Point-Set Topology 2

Tam

1 Connectedness and path-connectedness

Definition 1. Let X be a topological space. A *separation* of X is a pair of disjoint nonempty open sets U and V in X whose union is X. The space X is *connected* if there does not exist a separation of X.

Connected subsets of the real line are either one-point sets or intervals. Connected sets in $\mathbb{R}^n$, for $n \geq 2$, are not so nice. For example, the following set in $\mathbb{R}^2$ is connected.

$$S = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, y = \sin\frac{1}{x}, x > 0 \right\} \cup \left\{ (x,y) \in \mathbb{R}^2 \,|\, x = 0, y \in [-1,1] \right\}$$

Lemma 2. *The image of an connected set under a continuous map is connected.*

Definition 3. A metric space X is *path-connected* if for every $p, q \in X$, there exists a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. The map γ is called a *path* connecting p and q.

Theorem 4. *If A is path-connected, then A is connected. The converse is not true.*

The following theorem is an example how we can use connectedness to prove that two spaces are not homeomorphic.

Theorem 5. *None of the following space is homeomorphic to any of the others:* $(0,1)$*,* $[0,1)$*,* $[0,1]$ *and* $\mathbb{R}^2$*.*

The proof of the next theorem is an example of a common arguement using connectedness in topology, geometry and analysis. The general principle is that if you want to prove that a property P is true (or not true) on a connected space X, then try considering the subset of X where P is true and the subset where P is true. Then try to show both subsets are open. Since X is connected, one of them has to be empty.

Theorem 6. *An open set A in $\mathbb{R}^n$ is connected if and only if it is path-connected.*

Proof. Since path-connectedness implies connectedness we need to only show that A is path-connected if it is connected. Suppose A is nonempty and connected. Pick $p \in A$. Let U be the set of points in A that can be connected to p by a path in A. Let $V = A \setminus U$, so V is the set of points in A that cannot be connected to p by path in A. So $A = U \cup V$. We claim that U and V are open. To show that U is open, let $q \in U$ and γ be a path connecting p and q. Since A is open, there exists a open ball $B_r(q) \subset A$. For each $r \in B_r(q)$, there is a path λ in A connecting q and r. We can take λ to be the straight line connecting q and r. Thus, $B_r(q) \subset U$ and U is open. Similarly one can prove that V is open. Thus, either U or V is the empty set by the connectedness of A. □

Definition 7. If $x \in X$, then the *connected (path-connected) component* of X containing x is the union of all connected (path-connected) subsets of X that contain x.

Lemma 8. *Connected components are connected.*

Connected components are not generally open or closed. For examples, the connected component of 0 in $\mathbb{Q}$ is the set $\{0\}$, which is neither open or closed.

A metric space X is *locally connected at* $x \in X$ if for each open set U containing x there exists an connected open set V containing x that is contained in U. If X is locally connected at every point in X, then we say that X is *locally connected.*

Theorem 9. *A metric space X is locally connected if and only if for each open set U in X, each component of U is open in X.*

2 Metric spaces

Definition 10. A *metric space* is a set X with a function $d\colon X \times X \longrightarrow \mathbb{R}$ that satisfies the following properties

1. $d(x, y) = 0$ if and only if $x = y$ (Definiteness)

2. For all $x, y \in X$, $d(x, y) = d(y, x)$ (Symmetry)

3. For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle inequality)

It is not hard to see that the above three properties imply that the metric d is always non-negative. A subset of a metric space has an induced metric which makes the subset into a metric space.

Example 11. The following spaces are metric spaces.

1. $\mathbb{R}^n$ with the Euclidean distance.

2. Normed vector spaces and inner product vector spaces.

3. The space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the metric $d(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\}$.

Definition 12. Given $x \in X$ and $r > 0$, we define $B_r(x) = \{y \in X \mid d(x,y) < \epsilon\}$ called the *r–ball centered at x*. The collection of all ball $B_r(x)$ for $x \in X$ and $r > 0$ is a basis for the *metric topology* on X induced by d.

A sequence $x_n \in X$ converges to $x \in X$ if for each $\epsilon > 0$ there exists $N > 0$ such that $d(x_n, x) < \epsilon$ whenever $n > N$. We write $x_n \to x$.

