1. March 27

Let \( \Sigma \) be a smooth complete curve over \( \mathbb{F}_q \), and \( \Sigma^{\text{aff}} \) an affine open subvariety, and let the infinity be the set \( \infty = \Sigma - \Sigma^{\text{aff}} \). Then we have the following analogy:

\[
\begin{align*}
\mathbb{Z} \leftrightarrow & \mathbb{F}_q[\Sigma^{\text{aff}}], \\
\mathbb{Q} \leftrightarrow & \mathbb{F}_q(\Sigma), \\
\mathbb{Z}_p \leftrightarrow & \hat{\mathcal{O}}_x \text{ (completed local ring at } x \in \Sigma), \\
\mathbb{Z}_\infty := & \mathbb{R} \leftrightarrow \infty.
\end{align*}
\]

**Definition 1.1.** An *automorphic function* is a \( \mathbb{C} \)-valued function on 
\( K\backslash \text{GL}_n(\mathbb{A})/\text{GL}_n(\mathbb{Q}), \)
where \( K \) being the hyperspecial maximal compact subgroup of \( \text{GL}_n(\mathbb{A}). \)

For each \( x \in \Sigma \), denote \( F_x \) the field of fractions of \( \hat{\mathcal{O}}_x \), we have the similar setting for \( \Sigma \):

\[
\prod_{x \in \Sigma} \text{GL}_n(\hat{\mathcal{O}}_x) \bigg/ \prod_{x \in \Sigma} \text{GL}_n(F_x)/\text{GL}_n(F(\Sigma)),
\]

which is isomorphic to the set \( |\text{Bun}_n(\mathbb{F}_q)| \). The proof being easy (manipulating with trivializations of vector bundles, and double quotient corresponds to changing trivializations). Denote the space of automorphic functions associated to \( \Sigma \) by \( \mathcal{A}(\Sigma) \).

1.1. **Hecke Operators.** We want to construct many commuting operators on the space of automorphic functions. For each \( r = 0, \ldots, n \), define

\[
\mathcal{H}^{\operatorname{Heck}}_r = \left\{ (V', V, x) \mid \begin{array}{c}
V' \text{ isomorphic to a subsheaf of } V, \\
V/V' \text{ is a skyscraper sheaf of rank } r \text{ at } x
\end{array} \right\}/\sim,
\]

\[
\subset |\text{Bun}_n(\mathbb{F}_q) \times \text{Bun}_n(\mathbb{F}_q) \times \Sigma|.
\]

Also let the projections from \( \mathcal{H}^{\operatorname{Heck}}_r \) to the isomorphism classes of its three factors \( |\text{Bun}_n(\mathbb{F}_q)|, |\text{Bun}_n(\mathbb{F}_q)|, \) and \( \Sigma \) be \( pr_1, pr_2, p \) respectively.

We can describe \( \mathcal{H}^{\operatorname{Heck}}_r \) more explicitly. Fix an \( x \in \Sigma \), and choose a local uniformizer \( t \) at \( x \). Let \( V(x) \) be the fiber of \( V \) at \( x \), and \( E \subset V(x) \) an \( r \)-dimensional subspace. Note we have a short exact sequence of \( \hat{\mathcal{O}}_x \)-modules

\[
0 \to V_x \xrightarrow{t} V_x \to V(x) \to 0.
\]
Let $\widetilde{E}$ be the preimage of $E$ in $V_x$, and we define $V'$ to be the subsheaf of $V$ whose stalks are $V_y$ for any $y \neq x$ and $\widetilde{E}$ at $x$. Note $V'$ is locally free because $\Sigma$ is a curve. Therefore, the fiber of $pr_2 \times p$ over $(V, x)$ in $\mathcal{H}^{\text{heck}^r}$ is just the Grassmannian $\mathfrak{Gr}^r(V(x))$.

For any point $x \in \Sigma$, $r = 0, \ldots, n$, define operator

$$H_x^r : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma)$$

$$f \mapsto (pr_{2,x})^*(pr_{1,x})^*(f).$$

Note we regard $f$ as a function on $[\text{Bun}_n(F_q)]$, and $pr_i$ is viewed here as maps from $\mathcal{H}^{\text{heck}^r}$ to $[\text{Bun}_n(F_q)]$. Then it’s easy to see $H_x^r(f)$ at any point (a vector bundle) just sums up the value of $f$ at all subsheaves that have type $(1^r, 0^{n-r})$ at $x$ and are isomorphic elsewhere.

**Theorem 1.2.** For any $x, y \in \Sigma$ and any $r, r' = 0, \ldots, n$, the operators $H_x^r$ and $H_y^{r'}$ commute.

Now we work out some linear algebra for our settings. Let $F$ be a local field with ring of integers $\mathcal{O}$.

**Definition 1.3.** A lattice in $F^n$ is a finitely generated $\mathcal{O}$-submodule $L \subset F^n$ such that $L \otimes_{\mathcal{O}} F \cong F^n$.

An example of a lattice would be the standard lattice $L_0 = \mathcal{O}^n \subset F^n$.

**Lemma 1.4.** Let $L' \subset L$ lattices, then there exists an $\mathcal{O}$-basis $e_1, \ldots, e_n$ of $L$ and integers $m_1 \geq \cdots \geq m_n \geq 0$ such that $L' = t^{m_1} \mathcal{O} e_1 + \cdots + t^{m_n} \mathcal{O} e_n$.

**Proof.** This is just the structure theorem for free modules over a PID. □

**Corollary 1.5.** Let $L, L' \subset L_0$, then $L \in \text{GL}_n(\mathcal{O})L'$ if and only if $L_0/L \cong L_0/L'$.

**Corollary 1.6.** If $L \subset L_0$, then $\dim_{F_q} L_0/L = \sum_{i=1}^n m_i$, and if we write $L = gL_0$ for some $g \in \text{GL}_n(F) \cap \text{Mat}_n(\mathcal{O})$, then $\text{val} (\det g) = \dim_{F_q} L_0/L$.

All proofs are very easy. Thus we can describe $\mathcal{H}^{\text{heck}^r}$ in a slightly different way:

$$\mathcal{H}^{\text{heck}^r} = \left\{ (V', V, x) \mid \text{ there is } V' \to V \text{ injective, and isomorphic on } \Sigma \setminus \{x\}, V_x' \hookrightarrow V_x \text{ has } m_1 = \cdots = m_r = 1 \text{ and } m_{r+1} = \cdots = m_n = 0 \right\} / \sim.$$

An observation is that for any pair of lattices $L, L'$, there exists some $m \gg 0$ such that $t^m L' \subset L \subset t^{-m} L'$. Thus we can choose a basis $e_1, \ldots, e_n$ of $L$ such that $L' = t^{m_1} \mathcal{O} e_1 + \cdots + t^{m_n} \mathcal{O} e_n$ for some $m_1 \geq \cdots \geq m_n$ (i.e. dropping the condition $m_1 \geq 0$). We can also find $g \in \text{GL}_n(F)$ such that $L' = gL$ (no longer require $g \in \text{Mat}_n(\mathcal{O})$).

Let affine Grassmannian $\mathfrak{Gr}$ be the set of all lattices in $F^n$, which is isomorphic to $\text{GL}_n(F)/\text{GL}_n(\mathcal{O})$, by above we see that

$$\mathfrak{Gr} = \prod_{m_1 \geq \cdots \geq m_n} \text{GL}_n(\mathcal{O})t^{(m_1, \ldots, m_n)} \text{GL}_n(\mathcal{O})/\text{GL}_n(\mathcal{O}),$$

and equivalently,

$$\text{GL}_n(F) = \prod_{m_1 \geq \cdots \geq m_n} \text{GL}_n(\mathcal{O})t^{(m_1, \ldots, m_n)} \text{GL}_n(\mathcal{O}),$$

which is the Cartan decomposition for $\text{GL}_n(F)$.

Let $\mathcal{H}$ be the algebra of Hecke operators, $\mathbb{X}_s \cong \mathbb{Z}^n$ be the coroot lattice for $\text{GL}_n$, $\mathbb{C}[\mathbb{X}_s]$ the group ring for $\mathbb{X}_s$, $W \cong \mathfrak{S}_n$ the Weyl group of $\text{GL}_n$, then we have
Theorem 1.7 (Satake).

\[ \mathcal{H} \xrightarrow{\sim} \mathbb{C}[X_+]^W[q^\pm] = \mathbb{C}[T^\vee]^W[q^\pm] \cong \mathbb{C}[G^\vee]^G[q^\pm], \]

with basis of characters \( \chi_{(m_1, \ldots, m_n)}. \)

2. March 29

Let \( \Lambda = \mathbb{Z}^n \) with \( \mathfrak{S}_n \) acting by permutation. Let \( G = \text{GL}_n \), \( k \) be any field, \( F = k((t)) \) the field of Laurent series, and \( \mathcal{O} = k[[t]] \) the ring of power series. We have defined the affine Grassmannian \( \mathfrak{G}_r \) to be the set of all rank \( n \) \( \mathcal{O} \)-lattices in \( F^n \). The group \( G(F) \) acts transitively on \( \mathfrak{G}_r \) and the stablizer of the standard lattice \( L_0 = \mathcal{O}^n \) is \( G(\mathcal{O}) \). Therefore \( \mathfrak{G}_r = G(F)/G(\mathcal{O}). \)

Definition 2.1. Let \( (L, L') \) be a pair of lattices. A basis adapted to \( (L, L') \) is such \( v_1, \ldots, v_n \in L \) that \( L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_n \) and \( L' = t^{m_1}\mathcal{O}v_1 + \cdots + t^{m_k}\mathcal{O}v_n \). The adapted basis is said to have type \( \lambda = (m_1, \ldots, m_n). \)

Lemma 2.2. (1) Any pair \( (L, L') \) has an adapted basis, and its type \( \lambda \) is unique up to permutations. In this case we say \( (L, L') \) are in relative position \( \lambda \mod \mathfrak{S}_n \in \Lambda/\mathfrak{S}_n. \)

(2) Two pairs \( (L_1, L_2) \) and \( (L'_1, L'_2) \) are in the same relative position if and only if they belong to the same orbit of the action of \( G(F) \) in \( \mathfrak{G} \times \mathfrak{G}. \)

Remark 2.3. Note that for any two groups \( B \subseteq A \), we have natural bijection

\[ A/(A/B \times A/B) \to B/A/B \]

\[ (a, a') \mapsto a^{-1}a'. \]

Therefore \( G(F)\backslash(\mathfrak{G} \times \mathfrak{G}) \cong G(\mathcal{O})\backslash\mathfrak{G} \cong G(\mathcal{O})\backslash G(F)/G(\mathcal{O}). \)

Proof of Lemma 2.2. For the first part, note there exists \( r \gg 0 \) such that \( t^rL' \subseteq L \), then by Lemma 1.4 we can find an adapted basis for \( (L, t^rL'). \) Dividing by \( t^r \) on the coefficients we get an adapted basis for \( (L, L') \). The uniqueness can also be seen easily from Lemma 1.4.

For the second part, let \( v_1, \ldots, v_n \) be a basis adapted to \( (L_1, L_2) \), and \( v'_1, \ldots, v'_n \) one to \( (L'_1, L'_2) \). Assume they have the same relative position, then by permuting \( v'_i \) (possible by acting by an element in \( G(F) \)) we can assume they have the same type. Then we can define \( g \in G(F) \) to be the element sending \( v_i \) to \( v'_i \), then \( g(L_1) = L'_1 \). The same type assumption says that \( g(L_2) = L'_2 \) as well. The other direction is proved by running the argument backwards, which is more trivial.

