WOMP 2008: MEASURE THEORY AND FUNCTION SPACES

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1. Review of Integration

Recall the definition of Darboux integral:

Definition 1. *Let $f : [a,b] \mapsto \mathbf{R}$ be a bounded function. Let $P = (x_0, x_1, \ldots, x_n)$, with $a = x_0 < x_1 < \cdots < x_n = b$, be a partition of $[a,b]$.*

The upper and lower Darboux sums of f with respect to P *are*

$$U_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x),$$

$$L_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x).$$

The upper and lower Darboux integrals of f *are*

$$U_f = \inf\{U_{f,P}\}, \quad L_f = \sup\{L_{f,P}\}.$$

If $U_f = L_f$, then we say that f is Darboux-integrable *and*

$$\int_a^b f(t)\,dt = U_f = L_f.$$

The Darboux integral is equivalent to the Riemann integral.

Let

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then for any partition P of $[0,1]$, $U_{g,P} = 1$ and $L_{g,P} = 0$, and so the Darboux integral of g does not exist.

The rationals $\mathbf{Q}$ are countable. So let $\mathbf{Q} = \{q_1, q_2, \ldots\}$. Fix some $\epsilon > 0$, and let

$$g_k(x) = \begin{cases} 1, & \text{if } |x - q_k| < \epsilon/2^{k+1}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_0^1 g_k(x)\,dx = \epsilon/2^k$, and $0 \le g(x) \le \sum_{k=1}^{\infty} g_k(x)$. So we really would like to say that

$$0 \le \int_0^1 g(x)\,dx \le \sum_{k=1}^{\infty} \int_0^1 g_k(x)\,dx = \epsilon.$$

That is, $\int_0^1 g(x)\,dx$ really should be zero. We would like a definition of integral such that this is true.

2. The Lebesgue integral

If $I = (a,b), [a,b], (a,b]$, or $[a,b)$ is an interval, then its length is $b-a$. We define its *Lebesgue measure* $\lambda(I)$ as $b-a$.

If E is a union of countably many disjoint intervals, then we can define its Lebesgue measure $\lambda(E)$ to be the sum of the lengths of all those intervals. (This number may or may not be finite. Note that, in $\mathbf{R}$, all open sets qualify.)

Definition 2. *Let $f : [a,b] \mapsto \mathbf{R}$ be a nonnegative continuous function. Let $\tilde{P} = (y_0, y_1, y_2, \ldots y_n)$ with $0 = y_0 < y_1 < \ldots < y_n$. Let*

$$S_{f,\tilde{P}} = \sum_{i=1}^{n} (y_i - y_{i-1}) \lambda\{x : y_i < f(x)\}.$$

We define the Lebesgue integral of f *as*

$$\int_{[a,b]} f \, d\lambda = \sup_{\tilde{P}} S_{f,\tilde{P}}.$$

Now, if we could generalize the notion of measure, we could generalize the notion of Lebesgue integral.

Definition 3. *Let $F \subset \mathbf{R}$ be a set. Define $\lambda^*(F) = \inf\{\lambda(E) : E \supset F, E \text{ open}\}$. We call $\lambda^*(F)$ the* outer measure of F.

If for all $S \subset \mathbf{R}$ we have that $\lambda^*(S) = \lambda^*(S \cap F) + \lambda^*(S \cap F^C)$, we say that F is a measurable set *and define $\lambda(F) = \lambda^*(F)$.*

If $f : \mathbf{R} \mapsto \mathbf{R}$ is such that $\{x : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbf{R}$, then we say that f is a measurable function.

So $\lambda(\mathbf{Q}) = 0$, and thus the Lebesgue integral $\int_0^1 g(x) \, d\lambda = 0$. So there are some functions which are Lebesgue-integrable but not Darboux-integrable.

