1 \( L^p \) spaces

**Definition 1** Let \( f \) be a measurable function, and let \( 1 \leq p < \infty \). We say that the \( L^p(X, \mu) \) norm of \( f \) is

\[
||f||_{L^p} = \left( \int_X |f|^p \, d\mu \right)^{1/p}.
\]

We say that the \( L^\infty(X, \mu) \) norm of \( f \) is its upper bound except on sets of measure zero; that is,

\[
||f||_{L^\infty} = \text{ess sup}_{x \in X} |f(x)| = \inf \{ \alpha > 0 : |f(x)| \leq \alpha \text{ a.e.} \},
\]

where a.e. stands for “almost everywhere”, that is, “except on a set of measure zero”.

**Theorem 2 (Minkowski’s Inequality)** If \( ||f||_{L^p} < \infty \) and \( ||g||_{L^q} < \infty \) for \( 1 \leq p, q \leq \infty \), then

\[
||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p}.
\]

It is clear that \( ||\alpha f||_{L^p} = |\alpha|||f||_{L^p} \) for any scalar \( \alpha \). Thus, the set of all functions whose \( L^p \) norm is finite is almost a normed vector space. There are some nonzero functions with norm zero. (For example, let \( f(x) = 0 \) for \( x \neq 3 \), and let \( f(3) = 1 \). Then \( ||f||_{L^p} = 0 \) for any \( 1 \leq p \leq \infty \).)

To deal with this, we quotient out by functions which are zero a.e. We let \( L^p \) be the vector space whose elements are equivalence classes of functions with the equivalence relation \( f \equiv g \) if \( f(x) = g(x) \) almost everywhere, that is, \( \mu\{x : f(x) \neq g(x)\} = 0 \).

Usually, you can just think of \( L^p \) being a space of functions.

There is another very useful theorem (which you need to prove, among other things, Minkowski’s inequality):

**Theorem 3 (Hölder’s Inequality)** If \( 1 \leq p, q \leq \infty \) with \( 1/p + 1/q = 1 \), we say that \( p \) and \( q \) are conjugates or conjugate exponents. If \( ||f||_{L^p} \) and \( ||g||_{L^q} \) are finite, then

\[
\left| \int_X f(x)g(x) \, d\mu(x) \right| \leq ||f||_{L^p} ||g||_{L^q}.
\]

Note that 1 and \( \infty \) are conjugate exponents, and that 2 is its own conjugate.

It turns out that, if \( 1 \leq p < \infty \), and if \( F \) is a map from \( L^p \) to \( C \) such that

- \( F \) is linear; that is, \( F(f + g) = F(f) + F(g) \) and \( F(\alpha f) = \alpha F(f) \) for any scalar \( \alpha \) and \( f, g \in L^p \)
- \( F \) is bounded; that is, there exists a constant \( C > 0 \) such that \( |F(f)| \leq C||f||_{L^p} \) for all \( f \in L^p \)

then there is some \( g \in L^q \) such that

\[
F(f) = \int_X f(x)g(x) \, d\mu(x),
\]

where \( q \) is the conjugate to \( p \). We summarize this property by saying that \( L^q \) is the dual to \( L^p \).

Here are some examples:
1. Suppose that $\mu(X) = 1$, and $1 \leq p \leq r \leq \infty$. Then we can show (by Hölder’s inequality) that $\|f\|_{L^r} \leq \|f\|_{L^p}$ and so $L^r \subset L^p$. More generally, if $\mu(X)$ is finite, then

$$\left( \frac{1}{\mu(X)} \int_X |f|^p d\mu \right)^{1/p} \leq \left( \frac{1}{\mu(X)} \int_X |f|^r d\mu \right)^{1/r} \text{ or } \left( \frac{1}{\mu(X)} \int_X |f|^p d\mu \right)^{1/p} \leq \|f\|_{L^\infty}$$

and so $L^r \subset L^p$ again.

2. Suppose that $X = \mathbb{N}$, the natural numbers, and that $\mu$ is the counting measure, that is, $\mu(S) = |S|$ is the cardinality of the set $S$.

