CLASSIFYING COVERS
WOMP 2007

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ABSTRACT. A major theme in algebraic topology, algebraic geometry, and other subjects is the use of "classifying objects" and "classifying maps" to describe objects or constructions of interest. We explain this in the example of covering space theory. Along the way, we'll review group actions and some basic categorical concepts.

This outline was written quickly to accompany my WOMP talk, mostly in order to give the references in the first paragraph and the last section. The reader should beware of typos, misspellings, and misstatements in the rest.

OUTLINE

This talk is based heavily on the 2006 talk given by Megan Guichard and Mike Shulman. Their complete notes are available on the WOMP website at `http://www.math.uchicago.edu/~womp/2006/`. You may also find the 2004 or 2001 notes helpful, if you want to look at them. As the notes by Megan and Mike point out, you might also enjoy the description of covering space theory as related to Galois theory, in the first section of a paper based on talks given by John Baez. That paper is *Lectures on n-Categories and Cohomology*, by John Baez and Mike Shulman, available at `http://arxiv.org/abs/math/0608420`.

1. We assume that our spaces are "decent" in the following sense: they are locally path connected and semi-locally simply connected; this is the condition necessary to construct covering spaces of a given space.

2. Group actions: If a group G acts continuously on a space X, we topologize the cosets with the quotient topology.

(1) free actions
(2) properly discontinuous actions
(3) $\pi_1(X/G)$

3. Path-connected covers of a path-connected space X are classified by subgroups of $\pi_1(X)$. These subgroups are the stabilizers for transitive $\pi_1(x)$-actions on sets.

4. Covers of a path-connected space X are classified by actions of $\pi_1(X)$ on sets. (Given an action of $\pi_1(X)$ on a set F, construct a cover with fiber F.)

(1) Note: this classification includes something about maps of covers corresponding to maps of sets with a $\pi_1(X)$-action.
(2) Recall: groups as categories with one object, group actions as functors, equivariant maps as natural transformations

5. For non-path-connected spaces: The fundamental groupoid, $\Pi_1(X)$.

6. There is an equivalence of categories: $[\Pi_1(X), \mathbf{Set}] \simeq \mathscr{C}ov(X)$.

Vista

Other examples of the classifying map/object theme:

(1) Degree-n covers of a paracompact space, X, can also be classified by maps from X to $F_n(\mathbb{R}^\infty)/\Sigma_n$, the configuration space of n points in $\mathbb{R}^\infty$. (Note: $F_n(\mathbb{R}^\infty)/\Sigma_n$ is an Eilenberg-Mac Lane space of type Σ_n-1.) See [Aguilar, Gitler, Prieto] for the development of this classification.
(2) Real (rank-n) vector bundles over X are classified by maps from X to the rank-n *Grassmanian*, the "configuration space of hyperplanes".
(3) Principal G-bundles are classified by maps from X to BG. Sometimes called the classifying space of G, BG is an Eilenberg-Mac Lane space of type G-1.
(4) Cohomology groups of X with coeffieients in an Abelian group G are classified by maps to Eilenberg-Mac Lane spaces of type G-k for various natural numbers k.
(5) Examples from algebraic geometry: ...something about schemes and stacks (?)

Good references you probably won't hear about very often

- Marcelo Aguilar, Samuel Gitler, Carlos Prieto. *Algebraic topology from a homotopical viewpoint.* Universitext. Springer-Verlag, New York, 2002. [Eck Library call number QA612.A37]
  This book is known for its *very* different approach to the subject. The treatment of homology has even been called perverse! The authors deveelop fibrations and cofibrations early, and describe some classical constructions from a more modern (and more general) perspective.
- Akira Kono, Dai Tamaki. *Generalized cohomology.* American Mathematical Society. Providence, RI, 2006. [Eck Library call number QA612.3 .K66]
  This is another modern treatment of algebraic topology, possibly more appropriate as a second course in algebraic topology. It introduces modern tools such as spectra and spectral sequences, together with common applications.