Theorem 4. *If the Darboux integral of f exists, then f is measurable.*

Theorem 5. *If f is a nonnegative function on $[a,b]$ and the Darboux integral of f exists, then it is equal to its Lebesgue integral.*

3. More general measures

We can generalize the notion of measure from $\mathbf{R}$ to other spaces, which of course lets us generalize the notion of integral as well:

Definition 6. *Let X be a space. Let $\mathfrak{M}$ be a set of subsets of X, such that $X \in \mathfrak{M}$, if $S \in \mathfrak{M}$ then so is S^C, and if $S_i \in \mathfrak{M}$ for $1 \le i < \infty$, then so is $\cup_{i=1}^{\infty} S_i$. We call $\mathfrak{M}$ a* σ-algebra *for X.*

If $\mu : \mathfrak{M} \mapsto [0, \infty]$ is a function such that

$$\mu\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mu(S_i)$$

whenever $\{S_i\} \subset \mathfrak{M}$ and the S_i are pairwise-disjoint, then we call μ a positive measure.

We say that $f : X \mapsto [0,\infty]$ is measurable *if $\{x : f(x) > \alpha\} \in \mathfrak{M}$ for all $\alpha \in \mathbf{R}$. We define*

$$\int_X f \, d\mu = \sup_{0 < y_1 < \ldots < y_n} \sum_{i=1}^{n} (y_i - y_{i-1}) \mu\{x : y_i < f(x)\}.$$

For example, in $\mathbf{R}^n$, we can define an analogy to outer measure λ^* with "countable unions of rectangular boxes" replacing "open sets", and define the Lebesgue measure λ from λ^* as usual.

As another example, let $X = \mathbf{N}$, the natural numbers, and let $\mu(S) = |S|$, the number of elements of S. Then $\int_S f\, d\mu = \sum_{n\in S} f(n)$.

Finally, if we are studying physics, we can let X be an inhomogeneous object, and let $\mu(S)$ be the mass of the set $S \subset X$. This measure is useful for calculating the center of mass and the moment of inertia.

4. L^p SPACES

Definition 7. *Let f be a measurable function, and let $1 \le p < \infty$. We say that the $L^p(X,\mu)$* norm *of f is*

$$||f||_{L^p} = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

We say that the $L^\infty(X,\mu)$ norm *of f is its upper bound except on sets of measure zero; that is,*

$$||f||_{L^\infty} = \operatorname*{ess\,sup}_{x\in X} |f(x)| = \inf\big\{\alpha > 0 : |f(x)| \le \alpha \text{ a.e.}\big\},$$

where a.e. stands for "almost everywhere", that is, "except on a set of measure zero".

Theorem 8 (Minkowski's Inequality). *If $||f||_{L^p} < \infty$ and $||g||_{L^p} < \infty$ for $1 \le p \le \infty$, then*

$$||f+g||_{L^p} \le ||f||_{L^p} + ||g||_{L^p}.$$

It is clear that $||\alpha f||_{L^p} = |\alpha|\, ||f||_{L^p}$ for any scalar α. Thus, the set of all functions whose L^p norm is finite is *almost* a normed vector space. There are some nonzero functions with norm zero. (For example, let $f(x) = 0$ for $x \neq 3$, and let $f(3) = 1$. Then $||f||_{L^p} = 0$ for any $1 \le p \le \infty$.)

To deal with this, we quotient out by functions which are zero a.e. We let L^p be the vector space whose elements are *equivalence classes of functions* with the equivalence relation $f \equiv g$ if $f(x) = g(x)$ almost everywhere, that is, $\mu\{x : f(x) \neq g(x)\} = 0$.

Usually, you can just think of L^p being a space of functions.

There is another very useful theorem (which you need to prove, among other things, Minkowski's inequality):

Theorem 9 (Hölder's Inequality). *If $1 \le p, q \le \infty$ with $1/p + 1/q = 1$, we say that p and q are* conjugates *or* conjugate exponents. *If $||f||_{L^p}$ and $||g||_{L^q}$ are finite, then*

$$\left|\int_X f(x)g(x)\, d\mu(x)\right| \le ||f||_{L^p}||g||_{L^q}.$$

Note that 1 and ∞ are conjugate exponents, and that 2 is its own conjugate.

