\) is precisely the image of \(\lambda_T\).

My guess: the class \(\lambda_T\) is the obstruction of lifting \(T\) to an \(F\)-splitting.

5. Story on \(\hat{G}\)-side: \(\chi\)-data

5.1. On the dual side we have \(1^*G\) instead of \(G \times \Gamma_F\) and everything is dualized. In particular, we have the dual notion to \(a\)-data called \(\chi\)-data. We don’t need to assume \(G\) to be quasi-split, but we fix a splitting of \(G\) as before.

The abstract formulation of \(\chi\)-data is as follows. Recall we have \(\Sigma = \Gamma \times \langle \varepsilon \rangle\)-action on \(X\) and \(R\). Unlike \(a\)-data, since arithmetic duality will be used, \(\chi\)-data can only be formulated for \(\Gamma = \Gamma_F\) where \(F\) a local or global field, as we shall assume so. Suppose the action of \(\Gamma\) on \(X\) is continuous with respect to the profinite topology on \(\Gamma\) and discrete topology on \(X\). We use \(C\) to denote either the multiplicative group if \(F\) is local or the idele group if \(F\) is global. Let \(\hat{\bullet}\) denote Pontryagin dual.

Then for \(\alpha \in R\), we have splitting fields \(F_\alpha = F_{-\alpha}\) and \(F_{\pm \alpha}\), Galois groups \(\Gamma_\alpha\), \(\Gamma_{\pm \alpha}\), as well as groups \(C_\alpha, C_{\pm \alpha}\), etc. Note by continuity \(\Gamma_\alpha\) is finite over \(F\). Since \(F\) is local or global, we also have Weil groups \(W_F, W_\alpha = W_{F_\alpha}\), and \(W_{\pm \alpha} = W_{F_{\pm \alpha}}\). Then \(\Gamma\) acts on \(\prod_{\alpha \in O} \hat{C}_\alpha\) for any \(\Gamma\)-stable subset \(O \subset R\) by \(\sigma(\chi_\alpha) = \chi_\alpha \circ \sigma^{-1}\) for any \(\sigma \in \Gamma\).

**Definition 5.1.** A \(\chi\)-datum is defined to be a \(\Gamma\)-equivariant map

\[
\chi: R \to \prod_{\alpha \in R} \hat{C}_\alpha,
\]

such that \(\chi_{-\alpha} = \chi_\alpha^{-1}\), and \(\chi_\alpha\) is non-trivial on \(C_{\pm \alpha}\). A \(\zeta\)-datum is a map of the same definition as a \(\chi\)-datum except that \(\zeta_\alpha\) is trivial on \(C_{\pm \alpha}\).

Note that if \(|F_\alpha : F_{\pm \alpha}| = 2\), and let \(\sigma \in \Gamma_{\pm \alpha}\) be a non-trivial representative of \(\Gamma_{\pm \alpha}/\Gamma_\alpha\), then \(\sigma(\alpha) = \sigma^{-1}(\alpha) = -\alpha\), and \(\chi_\alpha^{-1} = \chi_{-\alpha} = \chi_\sigma^{-1}(\alpha) = \chi_\alpha \circ \sigma\). Thus \(\chi_\alpha\) must be trivial on \(\text{Nm}_{C_\alpha/C_{\pm \alpha}}(C_\alpha)\), hence must be an extension of the quadratic quasi-character of \(C_{\pm \alpha}\) associated with \(\hat{F}_\alpha/F_{\pm \alpha}\). In addition, we may regard \(\chi_\alpha\) as a character of \(W_\alpha\) via Artin reciprocity.

5.2. Recall that for any gauge \(p\) on \(R\) we have a 2-cocycle \(t_p\) of \(\Sigma\) with value in \(k^x \otimes_Z X\) where \(k^x\) is any field. Let \(k = C\), then we obtain a 2-cocycle of \(\Gamma\) in \(C^x \otimes_Z X\). This cocycle is in general cohomologically non-trivial, but becomes cohomologically trivial if inflated to \(W_F\). The splitting is given by any \(\chi\)-datum. Suppose \(O\) is a \(\Sigma\)-orbit in \(R\), and \(\alpha \in O\) a fixed element. Since \(|F_\alpha : F|\) is finite, we can choose a finite set of representatives of \(W_{\pm \alpha} \backslash W_F\), denoted by \(w_1, \ldots, w_n\), whose images \(\sigma_1, \ldots, \sigma_n\) is a set of representatives of \(\Gamma_{\pm \alpha}/\Gamma\). Then \(O = \{\pm \sigma_i^{-1} \alpha \mid 1 \leq i \leq n\}\). Define gauge \(p\) on \(O\) by declaring \(p(\sigma_i^{-1} \alpha) = 1\). We can then assemble \(p\) for all orbits \(O\) in \(R\) to obtain a gauge \(p\) on \(R\).