Since \( \mathfrak{G}_r \) is not of finite type, we now write it as a limit of (projective) varieties. Fix an integer \( r > 0 \), we certainly have \( t^rL_0 \subseteq t^{-r}L_0 \). Define

\[ \mathfrak{G}_r = \{ L \in \mathfrak{G} \mid t^rL_0 \subseteq L \subseteq t^{-r}L_0 \}, \]

we then have \( \mathfrak{G}_r = \varprojlim \mathfrak{G}_r. \) We can rewrite it as

\[ \mathfrak{G}_r \cong \{ L \subseteq t^rL_0 \cap t^{-r}L_0 \mid t^rL_0 \cong (k^n)^{2r} \mid L/t^rL_0 \text{ is } t\text{-stable} \}, \]

where the uniformizer \( t \) acts nilpotently on \((k^n)^{2r}\).

Remark 2.4. (1) When \( k = \mathbb{F}_q \), \( \mathfrak{G}_r \) is a finite set.

(2) When \( k = \mathbb{C} \), \( \mathfrak{G}_r \) is a projective variety.
Lemma 2.5.  

1. There is a bijection \( \Lambda \cong \mathfrak{gr}^{T(k)} \) identifying \( \lambda = (m_1, \ldots, m_n) \) with \( L^\lambda = t^{m_1}Oe_1 + \cdots + t^{m_n}Oe_n \), where \( e_i \) is the standard basis for \( L_0 \).

2. We have

\[
\mathfrak{gr} = \prod_{\lambda \in \Lambda} N(F)L^\lambda,
\]

or equivalently,

\[
G(F) = \prod_{\lambda \in \Lambda} N(F)t^\lambda G(O).
\]

The latter is called Iwasawa decomposition.

Proof. For the first part, first note that the map \( \lambda \mapsto L^\lambda \) is clearly injective. For surjectivity, note that the action of \( T(k) \) on \( \mathfrak{gr} \) induces a \( T(k) \)-action on \( V = t^{-r}L_0/t^rL_0 \), which commutes with the action of \( t \). It is clear that any \( T(k) \)-stable subspace \( E \subset V \) has a form \( E = E_1 \oplus \cdots \oplus E_n \) where \( E_i \subset \bigoplus_{j=-r}^{r-1} k(t^j e_i) \). If \( E \) is in addition \( t \)-stable, then \( E_i = \bigoplus_{j=m_i}^{r-1} k(t^j e_i) \) for some \( m_i \geq -r \).

For the second part, note any lattice can be transferred through Gaussian elimination with \( O \)-coefficients to some \( L^\lambda \).

Now we describe the Hecke operators for curves in a different way. Let \( \Sigma \) be a smooth projective curve over \( k \), and fix a \( k \)-point \( x \in \Sigma \). Let \( \mathcal{O} = \widehat{\mathcal{O}}_x \) be the completed local ring at \( x \), and \( F \) the fraction field of \( \mathcal{O} \). For convenience, let \( \mathcal{O}_{\text{out}} = \mathcal{O}_{\Sigma_-(x)} \). By \( \text{Bun}_n \) we mean the stack of vector bundles of rank \( n \) on \( \Sigma \). We claim that

\[
\mathfrak{gr} \cong \{ (V, \psi) \mid V \in \text{Bun}_n(\Sigma), \psi \text{ a trivialization of } V \text{ on } \Sigma - \{ x \} \}/\sim.
\]

The proof of the claim is straightforward: recall in last lecture we have that the double quotient \([\text{Bun}_n(k)]\) is isomorphic to \([\text{Bun}_n(k)]\), and those with a trivialization over \( \Sigma - \{ x \} \) correspond to the double coset represented by those with only one factor (i.e. at \( x \)) not in the local integral points. Now we put back the trivialization outside of \( x \), we get the result.

For any \( \lambda \in \Lambda/\mathfrak{S}_n \), we define

\[
\mathcal{H}eck^\lambda_x = \left\{ (V, V', \varphi) \mid \varphi : V|_{\Sigma-\{ x \}} \rightarrow V'|_{\Sigma-\{ x \}} \right\} / \sim.
\]
Denote still by \(\text{pr}_1, \text{pr}_2\) projections to \(V\) and \(V'\) respectively. Let \(\text{Bun}^0_n\) be the substack of bundles that are trivial on \(\Sigma - \{x\}\). Therefore by above discussion we have

\[
\text{pr}_1^{-1}(\text{Bun}^0_n) = G(O_{\text{out}}) \backslash (\mathfrak{g}_r \times \mathfrak{g}_r)^\lambda
= G(O_{\text{out}}) \backslash G(F) / \text{Stab}(L^0, L^\lambda),
\]

where the superscript \(\lambda\) denotes the pairs of relative position \(\lambda\).

In order to describe the alternative description for Hecke algebra, we first state some results in a more general setting. Let \(M\) be a locally compact topological group, and \(K\) a maximal compact subgroup. Choose a Haar measure \(\mu\) such that \(\mu(K) = 1\). Let \(C_c(K \backslash M/K)\) be the \(\mathbb{C}\)-valued \(K\)-biinvariant continuous functions with compact support on \(M\). It is an algebra under convolution. It can be shown to be naturally isomorphic to \(C_c(M \backslash (M/K \times M/K))\) which also has a natural convolution (with the support conditions for the functions properly defined).

**Remark 2.6.** For a space \(X\), and good enough functions \(f, g\) on \(X\). The convolution can be defined by \((f, g) \mapsto p_{13,*}(p'_{12} f \cdot p'_{23} g)\), where \(p_{ij}\) are the three projections from \(X \times X \times X\) to \(X \times X\).

When \(Y\) is a space with \(M\) acting on it, we have a natural action of \(C_c(K \backslash M/K)\) on \(C_c(Y/K)\):

\[
C_c(Y/K) \otimes C_c(K \backslash M/K) \to C_c(Y/K)
\]

\[
f \otimes g \mapsto (y \mapsto \int_M f(m) g(m^{-1}) dm).
\]

Now when \(Y = \{(V, v) \mid V \in \text{Bun}_n, v\) is an \(O\) basis of \(V_x\}, M = G(F), K = G(O),\) we have \(C_c(G(O) \backslash G(F) / G(O))\) acting on \(C_c(Y/K)\).

**Theorem 2.7.** The algebra \(C_c(G(O) \backslash G(F) / G(O))\) is commutative.

**Proof by Gelfand.** For general \(K \subset M\), suppose we have an anti-involution \(\tau\) on \(M\) such that \(\tau(K m K) = K m K\) for all \(m \in M\). We claim \(C_c(K \backslash M/K)\) is commutative: define \(\tau\) acting on \(C_c(M)\) by \((\tau f)(m) = f(\tau(m))\). Then it is easily verified \(\tau(f \ast g) = \tau(g) \ast \tau(f)\). On the other hand, by assumption on \(\tau\), we know \(\tau(f) = f\) for all \(f \in C_c(K \backslash M/K)\), we have \(\tau(f \ast g) = f \ast g\) and \(\tau(f \ast g) = g \ast f\), so \(C_c(K \backslash M/K)\) is commutative. Going back to our specific case, choose \(\tau\) to be the transposition of matrices which, by Cartan decomposition, satisfies the assumption on the anti-involution above, so we are done. \(\square\)

### 3. April 3

In this lecture we establish some facts about affine Grassmannian for a general reductive (actually semisimple only) group, analogous to the \(GL_n\) case (which is not semisimple hence it’s not a perfect analogy). Let \(k\) be an arbitrary field, \(F = k((\varpi))\) the field of Laurent series over \(k\), \(O = k[[\varpi]]\) the ring of formal power series. Let \(G\) be a split reductive group over \(k\), and we define

\[
\mathfrak{g}_r_G := G(F) / G(O).
\]

Let \(\mathfrak{g} = \text{Lie} G\), and fix a nondegenerate invariant symmetric bilinear form \((\cdot, \cdot)\) on \(\mathfrak{g}\), i.e. for all \(x, y, z \in \mathfrak{g}\), we have \(( [x, y], z) + (y, [x, z]) = 0\). We can extend it to a bilinear form on
\( \mathfrak{g}(F) \times \mathfrak{g}(F) \to F \). For example, when \( \mathfrak{g} = \mathfrak{sl}_n \), we can choose the Killing form \( (x, y) = \text{Tr}(\text{ad} x \circ \text{ad} y) \), and when \( G \) is reductive, we can extend the Killing form from its semisimple part by putting a nondegenerate symmetric bilinear form (automatically invariant) on its abelian part, and demanding the two parts orthogonal to each other.

Now assume \( G \) is semisimple.

**Definition 3.1.** A lattice in \( \mathfrak{g}(F) \) is a finitely generated \( \mathcal{O} \)-submodule \( L \subset \mathfrak{g}(F) \) such that \( L \otimes_{\mathcal{O}} F \cong \mathfrak{g}(F) \). The dual lattice \( L^\vee := \text{Hom}_{\mathcal{O}}(L, \mathcal{O}) \), identified as a lattice in \( \mathfrak{g}(F) \) as well through the bilinear form \( (\cdot, \cdot) \).

Then we can give another definition of affine Grassmannian:

\[
\mathfrak{G}r'_G := \{ L \subset \mathfrak{g}(F) \text{ lattice} \mid [L, L] \subset L, \text{ and } L = L^\vee \}.
\]

We want to connect \( \mathfrak{G}r_G \) with \( \mathfrak{G}r'_G \) (i.e. to show they are isomorphic). As usual, we introduce the standard lattice \( L_0 = \mathfrak{g}(\mathcal{O}) \), and let \( G(F) \) act on \( \mathfrak{G}r'_G \) by the adjoint action: \( g: L \mapsto \text{Ad}_g(L) \). Note that \( \text{Stab}_{G(F)} L_0 = G(\mathcal{O}) \), and the action of \( G(F) \) is transitive. Thus we get \( \mathfrak{G}r'_G \cong G(F)/G(\mathcal{O}) = \mathfrak{G}r_G \).

**Remark 3.2.** The two definitions can be seen in methodology parallel to the definitions of flag varieties. The definition of \( \mathfrak{G}r_G \) is similar to defining the flag variety by \( G/B \) with some chosen Borel \( B \), and that of \( \mathfrak{G}r'_G \) is choice-free, thus similar to defining the flag variety by the functor of points (set of all Borel subgroups in \( G \)).

**Lemma 3.3.** For all lattice \( L \subset \mathfrak{g}(F) \), there exists \( n \gg 0 \) such that \( \varpi^n L_0 \subset L \subset \varpi^{-n} L_0 \).

