Point-Set Topology II

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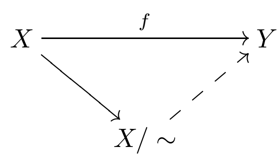
1 More on Quotients

Universal Property of Quotients. Let X be a topological space with equivalence relation $\sim$. Suppose that $f\colon X \to Y$ is continuous and $f(x) = f(y)$ whenever $x \sim y$. Then there exists a unique continuous map $X/\sim \to Y$ such that the diagram

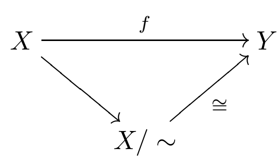


commutes.

Proposition 1. Let $f\colon C \to H$ be a continuous bijection, C compact, H Hausdorff. Then f is a homeomorphism.

Proof. Closed subsets of C are compact. Thus, their images in H are compact, hence closed. □

Corollary 2. Let X be compact, Y Hausdorff, and suppose that $f\colon X \to Y$ is continuous and surjective. Then Y is homeomorphic to the quotient $X/\sim$



where $x \sim y$ iff $f(x) = f(y)$.

Definition 3. A *closed (topological) n-manifold* is a compact Hausdorff space such that every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Proposition 4. Let M be a closed n-manifold, and $U \subset M$ an open set homeomorphic to $\mathbb{R}^n$. Let $C = M \smallsetminus U$, and let $\overline{M}$ be the quotient space obtained by identifying C to a point. Then $\overline{M}$ is homeomorphic to the n-sphere S^n.

Proof. Let $p \in S^n$ be a point. Identify $S^n \setminus \{p\}$ with $\mathbb{R}^n \cong U$ by stereographic projection. Sending $M \setminus U$ to p gives a surjective map g from M to S^n. By Corollary 2, it suffices to show that this map is continuous.

Let $V \subset S^n$ be open. If V does not contain p, then $g^{-1}(V)$ is open since $g|U$ is a homeomorphism. If V does contain p, then V^c is a compact set not containing p. Since $g|U$ is a homeomorphism, $g^{-1}(V^c)$ is compact, hence closed in M. Thus,

$$g^{-1}(V) = g^{-1}(V)^{cc} = g^{-1}(V^c)^c$$

is open in M. □

Theorem 5. Every closed manifold can be embedded as a subspace of $\mathbb{R}^N$ for $N \gg 0$.

Proof. Since M is compact, we can cover it by finitely many open sets $U_1, \ldots, U_k$ each homeomorphic to $\mathbb{R}^n$. Let $f_i \colon M \to S^n$ be the map given by collapsing U_i^c to a point. Then

$$f_1 \times \cdots \times f_k \colon M \to (S^n)^k \hookrightarrow \mathbb{R}^{(n+1)k}$$

is injective and continuous. Since M is compact and $\mathbb{R}^{(n+1)k}$ is Hausdorff, it is a homeomorphism onto its image. □

2 Connectedness

Notation. For this section, X will denote a topological space, and I the closed interval $[0, 1] \subset \mathbb{R}$.

Definition 6. A *path* in X is a continuous map $\gamma \colon I \to X$. If $x = \gamma(0)$ and $y = \gamma(1)$, we say that γ is a path from x to y.

Definition 7. X is *path-connected* if for every $x, y \in X$, there exists a path in X from x to y.

Definition 8. Let γ, δ be paths in X such that $\gamma(1) = \delta(0)$. The *inverse* $\overline{\gamma}$ of γ is the path

$$\overline{\gamma}(t) = \gamma(1 - t).$$

The *composite* $\gamma \cdot \delta$ is the path

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \delta(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Definition 9. We say that $x \sim_p y$ if there is a path in X from x to y. Using inverse and composite paths, it is easy to see that $\sim_p$ is an equivalence relation. The equivalence classes are called *path components*; they are the maximal path-connected subsets of X.

Example 10. For $n \geq 2$, $\mathbb{R}^n \smallsetminus \{0\}$ is path-connected. As we will see later, $\mathbb{R} \smallsetminus \{0\}$ has two path components.

Proposition 11. The continuous image of a path-connected space is path-connected.

Proof. Assume X is path-connected and $f\colon X \to Y$ is continuous. Let $x, y \in f(X)$. Then there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Let γ be a path from x' to y'. Then $f \circ \gamma$ is a path in $f(X)$ from x to y. □

It follows that, for $n \geq 1$, the n-sphere is path-connected, via the map

$$\begin{aligned} \mathbb{R}^{n+1} \smallsetminus \{0\} &\to S^n \\ x &\mapsto \frac{x}{\|x\|}. \end{aligned}$$

Definition 12. A *separation* of X is an expression of X as the disjoint union of two nonempty open subsets. X is *connected* if it has no separation.

Thus, $\mathbb{R} \smallsetminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is not connected.

Theorem 13. X is connected iff every continuous map $X \to \mathbb{R}$ taking both positive and negative values has a zero.

