Research talk on non-linear PDE

Timur Akhunov

September 22, 2008

By a non-linear PDE I mean the following differential equation (Cauchy Problem)

$$\begin{cases} F(x, u, \nabla u, \nabla^2 u, \ldots) = 0, \text{ on } \Omega \\ u(x) = u_0(x), \text{ on } \partial\Omega \end{cases}$$

where $x \in \overline{\Omega}$ and F is not linear. A PDE is linear if F has the following form:

$$F(x, u, \nabla u, \nabla^2 u, \ldots) = F_0(x) + F_1(x)u + F_2(x)\nabla u + \cdots$$

For a linear PDE solutions form a vector space up to the non-homogeneous factor F_0 as in ODE theory.

Linear vs Non-linear problems

PRO's of Linear problems:

1. More categorical - functional and Fourier analysis is far more applicable
2. Extensively studied

Good about non-linear problems:

1. Hard (that's good)
2. Plenty of open problems and it is a very active research area
3. Are better approximations of physical models

In truth, non-linear theory needs linear theory, because there are few other ways to deal with non-linear problems then perturb linear ones. So the two go in tandem.

A word of history/classification

In 2D it is possible to classify all linear PDE and when non-linear PDE were systematically studied in 30s many results were established in 2D using complex analysis that was hard to replicate in higher dimensions. Furthermore, given how numerous and vastly different phenomena are described with PDE it is

reasonable to expect there is no general theory possible.
The way modern theory is approached, people study classes of equations that come up in physical models, fashioning appropriate techniques for those classes.

Concrete example

Here is the problem that was extensively studied by Kenig and his collaborators in late 90s and beyond. My current research problem is quite similar.
Quasi-linear Schrödinger equation:

$$\begin{cases} \partial_t u = \sum_{j,k} a(x,t,u,\nabla u)\partial_{x_j}\partial_{x_k} u \\ +b(x,t,u,\nabla u)\nabla u + c(x,t,u)u + f \\ u(x,0) = u_0(x) \end{cases}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and u is complex valued. Then under certain assumptions on coefficients this problem has a unique solution for a short time. By a solution I mean $u \in C([0,T] : B) \cap C^1([0,T] : \tilde{B})$, where B is usually a Sobolev space, like

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}' : \int (1+|\xi|)^s \cdot \hat{f}^2(\xi) < \infty\}$$

Such equations appear in several fields of physics, such as plasma fluids, classical and quantum ferromagnetism, laser theory, etc. and also in complex geometry.

Let's consider a special case.

$$\begin{cases} \partial_t u = i\triangle u + |u|^2 \cdot u \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

Recall that $\triangle u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. I will outline the local existence result in the framework that generalizes to the QLSE.

A priori energy estimate: There is a $T_0 > 0$, such that a solution to (1) satisfies $\|u(x,t)\|_{H^s} \leq 2\|u_0(x)\|_{H^s}$ for $0 \leq t \leq T_0$. It means that a solution does not grow to much at least for some time. H^s norm is customary called energy from physical heuristics.

Viscous problem: We modify (1) to use off the shelf parabolic theory and prove existence for a modified problem

$$\begin{cases} \partial_t u = -\varepsilon\triangle^2 u + i\triangle u + |u|^2 \cdot u \\ u(x,0) = u_0(x) \end{cases}$$

$\varepsilon\triangle^2 u$ is called artificial viscosity from fluid dynamics. Then a contraction mapping argument shows there is a solution $u_\varepsilon \in C([0,T_\varepsilon] : B) \cap C^1([0,T_\varepsilon] : \tilde{B})$. The bad news is that $T_\varepsilon = O(\varepsilon)$.

Uniform time of existence We combine the previous two steps to show there is a u_ε that is a solution on $[0, T_0]$ and not just up to T_ε. This is called a pacing argument. We first solve a modified problem with initial data u_0. Then using a priori estimate u_ε is small up to T_0. So we can use $u_\varepsilon(T_\varepsilon)$ as initial data and so on until T_0.

Removing viscosity We now send $\varepsilon \to 0$ and use some functional analysis to show the limit is in the appropriate space. A priori estimate is important in this step and also in showing uniqueness.

A priori estimate is the key in this outline. It uses both the standard Sobolev theory and the algebraic structure of the equation at hand.

Relevant Sobolev theory is the following:

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)} \text{ for } s > \frac{n}{2}$$

And that for $s > \frac{n}{2}$, $H^s(\mathbb{R}^n)$ is an algebra, i.e.

$$\|f \cdot g\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \cdot \|g\|_{H^s(\mathbb{R}^n)}$$

The algebraic manipulations are to multiply (1) by $\overline{u}$, add it to conjugated (1) multiplied by u and integrate. Integration by parts kills the Laplacian.

To perform this outline for QLSE one has to get a lot more involved. Calculus of ΨDO is an essential and relevant tool in the a priori estimate.