In other words,

\[
\frac{L}{\varpi^n L_0} \subset \frac{\varpi^{-n} L_0}{\varpi^n L_0} =: V_n.
\]

Define a bilinear form

\[
\beta_n: V_n \times V_n \to k
\]

\[
(x, y) \mapsto \text{Res}_{\varpi = 0}(x, y),
\]

where the parentheses on the right hand side denote the bilinear form on \( \mathfrak{g}(F) \), and \( \text{Res}_{\varpi = 0} \) simply means taking the coefficient of \( \varpi^{-1} \). It is easy to check that \( \beta_n \) is nondegenerate, and we can describe \( \mathfrak{G}r_G \) as a limit of projective varieties:

\[
\mathfrak{G}r_n := \{ V \subset V_n \mid V \text{ is } \varpi\text{-stable, maximal isotropic w.r.t. } \beta_n, \text{ and the 3-form } \beta_n([\cdot, \cdot], \cdot) = 0 \text{ on } V \},
\]

and \( \mathfrak{G}r_n \) comes with a natural structure of a projective variety over \( k \). It is clear that we have the following isomorphism

\[
\{ L \in \mathfrak{G}r_G \mid \varpi^n L_0 \subset L \subset \varpi^{-n} L_0 \} \sim \mathfrak{G}r_n \quad L \mapsto \frac{L}{\varpi^n L_0},
\]

with the maximal isotropic condition on the right corresponding to the self-dual condition on the left, and after assuming this, the Lie subalgebra condition on the left is equivalent to the 3-form condition on the right (which instead of putting conditions on \( L \) only, translating some of those conditions to the dual via \( \beta_n \)).
Corollary 3.4. We have $\mathfrak{g}_G = \varinjlim G_r n$. The $G(\mathcal{O})$-action on lattice by conjugation descends to an action on $V_n$. Let $T \subset G$ be a split maximal torus, and let $\lambda \in \mathcal{X}_* = \mathcal{X}_*(T) = \text{Hom}_k(\mathcal{G}_m, T) \cong \mathbb{Z}^r \mathcal{G}_m$. We have an element $\varpi^\lambda \in T(F)$. Similar as before, we let $L^\lambda = \text{Ad} \varpi^\lambda(L_0)$. Let $t = \text{Lie} T$, $\Phi \subset t^*$ the root system of $(\mathfrak{g}, t)$, and (by choosing a Borel) fix a set of positive roots. Thus we have the decomposition

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = t \oplus \bigoplus_{\alpha \in \Phi} k e_\alpha,$$

where $e_\alpha$ is a nonzero vector in $\mathfrak{g}$. Therefore we have

$$\mathfrak{g}(\mathcal{O}) = t(\mathcal{O}) \oplus \bigoplus_{\alpha \in \Phi} \mathcal{O} e_\alpha,$$

and so

$$\text{Ad} \varpi^\lambda \mathfrak{g}(\mathcal{O}) = t(\mathcal{O}) \oplus \bigoplus_{\alpha \in \Phi} \varpi^{(\lambda, \alpha)} \mathcal{O} e_\alpha,$$

Assume $\lambda$ is dominant, then if $\langle \lambda, \alpha \rangle \leq n$ for all $\alpha > 0$, then we get $\varpi^n L_0 \subset L^\lambda$, and automatically $L^\lambda \subset \varpi^{-n} L_0$ by self-duality.

We define $\mathfrak{g}_\lambda$ to be the $G(\mathcal{O})$-orbit of $L^\lambda$, then $\mathfrak{g}_\lambda = G(\mathcal{O})/\text{Stab}_{G(\mathcal{O})} L^\lambda$. We can describe the Lie algebra of the stablizer explicitly if $\lambda$ is dominant and $\langle \lambda, \alpha \rangle \leq n$: since $\text{Stab}_{G(\mathcal{O})} L^\lambda = G(\mathcal{O}) \cap \text{Stab}_{G(F)} L^\lambda = G(\mathcal{O}) \cap \text{Ad} \varpi^\lambda(G(\mathcal{O}))$,

$$\text{Lie}(G(\mathcal{O}) \cap \text{Ad} \varpi^\lambda(G(\mathcal{O}))) = \mathfrak{g}(\mathcal{O}) \cap \text{Ad} \varpi^\lambda(\mathfrak{g}(\mathcal{O}))$$

$$= t(\mathcal{O}) \oplus \bigoplus_{\alpha \in \Phi^+} \varpi^{(\lambda, \alpha)} \mathcal{O} e_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} \mathcal{O} e_\alpha.$$ 

Thus we can calculate the dimension of $\mathfrak{g}_\lambda$:

$$\dim_k \mathfrak{g}_\lambda = \dim_k \frac{\mathfrak{g}(\mathcal{O})}{\text{Lie}(\text{Stab}_{G(\mathcal{O})} L^\lambda)} = \sum_{\alpha > 0} \langle \lambda, \alpha \rangle = 2\langle \lambda, \rho \rangle,$$

where $\rho$ is one half of the sum of all positive roots.

Lie theory tells us that the coroot lattice $Q^\vee \subset X_*$ as a finite index sublattice. We also know from Lie theory that $\langle \lambda, \rho \rangle \in \mathbb{Z}$ or $\frac{1}{2} \mathbb{Z}$ if $\lambda \in Q^\vee$ or $X_*$ respectively.

Corollary 3.5. The orbit $\mathfrak{g}_\lambda$ is even-dimensional if $\lambda \in Q^\vee$.

Now let $k = \mathbb{C}$. Choose a maximal compact subgroup $K \subset G$, and define the (polynomial) loop space of $K$

$$\Omega(K) := \{ \text{polynomial maps } f : S^1 \to K \text{ such that } f(1) = 1 \}.$$ 

We can view $G$ as embedded in $GL_n(\mathbb{C})$ for some $n$, and polynomial maps from $\mathbb{C}$ to $G$ are simply polynomial maps in coordinates. A polynomial map to $K$ is one that is the restriction (to $S^1$) of some polynomial map $\mathbb{C} \to G$ whose image of $S^1$ lies in $K$. We then have maps

$$\Omega(K) \hookrightarrow G(\mathbb{C}((\varpi))) \to \mathfrak{g}_r G = G(\mathbb{C}((\varpi)))/G(\mathbb{C}[[(\varpi)]]).$$

Theorem 3.6. The composition of the above map $\Omega(K) \to \mathfrak{g}_r G$ is an isomorphism.
The proof for $GL_n$ is essentially the Gram-Schmidt process.

Lastly, for a chosen $(G, K)$, we have the simply-connected as well as the adjoint type isogenus groups $K^{sc}$ and $K^{ad}$ for $K$, and the maps

$$K^{sc} \to K \to K^{ad}$$

being both group quotients and covering maps. Clearly $\pi_1(K^{ad}) = Z(K^{sc})$. So if $K$ is the simply-connected type, we have

$$\pi_0(\Omega(K^{sc})) \cong \pi_1(K^{sc}) = 1,$$

and $\mathfrak{gr}_{G^{sc}}$ is connected. If $K$ is the adjoint type, then

$$\pi_0(\Omega(K^{ad})) \cong \pi_1(K^{ad}) = Z(K^{sc}),$$

and moreover we have a natural map from $\mathfrak{gr}_{G^{sc}}$ to $\mathfrak{gr}_{G^{ad}}$ realizing the former as a connected component of the latter.

4. April 5

This lecture will define the Grassmannian as a scheme (more precisely an ind-scheme). We deal with the case $k = \mathbb{C}$ first, and mention a bit about the mixed characteristic case. As usual $F = k((\varpi))$ and $\mathcal{O} = k[[\varpi]]$.

Let $H$ be a linear algebraic group, then we can define the functor of arc space of $H$

$$\underline{H} (\mathcal{O}) : \text{Alg}/k \to \text{Sets}$$

$$R \mapsto H(R[[\varpi]]).$$

**Proposition 4.1.** This functor is representable by a scheme of infinite type $H(\mathcal{O})$.

Similarly, we can define the functor of loop space of $H$

$$\underline{H} (F) : \text{Alg}/k \to \text{Sets}$$

$$R \mapsto H(R((\varpi))).$$

**Proposition 4.2.** This functor is representable by an ind-scheme $H(F)$, i.e. it is a direct limit (as a functor) of closed embeddings of schemes.

**Definition 4.3.** The affine Grassmannian $\mathfrak{gr}_G$ is defined to be the quotient of functors (as fpqc sheaves of groups) $G(F)/G(\mathcal{O})$.

**Remark 4.4.** (1) The ind-scheme $G(F)$ is in general not reduced, even for the simplest groups like $\mathbb{G}_m$;

(2) The quotient map $G(F) \to \mathfrak{gr}_G$ is a $G(\mathcal{O})$-torsor in the fpqc topology.

4.1. $GL_n$-Case. Let $G = GL_n$, and $R$ a $k$-algebra.

**Definition 4.5.** A lattice is a finitely generated projective $R[[\varpi]]$-module $L \subset R((\varpi))^n$ such that $R((\varpi)) \otimes_{R[[\varpi]]} L \cong R((\varpi))^n$.

As usual we have the standard lattice $L_0 = R[[\varpi]]^n$, and for all lattice $L$, there exists a large $m$ such that $\varpi^m L_0 \subset L \subset \varpi^{-m} L_0$. To better connect the lattices to the geometric description of Grassmannians (i.e. through vector bundles), so that we can get vector bundles not just coherent sheaves, we need this lemma:

**Lemma 4.6.** The quotient $\varpi^{-m} L_0/L$ is a projective $R$-module.
4.2. **Case of mixed characteristics.** In this subsection only, we talk briefly about the number-theoretic settings. Let \( k = \mathbb{F}_p \), \( \mathcal{O} = \mathbb{Z}_p \), and \( F = \mathbb{Q}_p \). We want to define an ind-scheme \( \mathfrak{S}_r \) over \( k \) such that it represents the functor of \( \mathbb{Z}_p \)-lattices in \( \mathbb{Q}_p \).

The first step is to define the analogue of lattices. Note that \( \mathbb{Z}_p \) can be seen as the ring of Witt vectors over \( k \), denoted by \( W(k) \), and \( \mathbb{Q}_p = W(k)[p^{-1}] \). Let \( R \) be a \( k \)-algebra, it is tempting to define the \( R \)-points of the functor to be finitely generated projective \( W(R) \)-modules inside \( W(R)[p^{-1}]^n \) of generic rank \( n \). However, if \( R \) is non-reduced, then \( p \) may be a zero divisor in \( W(R) \), thus \( W(R) \) is not a subring of \( W(R)[p^{-1}] \).

To remedy this, we can restrict the domain category to all perfect \( k \)-algebras, i.e. those \( R \) such that the Frobenius \( x \mapsto x^p \) is bijective.

**Lemma 4.7.** If \( R \) is perfect then \( p \) is not a zero divisor in \( W(R) \).

Now consider the functor

\[
\text{PerfAlg}/k \to \text{Sets}
\]

\[
R \mapsto \text{lattices in } W(R)[p^{-1}]^n,
\]

then we have:

**Proposition 4.8.** This functor is representable by an ind-scheme \( \varprojlim_i X_i \) with each \( X_i \) a perfect scheme (affine-locally being the spectra of perfect \( k \)-algebras).

However, another problem arises since perfect \( k \)-algebras are in general not of finite type, because usually it is a perfect closure of some \( k \)-algebra by taking \( p \)-th root over and over again. Only until recently have people understood a deeper result:

**Theorem 4.9** (Bhatt-Scholze, 2015). With the notations above, \( X_i \) is a perfection of a projective variety over \( k \), and the embeddings \( X_i \hookrightarrow X_{i+1} \) come from the maps of those varieties.

However, for a general reductive group not much is known yet.