Proof. We show that X has a separation iff there exists a continuous map $f\colon X \to (-\infty, 0) \cup (0, \infty)$ taking both positive and negative values.
($\Longrightarrow$) If $X = U \cup V$ is a separation, define f to be 1 on U, -1 on V.
($\Longleftarrow$) $X = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ is a separation. □

Corollary 14. Every path-connected space is connected.

Proof. Let X be path-connected, and suppose $f\colon X \to \mathbb{R}$ is continuous and $f(x) < 0$, $f(y) > 0$. Let γ be a path from x to y; then $f \circ \gamma\colon I \to \mathbb{R}$ takes both positive and negative values. By the Intermediate Value Theorem, $f \circ \gamma$ has a zero, and so f has a zero. □

It follows that S^n is connected and $\mathbb{R} \smallsetminus \{0\}$ is not path-connected.

Example 15. Let $G \subset \mathbb{R}^2$ be the graph of the function $(0, \infty) \to \mathbb{R}$ given by $x \mapsto \sin(1/x)$. The *topologist's sine curve* is the closure of G in $\mathbb{R}^2$. It is connected but not path-connected.

Definition 16. If $x, y \in X$, we say that x is connected to y if there exists a connected subset $Y \subset X$ containing both x and y. (For instance, Y might be the image of a path.) This is an equivalence relation on X. Equivalence classes, called *components*, are the maximal connected subsets.

Example 17. The rationals $\mathbb{Q}$ and the Cantor set are topological spaces in which the components are all points, but the topology is *not* discrete.

3 The Product Topology

Definition 18. Given a family $\{X_\lambda \colon \lambda \in \Lambda\}$ of topological spaces, the *product topology* on $\prod_{\lambda\in\Lambda} X_\lambda$ is the coarsest topology such that the projection maps

$$\pi_\mu \colon \prod_\lambda X_\lambda \to X_\mu$$

are all continuous.

Proposition 19. A sequence in the product converges iff it converges componentwise. I.e., if x_n is a sequence of points in the product, then $x_n \to x$ iff for all λ, $\pi_\lambda(x_n) \to \pi_\lambda(x)$.

Remark 20. If $U \subset \prod_\lambda X_\lambda$ is a nonempty open subset, then $\pi_\lambda \colon U \to X_\lambda$ is surjective for all but finitely many λ. In particular, something like $(0,1)^\infty$ is *not* open as a subset of $[0,1]^\infty$.

Theorem 21. (Tychonoff) A product of compact spaces is compact.

Remark 22. The product topology on $\mathbb{R}^n$ is the same as the usual topology.

4 Spaces of Maps

In this section, we explore topologies on the set of maps from X to Y, denoted $\mathrm{Maps}(X,Y)$, and on the set of continuous maps from X to Y, denoted $\mathrm{Cont}(X,Y)$.

Definition 23. The *topology of pointwise convergence* is the coarsest topology on $\mathrm{Maps}(X,Y)$ such that the evaluation map

$$\begin{aligned} \mathrm{ev}_x \colon \mathrm{Maps}(X,Y) &\to Y \\ f &\mapsto f(x) \end{aligned}$$

is continuous for all $x \in X$.

The topology of pointwise convergence is equivalent to the product topology on

$$\mathrm{Maps}(X,Y) = \prod_{x\in X} Y.$$

A sequence of functions f_n converges to f in this topology iff it converges pointwise.

Note that ev_x is continuous iff for every open subset $U \subset Y$, $\{f \mid f(x) \in U\} = \mathrm{ev}_x^{-1}(U)$ is open. Thus, the topology of pointwise convergence is the coarsest topology on $\mathrm{Maps}(X,Y)$ such that for every $x \in X$ and every open $U \subset Y$, the set

$$\{f \colon f(x) \in U\}$$

is open. In this formulation, it is clear that the topology of pointwise convergence makes absolutely no reference to the topology on X.

One standard way to generalize statements to include more topology is to replace finite (or in this case, singleton) sets by compact sets.

Definition 24. The *compact-open topology* on $\mathrm{Cont}(X, Y)$ is the coarsest topology such that for every compact subset $C \subset X$ and every open $U \subset Y$,

$$\{f \colon f(C) \subset U\} \text{ is open.}$$

The compact-open topology is "the" standard topology to put on $\mathrm{Cont}(X, Y)$.

Remark 25. Assume Y is a metric space. Then $f_n \to f$ in the compact-open topology iff $f_n|_C \to f|_C$ uniformly on every compact subset $C \subset X$. This form of convergence, known as "uniform convergence on compact subsets," has very nice properties. For instance, if X is locally compact Hausdorff, f_n are continuous, and $f_n \to f$ uniformly on compact subsets, then f is necessarily continuous (the same does not hold for the topology of pointwise convergence).

Likewise, if $U \subset \mathbb{C}$ is open, $f_n \colon U \to \mathbb{C}$ are holomorphic, and $f_n \to f$ uniformly on compact subsets, then f is necessarily holomorphic.

The following remark gives a hint as to how the compact-open topology can be useful in algebraic topology.

Remark 26. If X is locally compact Hausdorff and $f, g \colon X \to Y$ are continuous, then a homotopy from f to g is precisely a path in $\mathrm{Cont}(X, Y)$ from f to g.