For metric spaces (X,d) and (X',d'), a map $f\colon X \longrightarrow X'$ is continuous iff for each $\epsilon > 0$, there exists $\delta > 0$ such that $d'(f(x), f(y)) < \epsilon$ whenever $d(x,y) < \delta$.

Theorem 13. *Let $f\colon X \longrightarrow X'$. Then f is continuous if and only if for every sequence $x_n \to x$ in X, $f(x_n) \to f(x)$.*

Definition 14. Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if for each $\epsilon > 0$ there exists $N > 0$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n > N$.

It is clear that a convergent sequence is Cauchy. The converse is not true. A metric space X is *complete* if every Cauchy sequence in X converges.

Definition 15. Let (X,d) and (X',d') be metric spaces.

1. A bijective map $f\colon X \longrightarrow X'$ is an *isometry* if $d'(f(x), f(y)) = d(x,y)$, i.e. f is distance preserving. We then say that X and X' are *isometric*.

2. A map $g\colon X \longrightarrow X$ is called a *contraction map* if there exists $0 < \alpha < 1$ such that for every $x, y \in X$, we have $d(g(x), g(y)) \le \alpha \cdot d(x,y)$.

Clearly isometries and contraction maps are continuous.

Theorem 16 (The Banach fixed point principle)**.** *Let (X,d) be a complete metric space, and let $g\colon X \longrightarrow X$ be a contraction map. Then there exists uniquely a point $x \in X$ that is fixed by g, i.e. $g(x) = x$.*

3 Function spaces

Let X and Y be topological spaces. We denote Y^X be the space of all functions from X to Y, $C(X,Y)$ the space of all continuous functions from X to Y. If Y is a metric space, we denote $B(X,Y)$ the space of bounded functions from X to Y.

3.1 Pointwise convergence topology

For $x \in X$ and U open in Y, let $S(x,U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$. The sets $S(x,U)$ form a basis for the *topology of pointwise convergence* of Y^X. A sequence f_n of functions from X to Y converges to a function f if for all $x \in X$, $f_n(x)$ converges to $f(x)$ in Y.

3.2 Uniform convergence topology

Now, for this subsection, suppose that Y is a metric space with a metric d. We define the *uniform metric* ρ on $B(X,Y)$ as

$$\rho(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}$$

for $f,g \in B(X,Y)$. The topology on $B(X,Y)$ induced by this metric is called the uniform convergence topology. A sequence f_n of functions from X to Y converges to a function f if for each $\epsilon > 0$, there exists $N > 0$ such that whenever $n > 0$, we have $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$.

The space $C(X,Y)$ is closed in $B(X,Y)$ with repest to the uniform convergence topology, but not in the pointwise convergence topology.

3.3 The compact-open topology

Definition 17. Let X and Y be topological spaces. The *compact-open topology* on $C(X,Y)$ is the topology generated by sets of the form $S(K,U)$, defined as follows: fix $K \subset X$ compact and $U \subset Y$ open, and let

$$S(K,U) = \{f \in C(X,Y) \mid f(K) \subset U\}.$$

If Y is a metric space, then convergence in the compact-open topology is the same as uniform convergence on compact sets, i.e. sequence f_n of functions from X to Y converges to a function f if for all compact $K \in X$, $f_n(x)$ converges uniformly to $f(x)$ for all $x \in K$.

For a "nice enough" metric space X, the set of isometries of X with the compact-open topology is a Lie group (you'll learn about these in Manifolds II).

Example 18. Consider $\mathrm{Isom}(\mathbb{R})$. Let Z be the set of elements of $\mathrm{Isom}(\mathbb{R})$ that are translations by an integral distance. Then Z is a discrete set in $\mathrm{Isom}(\mathbb{R})$ with respect to the compact-open topology.

In topology/geometry, if you take the quotient of "nice spaces" (i.e. manifolds) by a discrete group of Isom of the spaces, you will generally get a "nice spaces" (where "nice" here is not a mathematical term).

Example 19. Consider $\mathrm{Isom}(S^1)$. If θ is an irrational angle, then the set $\{\text{Rotation by } k\theta \mid k \in \mathbb{Z}\}$ is a non-discrete subset of $\mathrm{Isom}(S^1)$ with respect to the compact-open topology.