REVIEW OF GENERAL TOPOLOGY I
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1. Basic Definitions

Definition 1.1. A *topology* on a set X is a subset $\mathscr{T}$ of $\wp(X)$ subject to the following requirements:

(1) $\varnothing, X \in \mathscr{T}$;
(2) if $U, V \in \mathscr{T}$, then $U \cap V \in \mathscr{T}$;
(3) if I is a set and $\{U_i : i \in I\} \subseteq \mathscr{T}$, then $\bigcup_{i \in I} U_i \in \mathscr{T}$.

We refer to the elements of $\mathscr{T}$ as the *open sets of* X (for the topology $\mathscr{T}$), and to their complements as the *closed sets*. A *topological space* is a pair $(X, \mathscr{T})$ where X is a set and $\mathscr{T}$ a topology on X.

Example 1.2. The collection of all singleton subsets of a set X is called the *trivial* or *indiscrete topology*. The collection of all subsets of X is called the *discrete topology* on X.

Example 1.3. Let R be a ring, and let $\mathrm{Spec}(R)$ denote its spectrum (i.e., the collection of all its proper prime ideals). Call a subset U of $\mathrm{Spec}(R)$ *Zariski-closed* if there is a subset I of R such that $\mathfrak{p} \in U$ if and only if $I \subseteq \mathfrak{p}$. The collection of complements of all Zariski-closed sets is called the *Zariski topology* on $\mathrm{Spec}(R)$.

Definition 1.4. If $(X, \mathscr{T})$ is a topological space, a subset $\mathscr{B}$ of $\mathscr{T}$ is called a *basis for* $\mathscr{T}$ if every element of $\mathscr{T}$ is a union of elements in $\mathscr{B}$. A subset $\mathscr{S}$ of $\mathscr{T}$ is a *subbasis* if every element of $\mathscr{T}$ is a union of a finite intersection of elements in $\mathscr{S}$. The elements of $\mathscr{B}$ are called the *basis open sets*, and those of $\mathscr{S}$ the *subbasic open sets*.

Example 1.5. Let X be a set endowed with a linear order $<$, and let $\mathscr{S}$ be the collection of all subsets of X of the form $\{x \in X : x > a\}$ and $\{x \in X : x < b\}$, where a and b are elements of X. Then $\mathscr{S}$ is a subbasis for a topology on X called the *order topology*. If we let $\mathscr{B}$ consist of all the sets in $\mathscr{S}$ together with all sets of the form $\{x \in X : a < x < b\}$ for $a, b \in X$, then $\mathscr{B}$ is a basis for the order topology.

Example 1.6. Define $2^{<\omega}$ as the set of all functions $\sigma : \{0, \ldots, n\} \to \{0, 1\}$ for $n \in \mathbb{N}$, and 2^{ω} as the set of all functions $f : \mathbb{N} \to \{0, 1\}$). For $\sigma \in 2^{<\mathbb{N}}$, define $[\sigma] = \{f \in 2^{\mathbb{N}} : \sigma \subset f\}$ and let $\mathscr{B} = \{[\sigma] : \sigma \in 2^{<\mathbb{N}}\}$. Then $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on $2^{\mathbb{N}}$; the topological space $(2^{\mathbb{N}}, \mathscr{T})$ is called *Cantor space*.

Definition 1.7. If $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on a set X, we say the first is *finer* than (or is a *refinement* of) the second if $\mathscr{T}_2 \subseteq \mathscr{T}_1$.

Definition 1.8. Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces, and let f be a function $X \to Y$. We say f is

(1) *continuous* if $f^{-1}(V) \in \mathscr{T}$ for every $V \in \mathscr{S}$;
(2) *open (closed)* if $f(U)$ is open (closed) in Y for every set open (closed) in X.

Exercise 1.9. Show that not every continuous function is open.