4.3. **General reductive group case.** For the remaining of this section \( k = \mathbb{C} \). We start with the tori. Let \( G = T \cong \mathbb{G}_m^n \), and \( X_\ast = X_\ast(T) = \text{Hom}(\mathbb{G}_m, T) \), \( X^\ast = X^\ast(T) = \text{Hom}(T, \mathbb{G}_m) \) be the coweight and weight lattices respectively. We have a natural isomorphism \( X^\ast \cong \text{Hom}_\mathbb{Z}(X_\ast, \mathbb{Z}) \), since there exists a perfect pairing

\[
X^\ast \times X_\ast \to \mathbb{Z}
\]

\[
(\chi, \gamma) \mapsto n,
\]

where \( \chi \circ \gamma = (z \mapsto z^n) \). One can also recover \( T \) from the lattices by noticing \( T \cong \mathbb{G}_m \otimes_\mathbb{Z} X_\ast \). This suggests the definition \( T^\vee = \mathbb{G}_m \otimes_\mathbb{Z} X^\ast \), and it is clear that \( X_\ast(T^\vee) = X^\ast \) and \( X^\ast(T^\vee) = X_\ast \). One can also construct \( T^\vee \) directly from \( T \) in this case since topologically, \( X_\ast \cong \pi_1(T) \), and so \( T^\vee \cong \text{Hom}(\pi_1(T), \mathbb{G}_m) \), which is the same as rank-1 local systems on \( T \).

It turns out we can largely ignore the non-reducedness of the Grassmannian, thus we will focus on the underlying space of points. Since \( F^\times/\mathcal{O}^\times \cong \mathbb{Z} \) through valuation, we have as a space \( \mathfrak{S}_{Gr} = T(F)/T[[w]] \cong X_\ast(T) \).

Next we consider \( G = \text{SL}_n \subset \text{GL}_n \). We claim that \( \mathfrak{S}_{\text{SL}_n} \subset \mathfrak{S}_{\text{GL}_n} \). Indeed, we can identify the former with the lattices in \( F^n \) satisfying some additional conditions.
**Definition 4.10.** For two lattices $L, L' \subset F^n$, we define the relative length
\[ \text{Leng}(L, L') = \dim_{\mathbb{C}}(L/(L \cap L')) - \dim_{\mathbb{C}}(L'/(L \cap L')), \]

necessarily a finite number.

The following is straightforward.

**Lemma 4.11.** We have the isomorphism (as ind-schemes) $\mathfrak{Gr}_{\text{SL}_n} \cong \{ L \in \mathfrak{Gr}_{\text{GL}_n} \mid \text{Leng}(L, L_0) = 0 \}$.

We define the determinant bundle on $\mathfrak{Gr}_{\text{SL}_n}$ by letting the fiber over $L$ be $\det(L/(L \cap L_0)) \otimes_{\mathbb{C}} \det(L_0/(L \cap L_0))^\vee$.

**Proposition 4.12.** The determinant bundle is an ample line bundle on $\mathfrak{Gr}_{\text{SL}_n}$ hence gives a projective embedding of $\mathfrak{Gr}_{\text{SL}_n}$ (as ind-schemes).

The determinant bundle can be generalized to any reductive group $G$. The loop Lie algebra $\mathfrak{g}(F)$ admits an important central extension
\[ 0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}(F) \to 0, \]

where $\hat{\mathfrak{g}}$ is a Kac-Moody algebra. The construction is explicit: fix a symmetric nondegenerate invariant bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, the Kac-Moody algebra is defined as a vector space simply $\hat{\mathfrak{g}} = \mathbb{C} \oplus \mathfrak{g}(F)$, on which the cocycle $c_\beta \in H^2(\mathfrak{g}(F), \mathbb{C})$ for the central extension is given by $c_\beta(x, y) = \text{Res}_{\varpi = 0} \beta(x, y)$. The same can be done for groups as well, i.e. we have a central extension
\[ 1 \to \mathbb{G}_m \to \hat{G} \to G(F) \to 1, \]
in which the preimage of $G(\mathcal{O})$ splits. Therefore we have a $\mathbb{G}_m$-torsor
\[ \hat{G}/G(\mathcal{O}) \to G(F)/G(\mathcal{O}) = \mathfrak{Gr}_G. \]

The associated line bundle is the determinant bundle $\det$. For $\text{GL}_n$ or $\text{SL}_n$, the bilinear form $\beta$ can be chosen as the Killing form (extended to $\text{GL}_n$ as in Section 3).

For any simply-connected $G$, let $\bigvee_\beta = \text{Ind}_{\hat{\mathfrak{g}}(\mathcal{O})}^{\hat{\mathfrak{g}}(F)} 1_{c_\beta}$, for which the central character $c$ is some fixed one, we then have the identification (verify it)
\[ \hat{G} = \mathbb{P}(\bigvee_\beta). \]

4.4. **Another example:** $\text{PGL}_n$. Let $G = \text{PGL}_n = \text{GL}_n/\mathbb{G}_m$. It’s not hard to see
\[ \mathfrak{Gr}_{\text{PGL}_n} \cong \mathfrak{Gr}_{\text{GL}_n}/(L \sim \varpi L). \]

With this identification, any element in $\mathfrak{Gr}_{\text{PGL}_n}$ is then represented by some lattice of relative length $r \in \{0, \ldots, n-1\}$ to $L_0$, and any two elements are in the same connected component if and only if their relative lengths to $L_0$ are the same (mod $n$). For each $m = 0, \ldots, n-1$, let
\[ \mathfrak{Gr}^m = \{ L \subset L_0 \mid L \text{ is the preimage of some } m\text{-dimensional subspace of } L_0/\varpi L_0 \}
\cong \mathfrak{Gr}_m(\mathbb{C}^n), \]
the last term being the usual Grassmannian in $\mathbb{C}^n$.

**Proposition 4.13.** $\mathfrak{Gr}_{\text{PGL}_n}$ has $n$ connected components, and $\mathfrak{Gr}^m$ is the unique closed $G(\mathcal{O})$-orbit in the connected component corresponding to $m$. 

4.5. **Cartan and Iwasawa decompositions.** This subsection is probably going to be repeated in the next lecture. For each $\lambda \in X_*(T)$, we have $\varpi^\lambda \in T(F) \subseteq G(F)$, hence $\varpi^\lambda G(\mathcal{O})/G(\mathcal{O}) \subset \mathfrak{sr}_G$. Let $N$ be a maximal unipotent subgroup.

**Theorem 4.14** (Cartan Decomposition).

$$\mathfrak{sr}_G = \bigcup_{\lambda \in X_*(T)} G(\mathcal{O}) \varpi^\lambda.$$  

**Theorem 4.15** (Iwasawa Decomposition).

$$\mathfrak{sr}_G = \bigcup_{\lambda \in X_*(T)} N(F) \varpi^\lambda.$$  

To prove these we need a lemma.

**Lemma 4.16.**

$$(\mathfrak{sr}_G)^T \cong \{ \varpi^\lambda | \lambda \in X_* \}.$$  

Wrong sketch of the Proof (to be finished next lecture). Choose a sufficiently general one-parameter subgroup $\gamma: \mathbb{G}_m \to T$ such that $(\mathfrak{sr}_G)^\gamma = (\mathfrak{sr}_G)^T$. Any $G(\mathcal{O})$-orbit $X$ in $\mathfrak{sr}_G$ is a projective variety (why?), and is $\gamma$-stable. Fix a point $x \in X$, and look at the $\gamma$-orbit of $x$, we have an algebraic function $\mathbb{C}^X \to X$. But $X$ is projective, so this map extends to $\mathbb{C} \to X$. Let $x_0 = \lim_{z \to 0} \gamma(z) \cdot x$, then $x_0 \in (\mathfrak{sr}_G)^T \cap X$, which proves Theorem 4.14. □  

The “why?” part is wrong because it is only quasi-projective.

5. **April 10**

Let $G$ be a split reductive group and $T$ a maximal torus. Let $F = \mathbb{C}((\varpi))$ and $\mathcal{O} = \mathbb{C}[[\varpi]]$. Let $\lambda \in X_*(T)$, and then $\varpi^\lambda \in T(F) \subset G(F)$. Let $X^+_*(T)$ be dominant coweights.

The following part is fuzzy since I typed several days after this lecture and I did not care to look for details, also the lecture itself was a bit hand-waving during this part. The proof of Theorem 4.14 in the last lecture is wrong, but can be fixed by studying the neighborhood of the limit $x_0$ which only lies in the closure of the orbit (see the end of the last lecture), similar to a proof of Bruhat decomposition (which utilizes the similar idea as the wrong proof above). However, unlike Bruhat decomposition case, $G(F)/G(\mathcal{O})$ isn’t a locally compact space (it is infinite dimensional). So we can only morally choose a “neighborhood” of identity: consider $G[\varpi^{-1}]G(\mathcal{O})/G(\mathcal{O})$, where $G[\varpi^{-1}]$ means after choosing an embedding to $\text{GL}_n$, consider the elements with entries involving only $\varpi^{\geq -1}$ terms, and translate to $x_0$. This should be seen as pretending $F$ is a local function field (i.e. replacing $\mathbb{C}$ with a finite field), and choose a neighborhood there, and replacing finite field back with $\mathbb{C}$.

Now we present an alternative proof of Theorem 4.15 using algebraic geometry.

**Proof of Theorem 4.15 using algebraic geometry.** To show the theorem holds, it is the same as showing $G(F) = B(F)G(\mathcal{O})$, where $B$ is the Borel containing $N$, which in turn is the same as showing the following map is surjective:

$$B(\mathcal{O}) \backslash G(\mathcal{O}) \to B(F) \backslash G(F).$$
Since $B\backslash G$ is a projective variety, hence proper, by valuative criterion for properness, we know the map

$$(B \backslash G)(\mathcal{O}) \to (B \backslash G)(F)$$

is surjective. Now look at $B$-torsor $G \to G/B$, since the (non-abelian) cohomology $H^1(\text{Spec } R, B) = 0$ for both $R = \mathcal{O}$ and $R = F$, by long exact sequence of cohomology, we have

$$0 \to B(R) \to G(R) \to (B \backslash G)(R) \to H^1(\text{Spec } R, B) = 0,$$

to

hence $B(R) \backslash G(R) = (B \backslash G)(R)$ for $R = \mathcal{O}$ and $R = F$. Hence we are done. □

Remark 5.1. In the above proof, the sheaves we are taking cohomology are not abelian, however the “long exact sequence” still makes sense to at least $H^1$.


Theorem 5.2 (Hilbert-Mumford criterion). If $\partial Y \neq \emptyset$, then there exists a one-parameter subgroup $\gamma : \mathbb{C}^\times \to G$ such that

$$\lim_{z \to 0} \gamma(z)x \in \partial Y.$$

This theorem is readily seen equivalent to the following: suppose there exists $f : G \to X$ sending $g$ to $gx$, and there is an $\mathcal{O}$-point $y \in f(G)(\mathcal{O})$ such that

1. $f(g_F) = y \otimes \mathcal{O} F$, where $g_F \in G(F)$, and
2. $y \pmod {\varpi} \in \partial f(G),$

then we can take $g_F$ to be $\varpi^\lambda$ for some $\lambda \in X_s$.

