WOMP 2007: Function Spaces

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1 L^p spaces

Definition 1 *Let f be a measurable function, and let $1 \le p < \infty$. We say that the $L^p(X,\mu)$* norm of f *is*

$$||f||_{L^p} = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

We say that the $L^\infty(X,\mu)$ norm of f *is its upper bound except on sets of measure zero; that is,*

$$||f||_{L^\infty} = \operatorname*{ess\,sup}_{x\in X} |f(x)| = \inf\left\{\alpha > 0 : |f(x)| \le \alpha \text{ a.e.}\right\},$$

where a.e. stands for "almost everywhere", that is, "except on a set of measure zero".

Theorem 2 (Minkowski's Inequality) *If $||f||_{L^p} < \infty$ and $||g||_{L^p} < \infty$ for $1 \le p \le \infty$, then*

$$||f+g||_{L^p} \le ||f||_{L^p} + ||g||_{L^p}.$$

It is clear that $||\alpha f||_{L^p} = |\alpha|\,||f||_{L^p}$ for any scalar α. Thus, the set of all functions whose L^p norm is finite is *almost* a normed vector space. There are some nonzero functions with norm zero. (For example, let $f(x) = 0$ for $x \ne 3$, and let $f(3) = 1$. Then $||f||_{L^p} = 0$ for any $1 \le p \le \infty$.)

To deal with this, we quotient out by functions which are zero a.e. We let L^p be the vector space whose elements are *equivalence classes of functions* with the equivalence relation $f \equiv g$ if $f(x) = g(x)$ almost everywhere, that is, $\mu\{x : f(x) \ne g(x)\} = 0$.

Usually, you can just think of L^p being a space of functions.

There is another very useful theorem (which you need to prove, among other things, Minkowski's inequality):

Theorem 3 (Hölder's Inequality) *If $1 \le p, q \le \infty$ with $1/p + 1/q = 1$, we say that p and q are* conjugates *or* conjugate exponents. *If $||f||_{L^p}$ and $||g||_{L^q}$ are finite, then*

$$\left|\int_X f(x)g(x)\, d\mu(x)\right| \le ||f||_{L^p}||g||_{L^q}.$$

Note that 1 and ∞ are conjugate exponents, and that 2 is its own conjugate.

It turns out that, if $1 \le p < \infty$, and if F is a map from L^p to $\mathbf{C}$ such that

- F is linear; that is, $F(f+g) = F(f) + F(g)$ and $F(\alpha f) = \alpha F(f)$ for any scalar α and $f, g \in L^p$
- F is bounded; that is, there exists a constant $C > 0$ such that $|F(f)| \le C||f||_{L^p}$ for all $f \in L^p$

then there is some $g \in L^q$ such that

$$F(f) = \int_X f(x)g(x)\, d\mu(x),$$

where q is the conjugate to p. We summarize this property by saying that L^q is the *dual* to L^p.

Here are some examples:

1. Suppose that $\mu(X) = 1$, and $1 \leq p \leq r \leq \infty$. Then we can show (by Hölder's inequality) that $||f||_{L^p} \leq ||f||_{L^r}$ and so $L^r \subset L^p$. More generally, if $\mu(X)$ is finite, then

   $$\left(\frac{1}{\mu(X)}\int_X |f|^p \, d\mu\right)^{1/p} \leq \left(\frac{1}{\mu(X)}\int_X |f|^r \, d\mu\right)^{1/r} \quad \text{or} \quad \left(\frac{1}{\mu(X)}\int_X |f|^p \, d\mu\right)^{1/p} \leq ||f||_{L^\infty}$$

   and so $L^r \subset L^p$ again.

2. Suppose that $X = \mathbf{N}$, the natural numbers, and that μ is the counting measure, that is, $\mu(S) = |S|$ is the cardinality of the set S.

   We usually refer to $L^p(\mathbf{N}, \mu)$ as ℓ^p. Note that

   $$||f||_{\ell^p}^p = \sum_{n=0}^{\infty} |f(n)|^p, \quad ||f||_{\ell^\infty} = \sup_{n \in \mathbf{N}} |f(n)|.$$

   So $|f(n)| \leq ||f||_{\ell^p}$, and so if $1 \leq p \leq r < \infty$ and $f \in \ell^p$, then

   $$||f||_{\ell^r}^r = \sum |f(n)|^r \leq \sum |f(n)|^p ||f||_{\ell^p}^{r-p} \leq ||f||_{\ell^p}^r.$$

   Thus, in this case, we have that $L^p \subset L^r$ for $1 \leq p \leq r \leq \infty$, the opposite of the previous case.