Still fix \(\alpha\) and \(O\), we define contraction maps \(u_i: W \to W_{\pm \alpha}\) for \(1 \leq i \leq n\) by letting \(u_i(w) = W_{\pm \alpha}\) to be the element such that

\[
w_i w = u_i(w) w_j
\]

for appropriate \(1 \leq j \leq n\).
Choose representatives \( v_0 \in W_\alpha \) and if \([F_\alpha : F_{\pm \alpha}] = 2\) an element \( v_1 \in W_{\pm \alpha} - W_\alpha\). Define contraction \( v: W_{\pm \alpha} \to W_\alpha \) by

\[
v_0 u = v(u)v_j
\]

for \( j = 0 \) or \( 1 \) as appropriate. Note if we choose \( v_0 \) in the center of \( W_\alpha \), then \( v \) is identity when restricted to \( W_\alpha \).

Define \( 1 \)-cochains of \( W_F \) in \( \mathbb{C}^\times \otimes_{\mathbb{Z}} X \) by

\[
r_{O,p}(w) = \prod_{i=1}^n \chi_\alpha(v(u_i(w)))\sigma_i^{-1}\alpha,
\]

and

\[
r_p = \prod_{O \in R/\Sigma} r_{O,p}.
\]

For any gauge \( q \) on \( R \), we let

\[
r_q = s_{q/p}r_p.
\]

**Lemma 5.2.** We have \( \partial r_q = t_q \) as cocycles of \( W_F \) for any gauge \( q \). Moreover, for a fixed \( \chi \)-datum, all choices involved in constructing \( r_q \) only change it by a coboundary.

Similarly, we may replace \( \chi \)-data with \( \zeta \)-data and form \( 1 \)-cocycles (not just cochains) \( c_p \). Since \( c_p \) is already a cocycle, we don’t need to define \( c_q \) hence we use \( c = c_p \). Again for a fixed \( \zeta \)-datum, all choices made in the construction have no effect on the cohomology class of \( c \).

### 5.3.

Return to the concrete setting of reductive group \( G \). Again \( T \) is a maximal \( F \)-torus of \( G \). We have the fixed \( \Gamma_F \)-splitting \((\hat{T}, \hat{B}, \{X_\alpha\})\) of \( \hat{G} \).

An embedding \( \hat{\xi}: \hat{1}T \to \hat{1}G \) is called admissible if

1. it induces isomorphism \( \hat{T} \to T \) that is the same as the one induced by some choice of Borel \( B \) containing \( T \) and \( \hat{B} \),
2. it is a morphism of extensions

\[
\begin{array}{cccc}
1 & \longrightarrow & \hat{T} & \longrightarrow & \hat{1}T & \longrightarrow & W_F & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \parallel & & \parallel & & . \\
1 & \longrightarrow & \hat{G} & \longrightarrow & \hat{1}G & \longrightarrow & W_F & \longrightarrow & 1
\end{array}
\]

Note that \( \hat{T} \to \hat{T} \) is not \( W_F \)-equivariant in general, and the \( \hat{G} \)-conjugacy class of \( \hat{\xi} \) is independent of \( B \) or \( (T, B) \). We will attach to each \( \chi \)-datum for the \( \Gamma_F \)-action on \( R(G, T) \) an admissible embedding \( \hat{\xi}: \hat{1}T \to \hat{1}G \), whose \( \hat{G} \) conjugacy class is canonical.