Proof of Theorem 5.2. By Cartan decomposition, we know $G(F) = \bigcup \lambda G(\mathcal{O}) \varpi^\lambda G(\mathcal{O})$, therefore we know $g_F = g_1 \varpi^\lambda g_2$ for some $\lambda$ and some $g_1, g_2 \in G(\mathcal{O})$. After absorbing $g_2$ into $x$ (or equivalently, $f$) and $g_1^{-1}$ into $y$, we get the result. □

5.2. Classical Satake. Now we state the classical Satake’s theorem in more details. Now $F = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{M} = C_c(N(F)\backslash G(F)/G(\mathcal{O})) \cong \mathbb{C}[X_s]$ as vector spaces, the latter being the group ring of $X_s$, and $\mathcal{H} = C_c(B(F)/\mathcal{O})N(F) \cong \mathbb{C}[X_s]$ as algebras. The convolution defines a left action of $\mathcal{R}$ on $\mathcal{M}$, making the latter a free $\mathcal{R}$-module of rank 1. Similarly, let $\mathcal{H} = C_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ be the Hecke algebra, which acts on $\mathcal{M}$ on the right. Let $1_0 \in \mathcal{M}$ be the vector corresponding to the characteristic function of 0 in $\mathbb{C}[X_s]$, then $\mathcal{M} = \mathcal{R} \cdot 1_0$. Define the Satake map

$$S : \mathcal{H} \to \mathcal{R},$$

$$h \mapsto S(h), \text{ where } S(h) \cdot 1_0 = 1_0 \cdot h \in \mathcal{M}.$$

The Satake map is an algebra homomorphism.

Theorem 5.3 (Satake). The Satake map is injective, and the image is precisely $\mathcal{R}^W$, the Weyl group invariants in $\mathcal{R} \cong \mathbb{C}[X_s]$.

To demonstrate the non-trivialness of the Satake map, we present another theorem. Let $h^\lambda \in \mathcal{H}$ be the characteristic function of $G(\mathcal{O})p^\lambda G(\mathcal{O})$ in $G(F)$. Since $h^\lambda = h_{w(\lambda)}$ for any $w \in W$, we may assume $\lambda$ is dominant. For notational coherence, let $q = p$ below.
Theorem 5.4 (MacDonald). When $\lambda$ is dominant, we have
\[
S(h^\lambda) = \frac{q^{(\rho,\lambda)}}{W_\lambda(q^{-1})} \sum_{w \in W} \left( \prod_{\alpha > 0} \frac{1 - q^{-1}1_{-w(\alpha)}}{1 - 1_{-w(\alpha)}} \right) 1_{w(\lambda)},
\]
where $\rho$ is the half-sum of positive roots, $W_\lambda = \text{Stab}_W(\lambda)$ and $W_\lambda(q^{-1}) = \sum_{w \in W_\lambda} q^{-l(w)}$, and $l(w)$ is the length of $w$.

Remark 5.5. When $G = \text{GL}_n$, the part in large parentheses in (5.1) is called Hall polynomial.

6. April 12

Today is a digression class on the global Langlands conjecture for $\text{GL}_n$ in the function field case, which has been proved by Drinfeld and Lafforgue.

Let $F$ be a finite extension of $\mathbb{F}_q((\varpi))$ and $\mathcal{O}$ the ring of integers in $F$. Let $G$ be a reductive group over $\mathbb{Z}$. Let $\mathcal{H} = C^\infty_c(G(\mathcal{O}) \backslash G(F)/G(\mathcal{O}))$ be the Hecke algebra.

Definition 6.1. A spherical representation of $G(F)$ is a homomorphism $\rho: G(F) \to \text{GL}(V)$ for some (possibly infinite dimensional) vector space over $\mathbb{C}$ such that

1. for all $v \in V$, $\text{Stab}_G v$ contains an open subgroup,
2. $V^{G(\mathcal{O})} \neq 0$,
3. $V$ is irreducible.

Proposition 6.2. For any spherical representation $V$, we have $\dim_{\mathbb{C}} V^{G(\mathcal{O})} = 1$.

Proof. Note $V$ is also a simple $C^\infty_c(G(F))$-module. It then suffices to show $V^{G(\mathcal{O})}$ is simple as a $\mathcal{H}$-module, since $\mathcal{H}$ is commutative. Suppose $W \subset V^{G(\mathcal{O})}$ is a non-zero proper $\mathcal{H}$-submodule, then we claim $C^\infty_c(G(F)) \cdot W \cap V^{G(\mathcal{O})} = W$: let $v = \sum_i \varphi_i \cdot w_i \in W \cap V^{G(\mathcal{O})}$ (finite sum, $\varphi_i \in C^\infty_c(G(F))$), and $e$ be the characteristic function of $G(\mathcal{O})$, then we know $e \cdot w = w$ and $e \cdot v = v,$ hence $v = \sum_i (e \ast \varphi_i \ast e) \cdot w_i$. But $e \ast \varphi_i \ast e \in \mathcal{H}$, thus $v \in \mathcal{H} \cdot W \subset W$, hence the claim. This shows that $C^\infty_c(G(F)) \cdot W$ is a non-zero proper $C^\infty_c(G(F))$-submodule of $V$, which is a contradiction.

Corollary 6.3. For all spherical representation $V$, there exists a unique character $\chi_V: \mathcal{H} \to \mathbb{C}$ such that $h \cdot v = \chi_V(h) v$ for all $h \in \mathcal{H}$ and $v \in V^{G(\mathcal{O})}$.

The following is obvious.

Lemma 6.4. Let $X/\mathbb{F}_q$ be a projective variety, then $X(\mathbb{F}_q)$ is a finite set.

Let $\mathfrak{Gr}_G$ be the affine Grassmannian viewed as an ind-scheme defined over $\mathbb{F}_q$, and let $\mathfrak{Gr}_m$ be the system of projective varieties appearing in a chosen presentation of $\mathfrak{Gr}$ as the colimit. The above lemma then shows that any $G(\mathcal{O})$-orbit in $\mathfrak{Gr}_G(\mathbb{F}_q)$ is a finite set since any orbit is contained in $\mathfrak{Gr}_m(\mathbb{F}_q)$ for some $m$.

Let $T \subset G$ be a maximal torus, $X_*(T)$ the coweight lattice, and $W$ the Weyl group. Let $X^*(T)$ be the weight lattice, and $T^\vee$ the dual torus. Retain the notation from last lecture, we have $\mathcal{R} = \mathbb{C}[X_*(T)] \cong \mathbb{C}[X^*(T^\vee)] \cong \mathbb{C}[T^\vee]$, the middle two being group algebras and the last being the ring of regular functions. To see the last isomorphism simply note that a base vector in $\mathbb{F}[X^*(T^\vee)]$ can be identified with a rational monomial in $z_1, \ldots, z_n$, and $\mathbb{C}[T^\vee] = \mathbb{C}[z_1^\pm, \ldots, z_n^\pm]$. Now we state the function field version of the Satake theorem.
Theorem 6.5 (Satake). There exists an algebraic isomorphism $S: \mathcal{H} \rightarrow \mathbb{C}[T^\vee]^W \cong \mathbb{C}[T^\vee/W]$. Thus for a spherical representation $V$ there exists a unique $t_V \in T^\vee/W$ such that $h \cdot v = S(h)(t_V) \cdot v$ for all $v \in V^{G(\mathcal{O})}$.

Example 6.6. Let $G = \text{GL}_n$, $W = \mathfrak{S}_n$, and $\mathbb{C}[T^\vee]^W = \mathbb{C}[z_1^\pm, \ldots, z_n^\pm]^{\mathfrak{S}_n}$.

6.1. Global Langlands conjecture. Let $X$ be a connected smooth projective curve over $\mathbb{F}_q$. Let $F = \mathbb{F}_q(X)$ and $|X|$ the set of closed points of $X$. Let $A$ be the ring of adeles of $F$, and $O$ the integral adeles.

The definition for automorphic functions in this subsection is a simplified one for our specific case, and shouldn’t be remembered, because it easily follows from the general definition. Let $G = \text{GL}_n$, and fix $a \in A^\times$. Let $\deg(a) = \sum_{x \in |X|} \deg(x) \text{val}_x(a)$. Let $X_a = a^Z \backslash G(A)/G(F)$, and note $G(A)$ acts on the left on $X_a$.

Definition 6.7. (1) An unramified automorphic function is a function $f: X_a \rightarrow \mathbb{C}$ which is $G(O)$-invariant.

(2) $f$ is cuspidal if for all proper parabolic subgroup $P = LN$, we have

$$\int_{N(A)/N(F)} f(gn) dn = 0,$$

for all $g \in G(A)$.

Proposition 6.8. If $f$ is cuspidal, then $f$ is compactly supported in $X_a$.

Let $\mathcal{A}_0(X_a)$ be the space of cuspidal functions on $X_a$, then it is a subspace of $C_c^\infty(X_a)$. We present three standard results for cuspidal functions, two of them valid for general reductive groups.

Theorem 6.9. $\mathcal{A}_0(X_a)$ is a (pre-)unitary representation of $G(A)$, and it decomposes into a direct sum of finite dimensional irreducible subrepresentations

$$\mathcal{A}_0(X_a) = \bigoplus_{\rho} V_{\rho}.$$ 

Theorem 6.10 (Multiplicity one for $\text{GL}_n$, Piatetsky-Shapiro). Each irreducible representation of $\text{GL}_n$ occur in $\mathcal{A}_0(X_a)$ at most once.

Theorem 6.11 (Tensor product theorem). Any irreducible admissible representation of $G(A)$ decomposes into restricted tensor product of local components

$$V_{\rho} = \bigotimes'_{x \in |X|} V_{\rho,x}.$$ 

No explanation for the undefined terms will be provided here since they are not important in this course and are easily found in any text about automorphic representations.

The above is an oversimplified story for the automorphic side. Now we turn to the Galois side.

For each $x \in |X|$, fix a separable closure $F_x^s$ of $F_x$, and let $\Gamma_x = \text{Gal}(F_x^s/F_x)$, then we have the exact sequence

$$1 \rightarrow I_x \rightarrow \Gamma_x \rightarrow \hat{\mathbb{Z}} \rightarrow 1,$$
where $I_x$ is the inertia subgroup and $\widehat{\mathbb{Z}}$ is in fact the galois group $\text{Gal}(F^\text{ur}_x/F_x)$, generated topologically by the Frobenius $\text{Frob}_x$. Pick a decomposition group of $x$ in $\Gamma_F = \text{Gal}(F^*/F)$ and view $\Gamma_x$ as a subgroup of $\Gamma_F$. Let

$$\text{Irr}^\text{ur} = \{ \sigma : \Gamma_F \to \text{GL}_n(\overline{\mathbb{Q}_\ell}) \mid \sigma \text{ is continuous and trivial on } I_x \text{ for all } x \in |X| \}.$$ 

**Theorem 6.12** (Langlands conjecture for $\text{GL}_n$ in unramified case, Drinfeld for $n = 2$ and Lafforgue for general $n$). There exists a unique bijection

$$\{\text{unramified cuspidal irreducible representations}\} \to \text{Irr}^\text{ur}$$

$$\rho \mapsto \sigma(\rho)$$

such that for all $x \in |X|$, $\rho$ and $\sigma(\rho)$ have the same local $L$-factors (explained below).

For the local $L$-factors, fix any $x \in |X|$, and choose the maximal torus to be the standard one, hence we can view both $T$ and $T^\vee$ to be $\mathbb{G}_m^n$, and their $\mathbb{C}$-points $(\mathbb{C}^\times)^n$. Therefore $\mathbb{C}[T^\vee]^W \cong \mathbb{C}[z_1^{\pm1}, \ldots, z_n^{\pm1}]^{S_n}$. Let $t_{V,x}$ be the element in $T^\vee/W$ that is mentioned immediately after Theorem 6.5 which essentially is the character of $\mathcal{H}_x$ on $V_{xG(\mathcal{O}_x)}$. For spherical representations, the theory of Godement-Jacquet gives the local $L$-factor

$$L(\rho_x, s) = \prod_{i=1}^n \frac{1}{1 - \chi_i(\varpi_x)q_x^{-s}},$$

where $q_x$ is the cardinality of the residue field of $x$, and $\chi_i$ are unramified characters of $F^\times_x$ such that $\chi_i(\varpi_x) = z_i(t_{V,x})$. Since $\chi_i$ are unramified, their are completely determined by $z_i(t_{V,x})$.