3. Now let $X = (\mathbf{R}, dx)$, where dx is Lebesgue measure. We do not have *any* nice inclusion relations in this case. If $1 \leq p < r < \infty$, then

   $$f(x) = \begin{cases} |x|^{-1/r} & \text{on } [-1,1] \\ 0 & \text{elsewhere} \end{cases}$$

   is in L^p but not L^r, and

   $$f(x) = \begin{cases} 0 & \text{on } [-1,1] \\ |x|^{-1/p} & \text{elsewhere} \end{cases}$$

   is in L^r but not L^p.

Theorem 4 *Under these definitions, L^p is a complete normed vector space for $1 \leq p \leq \infty$.*

2 Convergence of sequences of functions

Definition 5 *Let $\{f_n\}$ be a sequence of functions defined on some measure space (X, μ), and let f be another function on X. If $f_n, f \in L^p(X, \mu)$ for some p and $\lim_{n\to\infty} ||f_n - f||_{L^p} = 0$, then we say that f_n* converges to f in $L^p(X,\mu)$, *or* $f_n \to f$ in L^p.

Note that if $f_n \to f$ in L^∞, then f_n converges to f uniformly almost everywhere. Often we care only about L^1 convergence and uniform convergence.

We can contrast this with two other definitions:

1. If $\lim_{x\to\infty} f_n(x) = f(x)$ for all $x \in X$, then we say that f_n *converges to f, or $f_n \to f$, pointwise.*
2. If $\lim_{x\to\infty} f_n(x) = f(x)$ for almost all $x \in X$, then we say that f_n *converges to f pointwise almost everywhere (a.e.)*

Note that if $f_n \to f$ in $L^1(X, d\mu)$ then $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

It is clear that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise and pointwise a.e. (If $f_n \to f$ in L^∞, then $f_n \to f$ pointwise a.e.) Some functions converge pointwise but not uniformly; for example, $f_n(x) = x^n$ converges to $f(x) = 0$ pointwise but not uniformly on $(0, 1)$.

However, we do have one positive result:

Theorem 6 (Egorov's Theorem) *Suppose that* $f_n \to f$ *pointwise a.e. on some measure space* X *with* $\mu(X) < \infty$. *Then for any* $\epsilon > 0$, *there is a measurable set* E *such that* $\mu(E) < \epsilon$ *and* $f_n \to f$ *uniformly on* $X \backslash E$.

We do need for $\mu(X) < \infty$. As an example, let $f(x) = 0$, $f_n(x) = 0$ on $[-n, n]$ and $f_n(x) = 1$ elsewhere. Then $f_n \to f$ pointwise, but Egorov's Theorem obviously cannot hold.

We would like to know how uniform and pointwise a.e. convergence relate to L^1 convergence. Let $f(x) = 0$ everywhere. Here are some examples:

- Let $f_n(x) = 1/n$ on $[0, n]$ and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ uniformly but not in $L^1(\mathbf{R}, dx)$.
- Let $f_n(x) = n$ on $[0, 1/n]$, and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ pointwise but not in $L^1([0,1], dx)$.
- Let $g_n(x) = 1$ on
$$\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \frac{1}{n+1}\right]$$
and be zero elsewhere. Let $f_n(x) = 1$ if $0 \le x \le 1$ and $g_n(x + k) = 1$ for some integer k, and let $f_n(x) = 0$ otherwise.

  Then $\int_0^1 |f_n(x)|\, dx = \frac{1}{n}$ for all $n \ge 1$, and so $f_n \to f$ in $L^1([0,1], dx)$. However, $f_n \not\to 0$ pointwise anywhere.

So pointwise convergence, uniform convergence, and L^1 convergence do not imply each other.

We do, however, have a few positive results:

Theorem 7 *If* $f_n \to f$ *in* L^1, *then there is a subsequence* f_{n_k} *such that* $f_{n_k} \to f$ *pointwise a.e.*

Theorem 8 (Dominated Convergence Theorem) *Suppose that* $f_n \to f$ *pointwise a.e., and that there is some function* g *such that* $\int_X |g|\, d\mu < \infty$ *and* $|f_n(x)| \le g(x)$ *for all* x *and* n. *Then* $f_n \to f$ *in* L^1.