2. Subset, Box, Product, and Quotient Topologies

Definition 2.1. Let $(X, \mathscr{T})$ be a topological space, and let $Y \subseteq X$. Define $\mathscr{S}$ as the collection of all sets $U \cap Y$ for $U \in \mathscr{T}$. Then $\mathscr{S}$ is a topology on Y, called the *subspace topology on Y induced by $\mathscr{T}$.*

Example 2.2. Consider the closed interval $[0, 1]$ as a subset of $\mathbb{R}$ under its usual topology. Then $[1, 0]$, $(a, 1]$, and $[0, b)$, for $0 < a, b < 1$, are all open in $[0, 1]$ under the subspace topology, being the intersections of $[0, 1]$ with, respectively, the open subsets $\mathbb{R}$, (a, ∞), and $(-\infty, b)$ of $\mathbb{R}$. Hence, the subspace topology of a set need not be a subset of the topology that induces it.

Definition 2.3. Let I be a set, and let $(X_i, \mathscr{T}_i)$ for each $i \in I$ be a topological space.

(1) The *box topology* on $\Pi_{i \in I} X_i$ is defined as the collection of all sets $\Pi_{i \in I} U_i$, where $U_i \in \mathscr{T}_i$ for all i.
(2) The *product topology* on $\Pi_{i \in I} X_i$ is defined as the collection of all sets $\Pi_{i \in I} U_i$, where $U_i \in \mathscr{T}_i$ for all i, and $U_i = X_i$ for cofinitely many i.

Example 2.4. Consider $\mathbb{R}$ under its usual topology. Then the product and box topologies on $\mathbb{R} \times \mathbb{R}$ coincide with the usual topology on $\mathbb{R}^2$, the product and box topologies on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ coincide with the usual topology on $\mathbb{R}^3$, and so on. The product topology on $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots = \{f : \mathbb{N} \to \mathbb{R}\}$ differs from the box topology on $\mathbb{R}^\omega$, since $(0, 1)^\omega = (0, 1) \times (0, 1) \times \cdots = \{f : \mathbb{N} \to (0, 1)\}$ is open in the latter but not the former.

Definition 2.5. Let $(X, \mathscr{T})$ be a topological space, Y a set, and $p : X \to Y$ a surjection. The *quotient topology on Y induced by p* is the collection $\mathscr{S}$ of all subsets U of Y such that $p^{-1}(U) \in \mathscr{T}$. We then call p the *quotient map* between the topological spaces $(X, \mathscr{T})$ and $(Y, \mathscr{S})$.

Exercise 2.6. Show that if $(X, \mathscr{T})$, $(Y, \mathscr{S})$, and p are as in the preceding definition, then $\mathscr{S}$ is the finest topology on Y for which p is continuous.

Example 2.7. Let $\sim$ be an equivalence relation on a set X with topology $\mathscr{T}$. Then X surjects onto the quotient space $X/\sim$ via the map $x \mapsto [x]_\sim$, and so we can endow $X/\sim$ with the quotient topology induced by this surjection.

3. Closure, Countability, and Separation

Definition 3.1. Let $(X, \mathscr{T})$ be a topological space and let Y be a subset of X.

(1) A point $x \in X$ is called a *limit point* of Y if every open set $U \in \mathscr{T}$ containing x contains a point $y \in Y - \{x\}$.
(2) The *closure* of Y is the set $\overline{Y}$ containing all points in Y and all limit points of Y.
(3) We say Y is *dense* in X if $\overline{Y} = X$.
(4) $(X, \mathscr{T})$ is called *separable* if it has a countable dense subset.

Exercise 3.2. Show that $\overline{Y}$ is equal to the smallest closed subset of X containing Y (i.e., the intersection of all closed subsets of X containing Y).

Example 3.3. $\mathbb{Q}$ is dense in $\mathbb{R}$ under the usual topologies since every real is the limit of a sequence of rationals. Hence, $\mathbb{R}$ is separable.

Example 3.4. Any indiscrete space is separable, as is any countable discrete space.

Exercise 3.5 (requires familiarity with ordinals)**.** Endow ω_1, the least uncountable ordinal, with the order topology (the ordering being inclusion, or equivalently, the standard ordering of the class of ordinals restricted to ordinals $< \omega_1$), and show that the resulting topological space is not separable. [*Hint:* Suppose S is a countable subset of ω_1. Then S is bounded by an ordinal $< \omega_1$ since ω_1 is regular and hence has no countable cofinal subset.]