On the Galois side, the Artin $L$-function is given by

$$L(\sigma(\rho)_x, s) = \frac{1}{\det(1 - q_x^{-s}\sigma_x(\text{Frob}_x))}.$$ 

Then the theorem claims that, after fixing an algebraic isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$ once and for all, then for all $x \in |X|$, $i = 1, \ldots, n$.

$$L(\rho_x, s) = L(\sigma(\rho)_x, s).$$

**Remark 6.13.**

1. A theorem of Deligne shows the choice of the isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$ is irrelevant to the theorem.
2. A Galois representation $\sigma : \Gamma_F \to \text{GL}_n(\overline{\mathbb{Q}_\ell})$ is the same as a compatible system of rank $n$ local systems at all points of $X$.
3. Another way of expressing the matching of the $L$-factors is that

$$\text{Tr}(\Lambda^i\text{diag}(z_1, \ldots, z_n)(t_{V,x})) = \text{Tr}(\Lambda^i\sigma_x(\text{Frob}_x)) \text{ for all } x \in |X|, i = 1, \ldots, n.$$ 

7. April 17

This lecture is meant to motivate the lectures that follow, and will be in many places imprecise. This lecture is a moral outline of the categorification of the Langlands conjecture, and how the conjecture can be expected conceptually. However, the content of this lecture shouldn’t be read into too deeply, since it is by no means mathematically correct.

First we try to categorify the Hecke algebra. Let $G$ be a reductive group, $X$ a smooth projective curve over $\mathbb{F}_q$, $F = \mathbb{F}_q((\varpi))$, $\mathcal{O} = \mathbb{F}_q[[\varpi]]$, and $\mathcal{H} = C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$. 


According to Grothendieck’s philosophy, which says interesting functions usually come from Frobenius actions on sheaves, we want to find a category that resembles Hecke algebra, i.e.
\[ \mathcal{H} \leftrightarrow \text{some monoidal category Sat of } G(\mathcal{O})\text{-equivariant sheaves on } \mathfrak{g}_{\mathfrak{r}G}. \]

The notion of monoidal category will be explained in the next lecture. Similarly, we want to have a module category of Sat that serves the same purpose as the space of automorphic functions to the Hecke algebra. In other words,
\[ C^\infty_c(G(F)\backslash G(\mathcal{A})/G(\mathcal{O})) = C^\infty_c(\text{Bun}_G) \leftrightarrow \text{a category of } \ell\text{-adic sheaves } \mathcal{F} \text{ on } \text{Bun}_G. \]

**Example 7.1.** An easier example of this monoidal-module-categories setting would be this: let \( G \) be a (topological) group, and \( C \) the category of all finite dimensional representations of \( G \), then \( C \) is a monoidal category with the multiplication given by tensor product. Let \( X \) be a \( G \)-set, and \( M \) be the category of \( G \)-equivariant vector bundles over \( X \), then \( M \) is a module category of \( C \), with the objects of \( C \) acting on \( M \) by twisting.

As the example suggests, the module category we want should be the one of certain \( G(\mathcal{O}) \)-equivariant sheaves on \( \text{Bun}_G \).

Now let \( G = \text{GL}_n \), define a Hecke correspondence
\[ \mathcal{H} \overset{Heck}{\to} p_1 \times p_2 \times q \overset{\to}{\text{Bun}_G \times \text{Bun}_G \times X}, \]
where \( \mathcal{H} \) is as defined before. Then we can categorify the spherical condition of \( \rho \) at each point \( x \in X \):
\[ h \cdot v_x = \chi_x(h)v_x \forall x \in X, h \in \mathcal{H}_x \overset{\to}{(p_2 \times q)_*p_1^*}\mathcal{F} = \mathcal{E}_i \boxtimes \mathcal{F}, \]
where \( \mathcal{E}_i \) some local system on \( X \).

Let \( G^\vee \) be the dual group of \( G \), the geometric Satake gives an equivalence of monoidal categories
\[ \mathcal{S}: \text{Rep}(G^\vee) \simeq \text{Sat}, \]
the latter being the category of finite dimensional (complex) representations of \( G^\vee \). For \( G = \text{GL}_n \), \( G^\vee = \text{GL}_n \) as well, and \( \text{Rep}(\text{GL}_n) \) is generated by wedge powers of the standard representation. If \( \mathcal{F} \) is a sheaf corresponding to spherical vectors, take \( \mathcal{E}_i \) to be that
\[ \mathcal{S}(\wedge^i \mathbb{C}^n) \ast \mathcal{F} = \mathcal{E}_i \boxtimes \mathcal{F}, \]
it is then natural to require that for all \( G^\vee \)-representations \( V \), there exists some local system \( \mathcal{E}_V \) on \( X \) that
\[ \mathcal{S}(V) \ast \mathcal{F} = \mathcal{E}_V \boxtimes \mathcal{F}. \]  
(7.1)
Think of local systems on \( X \) as representations of \( \pi_1(X) \), we hope that if \( \mathcal{F} \) is irreducible, then (7.1) would imply
\[ \mathcal{S}(V_1 \otimes V_2) \ast \mathcal{F} = \mathcal{S}(V_1) \ast \mathcal{S}(V_2) \ast \mathcal{F} = (\mathcal{E}_{V_1} \otimes \mathcal{E}_{V_2}) \boxtimes \mathcal{F}. \]
If this is true, then some theorem about monoidal categories shows that the monoidal functor \( \text{Rep}(G^\vee) \to \text{Rep}(\pi_1(X)) \) is induced by a functor \( \pi_1(X) \to G^\vee \), which is exactly what the Langlands conjecture says.

Again, this lecture is a lot of hand-waving, and I could not remember a lot of discussion happened in class. Many stuff here has been made precise in the following lectures, and I only typed this lecture for the sake of completeness.
8. April 19

Today we talk about monoidal categories and the module categories over a monoidal category.

Warning: based on Ginzburg’s performance in class, there could be errors here and there in the definitions.

8.1. **Monoidal Categories.** Let $k$ be a field, and $\mathcal{C}$ be an (essentially small) abelian $k$-linear category (its morphisms are $k$-vector spaces, and compositions are $k$-bilinear maps).

**Definition 8.1.** A *monoidal structure* on $\mathcal{C}$ is a $k$-linear biproduct

$$\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

equipped with

1. a functorial associativity constraint

$$\alpha_{x,y,z}: (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

for all objects $x, y, z$;

2. a unit object $1$ equipped with isomorphism $1 \otimes 1 \simeq 1$, such that the functors

$$x \mapsto 1 \otimes x, \quad x \mapsto x \otimes 1$$

are equivalences $\mathcal{C} \to \mathcal{C}$;

3. it satisfies the *pentagon axiom*, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
((x \otimes y) \otimes z) \otimes w & \overset{}{\longrightarrow} & (x \otimes (y \otimes z)) \otimes w \\
\downarrow & & \downarrow \\
x \otimes ((y \otimes z) \otimes w) & \overset{}{\longrightarrow} & x \otimes (y \otimes (z \otimes w))
\end{array}
\]

**Proposition 8.2.** The unit 1 is unique up to a unique isomorphism.

**Definition 8.3.** The monoidal category $(\mathcal{C}, \otimes)$ is called *symmetric* if there are isomorphisms $s_{x,y}: x \otimes y \to y \otimes x$ for all $x, y \in \mathcal{C}$ such that

1. $s_{y,x} \circ s_{x,y} = \text{Id}_{x \otimes y}$;

2. $s_{x,y}$ satisfies two hexagon axioms, i.e. the diagrams commute:

\[
\begin{array}{ccc}
(u \otimes (v \otimes w)) & \overset{}{\longrightarrow} & (v \otimes w) \otimes u \\
\downarrow & & \downarrow \\
(u \otimes v) \otimes w & \overset{}{\longrightarrow} & v \otimes (w \otimes u)
\end{array}
\]

\[
\begin{array}{ccc}
(u \otimes v) \otimes w & \overset{}{\longrightarrow} & w \otimes (u \otimes v) \\
\downarrow & & \downarrow \\
(u \otimes v) & \overset{}{\longrightarrow} & (w \otimes u) \otimes v
\end{array}
\]

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Example 8.4. Let $R$ be a commutative $k$-algebra, then the category $R$-mod is a symmetric monoidal category with $x \otimes y = x \otimes_R y$.

Example 8.5. If we change the map $s_{x,y}$ in $R$-mod to its negative, and call this category $R$-mod$^-$, this is not symmetric.

Remark 8.6 (Maclane coherence). The pentagon axiom implies that the products without parentheses $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ are well-defined. If the category is also symmetric, then the pantagon and the hexagon axioms ensure that the order of $x_i$ does not matter (the products are unique upto a unique isomorphism).

Definition 8.7. Let $\mathcal{C}, \mathcal{C}'$ be monoidal categories. A functor $F: \mathcal{C} \to \mathcal{C}'$ together with isomorphisms $\beta_F: F(x) \otimes F(y) \to F(x \otimes y)$ is called a monoidal functor if the diagrams of the form

$$ (F(x) \otimes F(y)) \otimes F(z) \simeq F(x \otimes y) \otimes F(z) \simeq F((x \otimes y) \otimes z) $$

as a whole fit into the vertices of the pentagon axiom.

Definition 8.8. A symmetric monoidal category $\mathcal{C}$ is called rigid if for all $x \in \mathcal{C}$, there exists $x^\vee \in \mathcal{C}$ and morphisms

$$ \text{coev}: 1 \to x \otimes x^\vee, $$

$$ \text{ev}: x^\vee \otimes x \to 1, $$

such that the composition

$$ x \simeq 1 \otimes x \xrightarrow{\text{coev} \otimes \text{Id}} (x \otimes x^\vee) \otimes x \to x \otimes (x^\vee \otimes x) \xrightarrow{\text{Id} \otimes \text{ev}} x \otimes 1 \simeq x $$

is the identity, and a similar one holds for $x^\vee$.

Proposition 8.9. (1) The map $x \mapsto x^\vee$ is a functor;

(2) $\text{ev}$ and $\text{coev}$ are morphism of functors.

Definition 8.10. A rigid symmetric monoidal category is called a tensor category.

Remark 8.11. (1) $1$ is simple if and only if the endomorphism ring $\text{End}(1)$ is a field. From now on we assume this is true.

(2) Let $f: x \to x$ be a morphism, then it induces

$$ 1 \xrightarrow{\text{coev}} x \otimes x^\vee \xrightarrow{f \otimes \text{Id}} x \otimes x^\vee \xrightarrow{\text{ev}} 1, $$

which we define to be $\text{Tr}(f) \in \text{End}(1)$.

Definition 8.12. For any $x \in \mathcal{C}$, let $\dim x := \text{Tr}(\text{Id}_x)$. 
Example 8.13. (1) Let \( \text{Vect} \) be the category of finite dimensional \( k \)-vector spaces, then it is a tensor category.

(2) \( R\text{-mod} \) is not rigid since the dual may not exist.

(3) The full subcategory of projective \( R \)-modules is rigid but not abelian.

(4) \( \text{Rep}(G) \) the category of finite dimensional \( G \)-representations is a tensor category.

Definition 8.14. A fiber functor on \((\mathcal{C}, \otimes)\) is an exact fully faithful monoidal functor \( \omega: \mathcal{C} \to \text{Vect} \).