To start we fix a Borel \( B \) containing \( T \). Thus we can transfer the action of \( \Gamma_F \) on \( X(T) \) to an action on \( \hat{X}(\hat{T}) \), through \( \Psi_0(G) \). Thus we obtain an embedding \( \Gamma_F \to \Omega_T \rtimes \Gamma_F \), hence a set-theoretic map \( \omega_T: \Gamma_F \to \Omega_T \) and we can inflate it to \( W_F \). Let \( W_T \subset \Omega_T \rtimes W_F \) be the subgroup of elements \( \omega_T(w) \times w \) where \( w \in W_F \). The construction \( n: \Omega_T \to N_G(T) \) also makes sense on the dual side, so we have

\[
n: \Omega_T \longrightarrow N_G(T)
\]

that is \( \Gamma_F \)-equivariant, hence also \( W_F \)-equivariant. Still use \( \hat{n} \) to denote the map \( \hat{n} \times \text{id}: \Omega_T \ltimes W_F \to N_{\hat{G}}(\hat{T}) \ltimes W_F \).
Let \( p = p_B \) be the gauge on \( \check{\mathcal{R}}(\check{G}, \check{T}) \) determined by \( \check{B} \). Then the cocycle of \( \Omega_T \times W_F \), with value in \( \check{T}(\mathbb{C}) \)

\[
t_p(w_1, w_2) = \check{\eta}(w_1) \check{\eta}(w_2) \check{\eta}(w_1 w_2)^{-1}
\]

is a coboundary when restricted to \( W_T \). A choice of a \( \chi \)-datum \((\chi_\alpha)\) of \( \Gamma_T \)-action (hence \( W_T \)-action) on \( R(G, T) \) transports to a \( \chi \)-datum of \( W_T \)-action on \( \check{\mathcal{R}}(\check{G}, \check{T}) \). Note \( \{\chi_\alpha\} \) is also a \( \chi \)-datum, and we use it to form 1-cochain \( r_p^{-1} \), so that \( \partial r_p^{-1} = t_p \). Thus we obtain homomorphism

\[
\xi: \quad L_T \longrightarrow L_G
\]

\[
t \times w \longmapsto t_{B, B} r_p(w) \check{\eta}(w),
\]

where \( t \mapsto t_{B, B} \) is the map induced by choice of \( B \), and \( \check{B} \).

5.4. To summarize the construction more simply: we are basically embedding the action of \( W_T \) on \( \check{T} \), which contains the same information as \( L_T \), into \( N_G(\check{T}) \times W_F \subset L_G \), an object formed using the “standard” (relative to a choice of splitting anyway) action of \( W_F \).

The effect of various choices and naturalities of \( \xi \) can be summarized as follows:

1. for fixed \( \Gamma_T \)-splitting, choice of \( B \), and choice of \( \chi \)-data, \( \xi \) is determined up to \( \check{T} \)-conjugacy,
2. change of \( \Gamma_T \)-splitting will change \( \xi \) by \( \text{Int} g \) for some \( g \in \check{G}^{\Gamma_T} \),
3. change of \( B \) into \( B' = vBv^{-1} \) where \( v \in N_G(T) \) will change \( \xi \) in the following way: \( \text{Int} v \) acts on \( T \) hence on \( \check{T} \), and thus on \( \check{T} \) using \( \xi \). Call this action \( \mu \). Let \( g \in N_G(\check{T}) \) acts on \( \check{T} \) as \( \mu \), then \( \xi' \) obtained using \( B' \) is equal to \( \text{Int} g^{-1} \circ \xi \).
4. change of \( \chi \)-data results in a \( \zeta \)-datum by taking quotient, then \( \xi \) is multiplied by the cocycle \( c \) obtained from that \( \zeta \)-datum, i.e. \( \xi'(t \times w) = c(w)\xi(t \times w) \),
5. if \( \text{Int} g \) transports \( (T, \{\chi_\alpha\}) \) to \( (T', \{\chi'_\alpha\}) \), then \( \xi' \) is simply the composition of \( \xi \) with canonical map \( L_T' \rightarrow L_T \) induced by \( \text{Int} g \).
6. finally, if \( F \) is global, there are two ways to pass from global to local an admissible embedding attached to a global \( \chi \)-datum: one is directly via map \( W_{F_v} \rightarrow W_F \) for any place \( v \), and the other is by naturally induce a local \( \chi \)-datum from the global one, and then attach a local admissible embedding to the \( \chi \)-datum. One can show there is a choice of those auxiliary data on the way so that these two ways coincide.