Here are some examples:

(1) Suppose that $\mu(X) = 1$, and $1 \le p \le r \le \infty$. Then we can show (by Hölder's inequality) that $||f||_{L^p} \le ||f||_{L^r}$ and so $L^r \subset L^p$. More generally,

if $\mu(X)$ is finite, then

$$\left(\frac{1}{\mu(X)}\int_X |f|^p\,d\mu\right)^{1/p} \leq \left(\frac{1}{\mu(X)}\int_X |f|^r\,d\mu\right)^{1/r} \text{ or } \left(\frac{1}{\mu(X)}\int_X |f|^p\,d\mu\right)^{1/p} \leq ||f||_{L^\infty}$$

and so $L^r \subset L^p$ again.

(2) Suppose that $X = \mathbf{N}$, the natural numbers, and that μ is the counting measure, that is, $\mu(S) = |S|$ is the cardinality of the set S.

We usually refer to $L^p(\mathbf{N},\mu)$ as ℓ^p. Note that

$$||f||_{\ell^p}^p = \sum_{n=0}^\infty |f(n)|^p, \quad ||f||_{\ell^\infty} = \sup_{n\in\mathbf{N}} |f(n)|.$$

So $|f(n)| \leq ||f||_{\ell^p}$, and so if $1 \leq p \leq r < \infty$ and $f \in \ell^p$, then

$$||f||_{\ell^r}^r = \sum |f(n)|^r \leq \sum |f(n)|^p ||f||_{\ell^p}^{r-p} \leq ||f||_{\ell^p}^r.$$

Thus, in this case, we have that $L^p \subset L^r$ for $1 \leq p \leq r \leq \infty$, the opposite of the previous case.

(3) Now let $X = (\mathbf{R}, d\lambda)$, where $d\lambda$ is the Lebesgue measure defined above. We do not have *any* nice inclusion relations in this case. If $1 \leq p < r < \infty$, then

$$f(x) = \begin{cases} |x|^{-1/r} & \text{on } [-1,1] \\ 0 & \text{elsewhere} \end{cases}$$

is in L^p but not L^r, and

$$f(x) = \begin{cases} 0 & \text{on } [-1,1] \\ |x|^{-1/p} & \text{elsewhere} \end{cases}$$

is in L^r but not L^p.

It turns out that, if $1 \leq p < \infty$, and if F is a map from L^p to $\mathbf{C}$ such that

- F is linear; that is, $F(f+g) = F(f) + F(g)$ and $F(\alpha f) = \alpha F(f)$ for any scalar α and $f, g \in L^p$
- F is bounded; that is, there exists a constant $C > 0$ such that $|F(f)| \leq C||f||_{L^p}$ for all $f \in L^p$

then there is some $g \in L^q$ such that

$$F(f) = \int_X f(x)g(x)\,d\mu(x),$$

where q is the conjugate to p. We summarize this property by saying that L^q is the *dual* to L^p.

Theorem 10. *Under these definitions, L^p is a complete normed vector space for $1 \leq p \leq \infty$.*

5. Convergence of sequences of functions

Definition 11. *Let $\{f_n\}$ be a sequence of functions defined on some measure space (X,μ), and let f be another function on X. If $f_n, f \in L^p(X,\mu)$ for some p and $\lim_{n\to\infty} ||f_n - f||_{L^p} = 0$, then we say that f_n* converges to f in $L^p(X,\mu)$, *or $f_n \to f$* in L^p.

Note that if $f_n \to f$ in L^∞, then f_n converges to f uniformly almost everywhere. Often we care only about L^1 convergence and uniform convergence.