Example 8.15. The forgetful functor \( \omega_G: \text{Rep}(G) \to \text{Vect} \) is a fiber functor.

Proposition 8.16. Let \( H, G \) be algebraic groups, and \( F: \text{Rep}(G) \to \text{Rep}(H) \) a monoidal functor such that \( \omega_H \circ F = \omega_G \), then it is induced by a group homomorphism \( f: G \to H \).

8.2. Module categories. For any abelian category \( \mathcal{M} \), consider the category \( \text{Fun}(\mathcal{M}, \mathcal{M}) \) the self-functors on \( \mathcal{M} \), then it has a natural monoidal structure by composition (it is obviously not symmetric).

Definition 8.17. A module category over a monoidal category \( \mathcal{C} \) is an abelian category \( \mathcal{M} \) with a monoidal functor \( \mathcal{C} \to \text{Fun}(\mathcal{M}, \mathcal{M}) \).

More explicitly, we have a bifunctor \( \mathcal{C} \times \mathcal{M} \to \mathcal{M} \) sending \((x, m)\) to \( x \otimes m \) such that

1. The analogous associativity (together with pentagon axiom) holds, by replacing the last elements in a pure tensor with elements in \( \mathcal{M} \);
2. \( 1 \otimes - = \text{Id}_\mathcal{M} \).

8.3. Connection to the categorification of Langlands conjecture. Let \( X \) be a smooth projective curve over \( k \), \( \Delta: X \to X \times X \) the diagonal map, and \( \text{Bun} = \text{Bun}_G \) the stack of \( G \)-bundles on \( X \) (by abuse of notations, we sometimes also use it to mean the \( k \)-points of the stack). Let \( \mathcal{S}(\mathcal{X}) \) to denoted a to-be-specified category of sheaves on an algebraic stack \( \mathcal{X} \), and \( \text{Sat} \) be a monoidal category of (to-be-specified) sheaves on \( \mathfrak{G}_G \).

Suppose for any \( h \in \text{Sat} \), we can define a Hecke operator \( \mathcal{H}_h: \mathcal{S}(\text{Bun}) \to \mathcal{S}(X \times \text{Bun}) \), and for any \( h_1, h_2 \in \text{Sat} \), the composition map\[
\mathcal{S}(\text{Bun}) \xrightarrow{\mathcal{H}_{h_1}} \mathcal{S}(X \times \text{Bun}) \xrightarrow{\mathcal{H}_{h_2}} \mathcal{S}(X \times X \times \text{Bun}) \xrightarrow{\Delta^*} \mathcal{S}(X \times \text{Bun})
\]
is isomorphic to \( \mathcal{H}_{h_1h_2} \), and such “composition” of Hecke operators satisfy an appropriate pentagon axiom.

We say \( \mathcal{E} \in \mathcal{S}(\text{Bun}) \) is an eigenobject if for all \( h \in \text{Sat} \), there exists some object denoted by \( \chi(h) \in \text{Rep}(\pi_1(X)) \) and an isomorphism\[
\gamma_h: \mathcal{H}_h \ast \mathcal{E} \xrightarrow{\sim} \chi(h) \boxtimes \mathcal{E},
\]
and \( \gamma_h \) satisfies appropriate associativity axioms.

Proposition 8.18. Suppose we have an eigenobject \( \mathcal{E} \) for which there exists an open substack \( U \subset \text{Bun} \) such that \( \mathcal{E}|_U \) is a constant sheaf, then \( \chi \) associated with \( \mathcal{E} \) is a monoidal functor.

(My note was too brief for this one; did I understand it wrong?)

Proof. Fix \( b \in U \), and consider the restrictions to \( X \times \{b\} \). Since \( (\mathcal{H}_h \ast \mathcal{E})|_{X \times \{b\}} \simeq (\chi(h) \boxtimes \mathcal{E})|_{X \times \{b\}} \), then the proposition seems to follow trivially. (Why do we need this locally-freeness?)}
Remark 8.19. Geometric Satake together with this proposition gives what we needed for Langlands conjecture as said in last lecture. The difficulty for \( GL_n \) is to define the Hecke functor for general representations, but we do know the Hecke operators for wedge powers of the standard representation are just the pull-push functor on \( \mathcal{H} \text{eck}^k \) as said last time.

8.4. Hopf algebra and representations.

**Definition 8.20.** A commutative Hopf algebra is a quadruple \((A, \Delta, \epsilon, S)\) where \( A \) is a commutative \( k \)-algebra and the maps

\[
\Delta: A \to A \otimes_k A,
\]
\[
\epsilon: A \to k,
\]
\[
S: A \to A
\]

satisfy certain diagrams so that they make \( \text{Spec} \, A \) an affine group scheme, with \( \Delta, \epsilon, S \) corresponding to multiplication, identity, and inverse respectively. One can write down all the diagrams needed for them.

**Example 8.21.** This is basically just the definition: for any group scheme \( G \) over \( k \), the ring \( \mathcal{O}_G(G) \) together with appropriate maps is a Hopf algebra.

**Proposition 8.22.** The functor \( A \mapsto \text{Spec} \, A \) is an equivalence of categories of the Hopf algebras over \( k \), and affine group schemes over \( k \).

**Definition 8.23.** A representation of \( G \) on a \( k \)-vector space \( V \) is an \( \mathcal{O}_G(G) \)-comodule structure on \( V: V \to V \otimes_k \mathcal{O}_G(G) \).

**Proposition 8.24.** Any representation \( V \) of \( G \) is equal to the direct limit of its finite dimensional subrepresentations.

**Proof.** The proof given in class seems to be wrong (missing some steps). A correct one can be found in Milne’s 2017 book Algebraic Groups (too lazy to put a citation here).

9. April 24

Let \( \mathcal{A} \) be a \( k \)-linear abelian category which is essentially small. Let \( x \in \mathcal{A} \). Let \( \text{Vect} \) be the category of finite dimensional \( k \)-vector spaces. Given \( V \in \text{Vect} \), we define \( V \otimes x \) and \( \text{Hom}(V, x) \) as objects in \( \mathcal{A} \).

**Lemma 9.1.** (1) The functor \( \mathcal{A} \to \text{Vect} \) sending \( y \) to \( \text{Hom}(V, \text{Hom}(x, y)) \) is representable by an object denoted by \( V \otimes x \in \mathcal{A} \).

(2) The functor \( \mathcal{A} \to \text{Vect} \) sending \( y \) to \( \text{Hom}(V \otimes y, x) \) is representable by an object \( \text{Hom}(V, x) \in \mathcal{A} \).

**Proof.** Suppose \( V \cong k^n \), then let \( V \otimes x = x \otimes k^n \), and one can show it represents the first functor. Same for the second one.

**Proposition 9.2.** Let \( \mathcal{C} \) be a tensor category.

(1) \( (x^\vee)^\vee \cong x \) since \( x \) satisfies the axioms for dual object of \( x^\vee \).

(2) For any \( x, y \in \mathcal{C} \), put \( \text{Hom}(x, y) = x^\vee \otimes y \in \mathcal{C} \), then \( \text{Hom}(1, \text{Hom}(x, y)) = \text{Hom}(x, y) \).

(3) If all objects of \( \mathcal{C} \) have finite length, then \( \dim \text{Hom}(x, y) < \infty \) for all \( x, y \in \mathcal{C} \).
Proof. We prove the last claim. Let \( f_1, f_2, \ldots \) be a sequence of linearly independent elements in \( \text{Hom}(x, y) \), then \( f_i \) induces a morphism \( F_i: 1 \to \text{Hom}(x, y) \). The simplicity of 1 and linear independence imply that \( \bigoplus_{i=1}^n F_i: 1^\oplus n \to \text{Hom}(x, y) \) is injective. If the sequence is infinite, then \( \text{Hom}(x, y) \) is then infinite dimensional, which is a contradiction to the assumption. So \( \text{Hom}(x, y) \) must be finite dimensional. \( \square \)

Let \( (A, \Delta, \epsilon, S) \) be a commutative Hopf algebra. We know the categories of commutative Hopf algebras and of affine group schemes are equivalent. We also defined in last lecture a representation \( V \) of affine group scheme \( G \) to be a comodule of \( k[G] \), which is a Hopf algebra.

**Lemma 9.3.** If \( V \) is a finite dimensional representation of \( G \) is the same as a homomorphism of group schemes \( G \to \text{GL}(V) \).

**Lemma 9.4.** Let \( A \) be a Hopf algebra.

1. Any \( A \)-comodule is the union of its finite dimensional subcomodules.
2. \( A \) is the union of its Hopf subalgebras that are finitely generated as a \( k \)-algebra. Equivalently, \( \text{Spec} A \) is the limit of a system of affine algebraic groups (groups schemes that are of finite type over \( k \)).

**Proof.** The proof in class is slightly problematic. For a correct proof see Milne’s 2017 book Algebraic Groups chapters 3 and 8. \( \square \)

**Lemma 9.5.** An affine group scheme \( G \) is algebraic if and only if there exists a faithful finite dimensional representation.

**Proof.** The “if” part is clear. For the other direction, note \( k[G] \) itself is a \( k[G] \)-comodule, therefore is the union of its finite dimensional subcomodules \( V_i \). Equivalently we have a homomorphism \( G \to \text{GL}(V_i) \), with kernel \( K_i \). Since the representation \( k[G] \) is faithful, we have \( \bigcap_i K_i = \{e\} \). Since \( K_i \) is closed in \( G \), and \( G \) is a noetherian scheme since it’s of finite type, there exists some \( i_0 \) such that \( K_{i_0} = \{e\} \). Thus \( V_{i_0} \) is a faithful finite dimensional representation of \( G \). \( \square \)

**Definition 9.6.** An affine algebraic group is called reductive if it contains no nontrivial normal unipotent subgroup.

**Proposition 9.7.** Let \( G \) be an affine group scheme over \( k \).

1. \( G \) is algebraic if and only if there exists \( x \in \text{Rep}(G) \) such that \( \text{Rep}(G) \) is generated by \( x^\otimes n \), \( x^\oplus n \), and their subquotients.
2. \( G \) is reductive if and only if \( \text{Rep}(G) \) is semisimple.

**Proof.**

1. If \( G \) is algebraic, we have \( G \subset \text{GL}_n \) for some \( n \), then by theory of algebraic groups, we can take \( x = k^n \). On the other hand, if such \( x \) exists, then \( x \) must be faithful, hence \( G \) is algebraic by our lemma above.

2. One direction is clear: if \( G \) is reductive, then its finite dimensional representations are semisimple. On the other hand, any algebraic group has a maximal normal unipotent subgroup \( R = R_u(G) \) and \( G/R \) is reductive. Fix any irreducible finite dimensional representation \( V \) of \( G \), then \( V^R \neq 0 \) by the general theory of algebraic groups. Since \( R \) is normal, we have \( V^R \) is \( G \)-stable, hence a subrepresentation of \( V \). Since \( V \) is irreducible, we have \( V^R = V \). Thus \( R \) acts on objects of \( \text{Rep}(G) \) by trivial actions.
since the category is semisimple. But there exists at least one faithful representation of \(G\), thus \(R\) must be trivial, hence \(G\) is reductive.