6. **Endoscopic Groups and Transfer Factors**

6.1. Recall we have for an \( F \)-group \( G \) the \( L \)-group \( L_G \). Let \( G^* \) a quasi-split inner form of \( G \), and \( \psi \): \( G \rightarrow G^* \) a fixed inner twist. Then \( \psi \) induces isomorphism of \( L \)-groups

\[
L\psi: \quad L_G^* \longrightarrow L_G.
\]

6.2. An endoscopic datum is a quadruple \((H, \mathcal{H}, s, \xi)\) where

1. \( s \in \check{G} \) is semisimple,
2. \( H \) is quasi-split reductive over \( F \), with \( L \)-group \( L_H \) and a fixed \( \Gamma_T \)-splitting \( S_{\mathcal{H}l} \),
3. \( \mathcal{H} \) is a split extension of \( W_F \) by \( \check{H} \), whose associated \( L \)-action (for \( S_{\mathcal{H}l} \)) is the same as the one for \( L_H \),
4. \( \xi: \mathcal{H} \rightarrow L_G \) is an \( L \)-embedding, i.e. a morphism of extensions

\[
\begin{array}{c}
1 \longrightarrow \check{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow \check{G} \longrightarrow L_G \longrightarrow W_F \longrightarrow 1
\end{array}
\]

\[
\]
such that the isomorphic image of \( \tilde{H} \) is equal to \( C_G(s)_0 \) (the connected centralizer), and
that \( \text{Int} s \circ \xi = a\xi \), where \( a \) is a cohomologically trivial 1-cocycle of \( W_F \) in \( Z(\tilde{G}) \), inflated to \( \mathcal{H} \).

In this notes \( \mathcal{H} \) will be limited to \( \Gamma^1 \) for simplicity. Maybe the general case will be added later.

6.3. Given \( G \) and endoscopic datum \((H, \mathcal{H}, s, \xi)\), one can construct a canonical map from the semisimple conjugacy classes of \( H(\tilde{F}) \) to those of \( G(\tilde{F}) \). Indeed, one chooses a Borel pair \((T, B)\) in \( G \) and one \((\mathcal{I}, \mathcal{B})\) in \( \tilde{G} \), which will identify the Weyl group \( \Omega(G, T) \) with \( \Omega(G, \mathcal{I}) \), through \( \Psi_0(G) = \Psi_0(\tilde{G}) \) (without concerning \( \Gamma_F \)-action at this point).

Similarly for \( H \) we have \((T_H, B_H)\) and \((\mathcal{I}_H, \mathcal{B}_H)\), and identification \( \Omega(H, T_H) \sim \Omega(H, \mathcal{I}_H) \). The embedding \( \xi: \mathcal{H} \to \Gamma^1 \) gives embedding \( \xi: \tilde{H} \to \tilde{G} \). We can find \( x \in \tilde{G} \) such that \( \text{Int} x \circ \xi \), maps \( \mathcal{I}_H \) isomorphically onto \( \mathcal{I} \). Thus we have isomorphisms (depending on a lot of choices, and doesn’t play with \( \Gamma_F \) in general)

\[
\tilde{T}_H \xrightarrow{\sim} \mathcal{I}_H \xrightarrow{\sim} \mathcal{I} \xrightarrow{\sim} \tilde{T},
\]

and thus an isomorphism \( T_H \sim T \). On the other hand, \( \text{Int} x \circ \xi \) also embeds \( \Omega(\tilde{H}, \mathcal{I}_H) \) into \( \Omega(\tilde{G}, \mathcal{I}) \), hence \( \Omega(\tilde{H}, T_H) \) into \( \Omega(G, T) \). Therefore we have a map

\[
T_H/\Omega(H, T_H) \to T/\Omega(G, T).
\]

By a well-known result of Steinberg, it induces a map

\[
\mathcal{A}_{H/G} : \text{Cl}_{ss}(H(\tilde{F})) \to \text{Cl}_{ss}(G(\tilde{F})).
\]

In the case \( G = G^* \) and \( T_H \) is defined over \( F \), we may in fact choose \((T, B)\) (without affecting other choices) so that \( T \), and \( T_H \sim T \) are both defined over \( F \). In this case we call \( T_H \to T \) admissible.