We can contrast this with two other definitions:

(1) If $\lim_{x\to\infty} f_n(x) = f(x)$ for all $x \in X$, then we say that f_n *converges to f, or $f_n \to f$, pointwise.*
(2) If $\lim_{x\to\infty} f_n(x) = f(x)$ for almost all $x \in X$, then we say that f_n *converges to f pointwise almost everywhere (a.e.)*

Note that if $f_n \to f$ in $L^1(X, d\mu)$ then $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

It is clear that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise and pointwise a.e. (If $f_n \to f$ in L^∞, then $f_n \to f$ pointwise a.e.) Some functions converge pointwise but not uniformly; for example, $f_n(x) = x^n$ converges to $f(x) = 0$ pointwise but not uniformly on $(0, 1)$.

However, we do have one positive result:

Theorem 12 (Egorov's Theorem)**.** *Suppose that $f_n \to f$ pointwise a.e. on some measure space X with $\mu(X) < \infty$. Then for any $\epsilon > 0$, there is a measurable set E such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash E$.*

We do need for $\mu(X) < \infty$. As an example, let $f(x) = 0$, $f_n(x) = 0$ on $[-n, n]$ and $f_n(x) = 1$ elsewhere. Then $f_n \to f$ pointwise, but Egorov's Theorem obviously cannot hold.

We would like to know how uniform and pointwise a.e. convergence relate to L^1 convergence. Let $f(x) = 0$ everywhere. Here are some examples:

- Let $f_n(x) = 1/n$ on $[0, n]$ and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ uniformly but not in $L^1(\mathbf{R}, d\lambda)$.
- Let $f_n(x) = n$ on $[0, 1/n]$, and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ pointwise but not in $L^1([0, 1], d\lambda)$.
- Let $g_n(x) = 1$ on
$$\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \frac{1}{n+1}\right]$$
and be zero elsewhere. Let $f_n(x) = 1$ if $0 \le x \le 1$ and $g_n(x + k) = 1$ for some integer k, and let $f_n(x) = 0$ otherwise.
Then $\int_0^1 |f_n(x)| \, d\lambda = \frac{1}{n}$ for all $n \ge 1$, and so $f_n \to f$ in $L^1([0, 1], d\lambda)$. However, $f_n \not\to 0$ pointwise anywhere.

So pointwise convergence, uniform convergence, and L^1 convergence do not imply each other.

We do, however, have a few positive results:

Theorem 13. *If $f_n \to f$ in L^1, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise a.e.*

Theorem 14 (Dominated Convergence Theorem)**.** *Suppose that $f_n \to f$ pointwise a.e., and that there is some function g such that $\int_X |g| \, d\mu < \infty$ and $|f_n(x)| \le g(x)$ for all x and n. Then $f_n \to f$ in L^1.*

This is proven using Fatou's Lemma. Note that if $\mu(X) < \infty$, then $g(x) = |f(x)| + 1 \in L^1(X, d\mu)$ whenever $f \in L^1(X, d\mu)$; thus, on a set of finite measure, uniform convergence *does* imply L^1 convergence.

6. Subspaces

In a metric space, it is occasionally easy to define or prove things on dense subspaces and then extend them to the entire space; for example, 2^x can be defined for all rational x using nth roots, and can then be extended to all real x by continuity.

L^p spaces have a number of useful dense subspaces.

Theorem 15. *If $1 \leq p < \infty$, and (X, μ) is a measure space, then*

- *The set of all simple functions whose support has finite measure is dense in $L^p(X, \mu)$. (A function is called simple if its range is a finite set; a function's support is the closure of the set where it is nonzero.)*
- *The set of all continuous functions whose support is compact is dense in $L^p(\mathbf{R}^n, d\lambda)$. (This is true for other spaces as well.)*

Neither of these sets are dense in $L^\infty(\mathbf{R}^n, d\lambda)$. The set of all simple functions with no restrictions on support is dense in $L^\infty(X, \mu)$, but the set of all continuous functions is not.

Since the first set is contained in L^p for all p, we have that $L^p \cap L^q$ is dense in L^p for all $1 \leq p < \infty$ and all $1 \leq q \leq \infty$.