Let \(G\) be an affine group scheme, and \(\omega: \text{Rep}(G) \to \text{Vect}\) a fiber functor (i.e. exact, faithful, and monoidal). Define a functor \(\text{Aut} \otimes \omega: \text{Alg} / k \to \text{Grps}\) by

\[
R \mapsto \left\{ (\lambda_x \in \text{Aut}_R(R \otimes \omega(x)))_{x \in \text{Rep}(G)} \left| \begin{array}{l}
\lambda_1 = \text{Id}, \lambda_x \otimes_R \lambda_y = \lambda_{x \otimes y}, \\
\text{and for all } \alpha: x \to y, \\
\text{we have } \lambda_y(1_R \otimes \omega(\alpha)) = (1_R \otimes \omega(\alpha))\lambda_x
\end{array} \right. \right\}.
\]

We also have the functor of points associated with \(G\), also denoted by \(G\).

**Proposition 9.8.** There is an isomorphism of functors \(G \simeq \text{Aut} \otimes \omega\).

10. April 26

We continue from last lecture. We have a corollary to the last proposition we had.

**Corollary 10.1.** Let \(G, H\) be two affine group schemes, and \(F: \text{Rep} H \to \text{Rep} G\) be a monoidal functor such that \(\omega_G \circ F = \omega_H\), then there exists a homomorphism \(f: G \to H\) such that \(F \simeq f^*\).

**Proof.** Given \(F\), define

\[
F^*: \text{Aut} \otimes \omega_G \to \text{Aut} \otimes \omega_H
\]

\[
\lambda = (\lambda_x) \mapsto (F^* \lambda)_y := \lambda_{F(y)}.
\]

This induces a group homomorphism \(f: G \to H\) by our proposition above. Then one can verify \(f^* \simeq F\). \(\square\)

Now we state the main theorem for the whole tensor categories exposition and hopefully we can forget about them soon after.

**Theorem 10.2.** Let \(C\) be a tensor category over an algebraically closed field \(k\) with characteristic 0, and in which the object 1 is simple. Let \(\omega: C \to \text{Vect}\) be a fiber functor. Then the functor \(\text{Aut} \otimes \omega\) is representable by an affine group scheme \(G\) such that there exists a functor \(\overline{\omega}: C \to \text{Rep} G\) such that \(\omega = \omega_G \circ \overline{\omega}\), where \(\omega_G\) is the forgetful functor.

A slightly stronger theorem with slightly weaker condition is as follows.

**Theorem 10.3.** Let \(C\) be an abelian category that is \(k\)-linear with functor \(\otimes\) and object 1, and isomorphisms

\[
\alpha_{x,y,z}: x \otimes (y \otimes z) \to (x \otimes y) \otimes z,
\]

\[
\beta_{x,y}: x \otimes y \to y \otimes x,
\]

\[
1 \otimes x \to x \to x \otimes 1,
\]

but they may not have pentagon/hexagon axioms. Let \(\omega: C \to \text{Vect}\) be an exact, faithful, and monoidal functor such that

1. there is an isomorphism \(\gamma: k \to \omega(1)\);
2. there is isomorphisms \(\tau_{x,y}: \omega(x) \otimes \omega(y) \to \omega(x \otimes y)\);
3. via \(\gamma\) and \(\tau_{x,y}\), the usual properties of \(\otimes\) in \(\text{Vect}\) holds;
(4) for any $x \in C$ such that $\dim \omega(x) = 1$, there is $x^\vee \in C$ such that $x \otimes x^\vee \simeq 1$.

Then the conclusion in the Theorem \ref{thm:main} holds.

**Example 10.4.**

(1) The category $\text{Vect}^k$ satisfies the assumptions, and $G = \mathbb{G}_m$.

(2) Let $\Gamma$ be an abstract group, then the theorem implies that there is an affine group scheme $G_\Gamma$ together with a homomorphism $\Gamma \to G_\Gamma$ such that it induces $\text{Rep} G_\Gamma \simeq \text{Rep} \Gamma$. Note the morphism $\Gamma \to G_\Gamma$ should be interpreted as first forming the $\text{Aut}(\omega_\Gamma)$ for $\Gamma$ similar to group schemes, and treat this as a functor associated with $\Gamma$, and then take the morphism from this functor to $G_\Gamma$.

(3) Let $X$ be a topological space, $C$ be the category of finite dimensional local systems on $X$. For all $x \in X$, the functor $\mathcal{L} \mapsto \mathcal{L}|_x$ is a fiber functor, hence $C \simeq \text{Rep} \pi_1(X, x)$ for any $x \in X$. This is a special case of the previous item.

**Non-Example 10.5.** Let $C$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, then $C \ncong \text{Rep} G$ for any $G$.

10.1. **Sketch of the proof of Theorem \ref{thm:main}**

The rest of the lecture will be a sketch of the proof Theorem \ref{thm:main}, which will extend to the next lecture.

**Lemma 10.6.** Under the assumption of the main theorem, we have

(1) $\dim x = \dim \omega(x)$;

(2) any object of $C$ has finite length;

(3) $\text{Hom}(x, y)$ is finite dimensional for all $x, y \in C$.

**Proof.**

(1) By definition, $\dim x = \text{Tr}(\text{Id}_x)$, and the right-hand side is calculated by a chain of maps in $C$. After taking $\omega$, we get the exactly same diagram for calculating $\dim \omega(x)$ in $\text{Vect}$. Since $\omega$ is $k$-linear, we get the result. A corollary of this is that if $\dim x = 0$, then $\dim \omega(x) = 0$ hence $\omega(x) = 0$. By faithfulness, we have $x = 0$ (faithfulness is a characterization of the morphisms, but in an abelian category, it’s easy to get the result for objects).

(2) Since the length of any object is not larger than its dimension (by that in any strictly decreasing chain of objects, the dimension must be strictly decreasing, by the simplicity of 1), we get the result.

(3) This is the previous claim and Proposition \ref{prop:simple}.

□

**Corollary 10.7.** Given a family of subobjects of $x$, it makes sense to talk about their intersections in $x$.

**Proof.** By finite-length-ness of $x$, the intersection of all those subobjects stabilizes after finite steps, hence the claim. □

Let $A$ be a finite dimensional $k$-algebra, and by $A\text{-Mod}$ we mean finitely generated $A$-modules. We state a motivational lemma.

**Lemma 10.8.** Let $X \in A\text{-Mods}$, $\alpha : X \to X$ a $k$-linear map such that for all $n \geq 0$, and all $A$-submodule $Y \subset X^\oplus n$, $\alpha^\oplus n(Y) \subset Y$. Then there exists $a \in A$ such that $\alpha(x) = ax$ for all $x \in X$.

**Proof.** Let $x_1, \ldots, x_n$ be a $k$-basis of $X$. Let $Y = A(x_1 \oplus \cdots \oplus x_n) \subset X^\oplus n$. Then $\alpha^\oplus n(x_1 \oplus \cdots \oplus x_n) = a(x_1 \oplus \cdots x_n)$ for some $a \in A$. Thus $\alpha(x_i) = ax_i$ for all $i$, hence the result. □
Let $\mathcal{C}$ be an abelian $k$-linear category, and let $X \in \mathcal{C}$. Let $\langle X \rangle$ be the full subcategory of all subquotients of $X^{\oplus n}$ for all $n$. Let $\omega : \mathcal{C} \to \text{Vect}$ be an exact, faithful functor. Let
\[
A_X := \{ \alpha \in \text{End}_k(\omega(X)) \mid \text{for all } n \geq 0, Y \subset X^{\oplus n}, \alpha^{\oplus n}(\omega(Y)) \subset \omega(Y) \}. \]
Note $A_X$ is a finite dimensional $k$-algebra.

**Proposition 10.9.** There exists an equivalence $\varpi_X : \langle X \rangle \to A_X \text{-}\text{Mods}$ such that the diagram commutes:
\[
\begin{array}{ccc}
\langle X \rangle & \xrightarrow{\varpi_X} & A_X \text{-}\text{Mods} \\
\omega|_{\langle X \rangle} & \downarrow & \\
\text{Vect} & \end{array} \]

**Proof.** Let $a \in A_X$, $Y' \subset Y \subset X^{\oplus n}$, then $a(\omega(Y)) \subset \omega(Y)$ and $a(\omega(Y')) \subset \omega(Y')$. So $a$ acts on $\omega(Y)/\omega(Y') \cong \omega(Y/Y')$. So we see for all $Z \in \langle X \rangle$, $a$ acts on $Z$, thus makes $\omega(Z)$ an $A_X$-module. In this way we define the functor $\varpi_X : \langle X \rangle \to A_X \text{-}\text{Mods}$. We want to show $\varpi_X$ is an equivalence.

Let $W \subset V \in \text{Vect}$, and $Z \subset Y \in \mathcal{C}$, define objects in $\mathcal{C}$:
\[
(Z : W) := \text{Ker}(\text{Hom}(V,Y) \to \text{Hom}(W,Y/Z)),
\]
\[
P_X := \bigcap_{Y \subset X^{\oplus n}} (\text{Hom}(\omega(X), H) \cap (Y : \omega(Y))),
\]
where in the second definition, the inside intersection happens in $\text{Hom}(\omega(X^{\oplus n}), X^{\oplus n})$ with $X$ diagonally embedded into $X^{\oplus n}$, and the outer intersection is done by taking the system of embeddings $X^{\oplus n} \hookrightarrow X^{\oplus(n+1)}$ into the first $n$ summands (in fact the choice of $n$ summands of the $n + 1$ doesn’t matter since the inner intersection is symmetric with respect to those direct summands).

In general, we have canonical maps $\text{Hom}_k(W,V) \to \text{Hom}(\text{Hom}(V,X), \text{Hom}(W,X))$. When $W = V = \omega(X)$, we have the diagram
\[
\begin{array}{ccc}
\text{End}_k(\omega(X)) & \xrightarrow{\text{can}} & \text{End}_\mathcal{C}(\text{Hom}(\omega(X), X)) \\
\uparrow & & \uparrow \\
A_X & \xrightarrow{\text{can}} & \text{End}_\mathcal{C} P_X \\
\end{array}
\]
Therefore we obtain an action of $A_X$ on $\text{End}_\mathcal{C} P_X$, so we can define the following functor
\[
A_X \text{-}\text{Mod} \to \langle X \rangle \\
M \mapsto P_X \otimes_{A_X} M,
\]
where the definition of the “tensor over $A_X$” is described below.

From the morphism $A_X \to \text{End}_\mathcal{C} P_X$ we get a morphism $A_X \otimes P_X \to P_X$. On the other hand, for any $A_X$-module $M$, we have the action map $A_X \otimes M \to M$. We then have two maps by tensoring the respective identity maps:
\[
\alpha_1 : P_X \otimes (A_X \otimes M) \to P_X \otimes M, \\
\alpha_2 : (P_X \otimes A_X) \otimes M \to P_X \otimes M.
\]
Then we define $P_X \otimes_{A_X} M$ to be the coequalizer of $\alpha_1$ and $\alpha_2$. 

The claim is that $P_X \otimes_{A_X} -$ is the quasi-inverse to the functor $\mathfrak{m}_X$, which we do not go into details.

If we have $X \subset Y \in \mathcal{C}$, we then have $\langle X \rangle \subset \langle Y \rangle$ and an induced map $A_Y \to A_X$. One wishes that by taking $A := \lim_{\leftarrow X} A_X$, one can show $\mathcal{C} \simeq A$-Mods. However, it doesn’t work, since the limit and the “-modules” don’t commute. Next time, we will show the “dual” of this idea will work, i.e. by taking the categories of comodules of some Hopf algebras. This is because over Hopf algebras, any module is the limit of its finite subcomodules. And after taking the spectrum of the Hopf algebra, one gets an affine group scheme, hence proving Theorem 10.2.