Definition 3.6. A topological space $(X, \mathscr{T})$ is called

(1) *first countable* if, for each $x \in X$, there exists a sequence $U_0^x, U_1^x, \ldots$ of sets open in X, such that for every open set U of X there exists an $i \in \mathbb{N}$ with $U_i^x \subseteq U$.
(2) *second countable* if it has a countable basis.

Exercise 3.7. Show that every second countable space is first countable.

Example 3.8. $\mathbb{Q}$, under its usual topology, is second countable. Indeed, this topology is an order topology, and hence we can give it the countable basis of Example 1.5. The same basis can be used to generate the usual topology on $\mathbb{R}$, showing that that topological space is second countable as well.

Definition 3.9 (separation axioms)**.** A topological space $(X, \mathscr{T})$ is called a

(1) T_0 *space* if for every pair of distinct points in X there exists an open subset of X containing one but not the other;
(2) T_1 *space* if for every pair of distinct points $x, y \in X$ there exist open subsets U, V of X such that $x \in U - V$ and $y \in V - U$;
(3) a *Hausdorff* (or T_2) *space* if for every pair of distinct points $x, y \in X$ there exist disjoint open subsets U, V of X such that $x \in U$ and $y \in V$;

(4) a *regular space* if for every closed subset C of X, and every point $x \in X - C$, there exist disjoint open subsets U, V of X such that $C \subseteq U$ and $x \in V$;
(5) a *normal space* if for every pair of distinct closed subsets C, D of X there exist disjoint open subsets U, V of X with $C \subseteq U$ and $D \subseteq V$.

Exercise 3.10. Show that every T_2 space is T_1, and every T_1 space is T_0.

Exercise 3.11. Show no set under the trivial topology is T_0 if it has more than one point.

Exercise 3.12. Show that a topological space $(X, \mathscr{T})$ is T_1 if and only if every singleton subset of X is closed, if and only if, for every $x \in X$ and every $Y \subseteq X$, x is a limit point of Y just in case every open subset of X containing x contains infinitely many points of X.

Exercise 3.13. Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces, the former Hausdorff, and let f be a continuous map $X \to Y$. Show that $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$ under the product topology.

Example 3.14 (line with two origins)**.** Let $X = (\mathbb{R} - \{0\}) \cup \{a, b\}$, where a and b are not elements of $\mathbb{R}$. Let $\mathscr{B}$ consist of all open subsets of $\mathbb{R}$ not containing 0, along with all sets of the form $(-x, 0) \cup \{y\} \cup (0, x)$ where $x \in \mathbb{R}^+$ and $y \in \{a, b\}$. Then $\mathscr{B}$ is the basis for a topology which turns X into a T_1 space that is not Hausdorff.

4. Homeomorphisms and Topological Properties

Definition 4.1. If $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces, a function $f : X \to Y$ is called a *homeomorphism* if it is bijective, continuous, and open. We say $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ are *homeomorphic* if there exists a homeomorphism $X \to Y$.

Exercise 4.2. Show that f as above is a homeomorphism if and only if it is bijective, continuous, and the inverse function $f^{-1} : Y \to X$ is continuous.

Example 4.3. The Cantor set, under the subspace topology inherited from $\mathbb{R}$, is homeomorphic to $\Pi_{i \in \mathbb{N}}\{0, 1\}$, under the product topology inherited from $\{0, 1\}$ as a discrete topological space.

Example 4.4. The unit disc and unit square in $\mathbb{R}^2$, under the subspace topologies inherited from the usual topology of $\mathbb{R}^2$, are homemorphic.

Example 4.5. The open interval $(0, 1)$, as a subspace of $\mathbb{R}$, is homeomorphic with $\mathbb{R}$.

Definition 4.6. A property is called a *topological property* if whenever it holds of one topological space, it holds of every topological space homeomorphic to the first.

Example 4.7. All cardinal invariant properties (e.g., finiteness, countability) are clearly topological properties.

Example 4.8. Separability and density (cf. Definition 3.1) are topological properties, as are all the separation properties of Definition 3.6.