Since all choices made in constructing \( \mathcal{A}_{H/G} \) can only be changed using inner automorphisms, we see \( \mathcal{A}_{H/G} \) is canonical. In fact, \( \mathcal{A}_{H/G} \) is defined over \( F \), or equivalently \( \Gamma_F \)-equivariant. So if a class \([\gamma_H] \in \text{Cl}_{ss}(H(\tilde{F}))\) is represented by \( \gamma_H \in H(\tilde{F}) \), then \( \mathcal{A}_{H/G}(\gamma_H) = [\gamma] \) for some \( \gamma \in G(\tilde{F}) \).

**Definition 6.1.** An element in either \( H(F) \) or \( G(F) \) is called strongly regular semisimple if its centralizer is a torus. An element \( \gamma_H \in H(\tilde{F}) \) is called (resp. strongly) \( G \)-regular semisimple if it is semisimple, and \( \mathcal{A}_{H/G}(\gamma_H) \) is a (resp. strongly) regular semisimple class.

A (strongly) \( G \)-regular semisimple element is necessarily (strongly) regular semisimple.

6.4. Let \( F \) be local. We will now describe the transfer factors for \( G \) and endoscopic datum \((H, \Gamma^1, s, \xi)\). From now on subscript \( \bullet_H \) denotes objects related to \( H \), \( \bullet_G \) their counterparts related to \( G \), and no subscript means those for \( G^* \). Let \( G_{SC} \) be the simply-connected cover of \( G^* \), and subscript \( \bullet_{SC} \) denotes liftings to \( G_{SC} \) of objects related to \( G^* \).

Let \( \gamma_H \in H(\tilde{F}) \) be strongly \( G \)-regular semisimple, and \( \gamma_G \in G(F) \) strongly regular semisimple, and \( \mathcal{A}_{H/G}(\gamma_H) = [\gamma_G] \). Let \( T_H = C_H(\gamma_H) \), and \( T_H \to T \subset G^* \) an admissible embedding. Let \( \gamma \) be the image of \( \gamma_H \) in \( T \). We fix a \( \alpha \)-datum and a \( \chi \)-datum for \( \Gamma_F \)-action on \( R(G^*, T) \) (equivalently \( \tilde{R}(G^*, T) \)).

Without loss of generality, we may assume the choice of \((\mathcal{I}, \mathcal{B})\) and \((\mathcal{I}_H, \mathcal{B}_H)\) in constructing embedding \( T_H \to T \) is the same as the one that is part of a fixed \( \Gamma_F \)-splitting of \( \tilde{G} \) and \( \tilde{H} \) respectively. We may even assume \( \xi \) maps \( \mathcal{I}_H \) to \( \mathcal{I} \) and \( \mathcal{B}_H \) into \( \mathcal{B} \). In this way we have \( s \in \mathcal{I} \), whose image in \( \tilde{I} \) is denoted \( s_T \). Since \( s \) is central in \( \xi(\tilde{H}) \), \( s_T \) depends only on \( T_H \to T \) (in particular, independent of \( B_H \) after the explicit choices made on the dual side).

The embedding \( \text{Int}(\tilde{G}) \to \tilde{T} \) is canonical, thus allows us to define \( \tilde{T}_{AD} = \tilde{T}/\text{Int}(\tilde{G}) \), which is canonically isomorphic to the dual torus of \( T_{SC} \). By definition, the image of \( s_T \) in \( \tilde{T}_{AD} \) is \( \Gamma \)-invariant, hence gives a well-defined element \( s_T \in \pi_0(\tilde{T}_{AD}) \).
6.5. Fix once and for all an F-splitting $S_{\text{pl}}$ of $G^*$, which is also regarded as an F-splitting of $G_{SC}$. The first term in the transfer factor is

$$\Delta_I(\gamma_H, \gamma_G) = (\lambda_T, s_T),$$

where $\lambda_T$ is computed using $S_{\text{pl}}$, and the pairing is Tate-Nakayama duality.