This is proven using Fatou's Lemma. Note that if $\mu(X) < \infty$, then $g(x) = |f(x)| + 1 \in L^1(X, d\mu)$ whenever $f \in L^1(X, d\mu)$; thus, on a set of finite measure, uniform convergence *does* imply L^1 convergence.

3 Subspaces

In a metric space, it is occasionally easy to define or prove things on dense subspaces and then extend them to the entire space; for example, 2^x can be defined for all rational x using nth roots, and can then be extended to all real x by continuity.

L^p spaces have a number of useful dense subspaces.

Theorem 9 *If* $1 \le p < \infty$, *and* (X, μ) *is a measure space, then*

- *The set of all simple functions whose support has finite measure is dense in* $L^p(X, \mu)$. *(A function is called simple if its range is a finite set; a function's support is the closure of the set where it is nonzero.)*
- *The set of all continuous functions whose support is compact is dense in* $L^p(\mathbf{R}, dx)$. *(This is true for other spaces as well.)*

Neither of these sets are dense in $L^\infty(\mathbf{R}, dx)$. The set of all simple functions with no restrictions on support is dense in $L^\infty(X, \mu)$, but the set of all continuous functions is not.

Since the first set is contained in L^p for all p, we have that $L^p \cap L^q$ is dense in L^p for all $1 \le p < \infty$ and all $1 \le q \le \infty$.

4 Arzela-Ascoli

Arzela and Ascoli proved a very important theorem:

Theorem 10 (Arzela-Ascoli) *Let $\{f_n\}$ be a sequence of functions defined on $\mathbf{R}$. Suppose that the sequence is uniformly bounded, that is, there is a constant C such that $|f_n(x)| < C$ for all n and x. Suppose furthermore that the f_n are continuous and differentiable, and that the derivatives are also uniformly bounded.*

Then there is a subsequence f_{n_k} which converges uniformly to some continuous function f.

Proof. Pick your favorite countable dense subset Q of $\mathbf{R}$, for example, the rationals. Let its elements be $q_1, q_2, \ldots$

Then $\{f_n(q_1)\}$ is a sequence of points in $[-C, C]$, a compact space; thus, there is some increasing sequence $\{n_k^1\}$ such that the subsequence $f_{n_k^1}(q_1)$ converges as $k \to \infty$. Define $f(q_1)$ to be the number it converges to.

Now, $\{f_{n_k^1}(q_2)\}$ is also a sequence of points in $[-C, C]$. So we may pick out another subsequence n_k^2 of n_k^1 such that $f_{n_k^2}(q_2) \to f(q_2)$ for some $f(q_2)$. Note that since n_k^2 is a subsequence of n_k^1, $f_{n_k^2}(q_1) \to f(q_1)$.

Repeat this for each q_j. Then let $n_k = n_k^k$, that is, the kth element of the subsequence for q_k. Then if $k \geq j$, there is some $h \geq k$ with $n_k = n_h^j$; thus, $f_{n_k}(q_j) \to f(q_j)$ for all j.

So we have picked out a subsequence that converges on Q, using nothing more than the fact that the f_ns are uniformly bounded. We can easily require that $|f_{n_j^k}(q_k) - f(q_k)| < 2^{-j}$; thus, $f_{n_k} \to f$ uniformly on $Q = \{q_j\}$.

Fix $\epsilon > 0$. Let $r \in \mathbf{R}$. Then there is some q_j with $|r - q_j| < \epsilon/2C$, so that $|f_n(r) - f_n(q_j)| < \epsilon/4$ for all n.

There is some N such that $|f_{n_k}(q_j) - f(q_j)| < \epsilon/4$ for all $k > N$ and all j. Thus, if $k, h > N$, then

$$|f_{n_k}(r) - f_{n_h}(r)| < |f_{n_k}(r) - f_{n_k}(q_j)| + |f_{n_k}(q_j) - f(q_j)| + |f(q_j) - f_{n_h}(q_j)| + |f_{n_h}(q_j) - f_{n_h}(r)| < \epsilon$$

Thus, $f_{n_k}(r)$ is a Cauchy sequence for all r, and so it converges to some number $f(r)$.

So $\{f_{n_k}\}$ converges uniformly on $\mathbf{R}$. Any sequence of uniformly convergent continuous functions converges to a function which is itself continuous; thus we are done. ■