**Lemma 6.2.** For any two pairs $(\gamma_H, \gamma_G)$ and $(\Gamma', \gamma'_G)$, their quotient

$$\Delta_I(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := \Delta_I(\gamma_H, \gamma_G)/\Delta_I(\gamma'_H, \gamma'_G)$$

is independent of $S_{\text{pl}}$.

6.6. It makes sense to regard $R(H, T_H)$ as a $\Gamma$-stable subset of $R(G^*, T)$ via (the construction of) admissible embedding $T_H \to T$. With this note, the second term is

$$\Delta_{II}(\gamma_H, \gamma_G) = \prod_{[\alpha] \in [R(G^*, T) - R(T_H)]/\Gamma} \chi_\alpha \left( \frac{\alpha(\gamma) - 1}{a_\alpha} \right),$$

which can be verified to be well defined. We also define

$$\Delta_{II}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := \Delta_{II}(\gamma_H, \gamma_G)/\Delta_{II}(\gamma'_H, \gamma'_G).$$

6.7. For the third term we first deal with when $G = G^*$, and $\psi = \text{id}$. Then we can find $h \in G_{SC}$ such that $h\gamma h^{-1} = \gamma$, and the cohomology class of cocycle $\nu: \sigma \mapsto h\sigma(h)^{-1}$ in $H^1(F, T)$ is independent of $h$. We use $\text{inv}(\gamma_H, \gamma_G)$ for this class. Then the first part of the third term is

$$\Delta_{III}(\gamma_H, \gamma_G) = (\text{inv}(\gamma_H, \gamma_G), s_T)^{-1},$$

and

$$\Delta_{III}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := \Delta_{III}(\gamma_H, \gamma_G)/\Delta_{III}(\gamma'_H, \gamma'_G).$$

In general case where $G$ is not necessarily quasi-split, one cannot define $\Delta_{III}$ for a pair $(\gamma_H, \gamma_G)$ only, and has to define the relative term to another pair $(\gamma'_H, \gamma'_G)$, as follows. Let $u(\sigma) \in G_{SC}$ be such that $\psi(\sigma)^{-1} = \text{Int}(u(\sigma))$ for $\sigma \in \Gamma$, and find $h, h' \in G_{SC}$ such that

$$h\psi(\gamma_G)h^{-1} = \gamma,$$

$$h'\psi(\gamma'_G)h'^{-1} = \gamma',$$

and set

$$\nu(\sigma) = hu(\sigma)\sigma(h)^{-1},$$

$$\nu'(\sigma) = h'u(\sigma)\sigma(h')^{-1},$$

well-defined up to coboundaries. Since $\partial u = \partial h = \partial h'$, all of which taking values in $Z_{SC}$, if we let $\mathbb{U}$ to be the torus

$$T_{SC} \times T_{SC}'/(\{z, z^{-1}\} | z \in Z_{SC}),$$

then $(\nu, \nu'^{-1})$ induces a well-defined class independent of $u$, $h$ and $h'$

$$\text{inv} \left( \frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G} \right) \in H^1(F, \mathbb{U}).$$

Note that our notation here is the reciprocal of that in Langlands-Shelstad, because I want to be more consistent in notations with quasi-split case.

On the other hand, we have simply-connected cover $\tilde{G}_{SC}$ of the derived group of $G$, and $\mathcal{R}_{SC}$ the preimage of $\mathcal{T}$. Let $\tilde{s} \in \mathcal{R}_{SC}$ be an element that has the same image as $s$ in $\mathcal{T}$ for $s$ in $\mathcal{J}_{AD}$, then the isomorphism $\mathcal{T} \to \mathcal{\tilde{T}}$ constructed on the way of choosing an admissible embedding (again, choice
of \( B_H \) doesn’t matter) induces an isomorphism \( \mathcal{R}_{SC} \to \tilde{T}_{SC} \), where the latter is the dual torus of \( T_{AD} = T/Z(G) \). The image of \( \tilde{s} \) in \( \tilde{T}_{SC} \) is denoted by \( \tilde{s}_T \). Similarly we have \( \tilde{s}'_T \in \tilde{T}'_{SC} \). They both depend only on the admissible embeddings \( T_H \to \tilde{T} \) and \( T'_H \to T' \) (after fixing choices on the dual side at the beginning anyway).

The dual torus of \( U \) may be canonically identified with \( \tilde{U} \cong \tilde{T}_{SC} \times \tilde{T}'_{SC} / \{(z, z) \mid z \in Z(\check{G}_{SC}) \} \).

Let \( s_U \) be the image of \((\tilde{s}_T, \tilde{s}'_T)\) in \( \tilde{U} \), then it is independent of choice of \( \tilde{s} \). Then \( s_U \) is also \( \Gamma_f \)-invariant, hence defines an element \( s_U \in \pi_0(\tilde{U}^{\Gamma_f}) \). Then Tate-Nakayama duality enables us to define

\[
\Delta_{III}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) = \left\langle \mathrm{inv} \left( \frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G} \right), s_U \right\rangle^{-1}.
\]

This is consistent with quasi-split case.

6.8. Continuing with the third term. It is the only part \( \mathcal{M} = \mathcal{L}_H \) will be used. Here we need to use the choice of \( B_H \) and \( B \) explicitly, and it has no effect on the end product. Such choices together with the \( \chi \)-datum gives us admissible embeddings

\[
\xi_{T_H} : \mathcal{L}_T \to \mathcal{L}_H,
\]

\[
\xi_T : \mathcal{L}_T \to \mathcal{L}_G.
\]

Thus we obtain a 1-cocycle \( a : W_F \to \mathcal{T} \) (with the \( W_F \)-action on \( \mathcal{T} \) transported to \( \mathcal{T} \) via embedding \( \xi_T \), instead of the “original” one), inflated to \( \mathcal{L}_T \) such that

\[
\xi \circ \xi_{T_H} = a \xi_T.
\]

Its class \( a \in H^1(W_F, \tilde{T}) \) is independent of the choices of \( B_H \), \( B \), nor splittings on \( \check{H} \) or \( \check{G} \). Then we define

\[
\Delta_{III}(\gamma_H, \gamma_G) = (a, \gamma),
\]

where the pairing is the canonical isomorphism

\[
H^1(W_F, \tilde{T}) \cong \text{Hom}_{cont}(T(F), C^\times).
\]

We also define as before

\[
\Delta_{III}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := \Delta_{III}(\gamma_H, \gamma_G) / \Delta_{III}(\gamma'_H, \gamma'_G).
\]

6.9. The final term of transfer factor is essentially just the discriminant function. For \( \gamma \in T(F) \), we define

\[
D_{G^*}(\gamma) = \prod_{\alpha \in R(G^*, T)} |\alpha(\gamma) - 1|_{F}^{1/2}.
\]

Similarly we can define \( D_{H}(\gamma_H) \). Then

\[
\Delta_{IV}(\gamma_H, \gamma_G) = D_{G^*}(\gamma)D_{H}(\gamma_H)^{-1}.
\]

Again we let

\[
\Delta_{IV}(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := \Delta_{IV}(\gamma_H, \gamma_G) / \Delta_{IV}(\gamma'_H, \gamma'_G).
\]
6.10. Finally, we can define the relative transfer factor
\[ \Delta(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) := (\Delta_I \Delta_{III} \Delta_{III_2} \Delta_{IV})(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G). \]

If \( G \) is quasi-split, then we define the absolute transfer factor
\[ \Delta(\gamma_H, \gamma_G) = \Delta_0(\gamma_H, \gamma_G) := (\Delta_I \Delta_{III} \Delta_{III_2} \Delta_{IV})(\gamma_H, \gamma_G). \]

In general we have to fix a pair \((\gamma'_H, \gamma'_G)\) and define \(\Delta(\gamma'_H, \gamma'_G)\) arbitrarily (but nonzero), then define
\[ \Delta(\gamma_H, \gamma_G) = \Delta(\gamma_H, \gamma_G, \gamma'_H, \gamma'_G) \Delta(\gamma'_H, \gamma'_G). \]

**Theorem 6.3.** The transfer factor \(\Delta(\gamma_H, \gamma_G)\) is independent of choice of admissible embedding \(T_H \to T\), \(\alpha\)-data, or \(\chi\)-data.

7. **Loose Ends**

I didn’t include all the properties of transfer factors, how they patch together globally, or how they extend to non-strongly \(G\)-regular elements.

**References**