We usually refer to $L^p(\mathbb{N}, \mu)$ as $\ell^p$. Note that

$$\|f\|_{\ell^p} = \sum_{n=0}^\infty |f(n)|^p, \quad \|f\|_{\ell^\infty} = \sup_{n \in \mathbb{N}} |f(n)|.$$  

So $|f(n)| \leq \|f\|_{\ell^p}$, and so if $1 \leq p < r < \infty$ and $f \in \ell^p$, then

$$\|f\|_{\ell^r} = \sum |f(n)|^r \leq \sum |f(n)|^p |f|_{\ell^p} = \|f\|_{\ell^p}.$$  

Thus, in this case, we have that $L^p \subset L^r$ for $1 \leq p \leq r \leq \infty$, the opposite of the previous case.

3. Now let $X = (\mathbb{R}, dx)$, where $dx$ is Lebesgue measure. We do not have any nice inclusion relations in this case. If $1 \leq p < r < \infty$, then

$$f(x) = \begin{cases} |x|^{-1/r} & \text{on } [-1, 1] \\ 0 & \text{elsewhere} \end{cases}$$

is in $L^p$ but not $L^r$, and

$$f(x) = \begin{cases} 0 & \text{on } [-1, 1] \\ |x|^{-1/p} & \text{elsewhere} \end{cases}$$

is in $L^r$ but not $L^p$.

**Theorem 4** Under these definitions, $L^p$ is a complete normed vector space for $1 \leq p \leq \infty$.

### 2 Convergence of sequences of functions

**Definition 5** Let $\{f_n\}$ be a sequence of functions defined on some measure space $(X, \mu)$, and let $f$ be another function on $X$. If $f_n, f \in L^p(X, \mu)$ for some $p$ and $\lim_{n \to \infty} \|f_n - f\|_{L^p} = 0$, then we say that $f_n$ converges to $f$ in $L^p(X, \mu)$, or $f_n \to f$ in $L^p$.

Note that if $f_n \to f$ in $L^\infty$, then $f_n$ converges to $f$ uniformly almost everywhere. Often we care only about $L^1$ convergence and uniform convergence.

We can contrast this with two other definitions:

1. If $\lim_{x \to \infty} f_n(x) = f(x)$ for all $x \in X$, then we say that $f_n$ converges to $f$, or $f_n \to f$, pointwise.

2. If $\lim_{x \to \infty} f_n(x) = f(x)$ for almost all $x \in X$, then we say that $f_n$ converges to $f$ pointwise almost everywhere (a.e.)

Note that if $f_n \to f$ in $L^1(X, d\mu)$ then $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

It is clear that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise and pointwise a.e. (If $f_n \to f$ in $L^\infty$, then $f_n \to f$ pointwise a.e.) Some functions converge pointwise but not uniformly; for example, $f_n(x) = x^n$ converges to $f(x) = 0$ pointwise but not uniformly on $(0, 1)$.

However, we do have one positive result:
Theorem 6 (Egorov’s Theorem) Suppose that $f_n \to f$ pointwise a.e. on some measure space $X$ with $\mu(X) < \infty$. Then for any $\epsilon > 0$, there is a measurable set $E$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $X \setminus E$.

We do need for $\mu(X) < \infty$. As an example, let $f(x) = 0$, $f_n(x) = 0$ on $[-n, n]$ and $f_n(x) = 1$ elsewhere. Then $f_n \to f$ pointwise, but Egorov’s Theorem obviously cannot hold.

We would like to know how uniform and pointwise a.e. convergence relate to $L^1$ convergence. Let $f(x) = 0$ everywhere. Here are some examples:

- Let $f_n(x) = 1/n$ on $[0, n]$ and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ uniformly but not in $L^1(\mathbb{R}, dx)$.
- Let $f_n(x) = n$ on $[0, 1/n]$, and let $f_n(x) = 0$ elsewhere. Then $f_n \to f$ pointwise but not in $L^1([0, 1], dx)$.
- Let $g_n(x) = 1$ on
  $$\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \frac{1}{n+1}\right]$$
  and be zero elsewhere. Let $f_n(x) = 1$ if $0 \leq x \leq 1$ and $g_n(x + k) = 1$ for some integer $k$, and let $f_n(x) = 0$ otherwise.

  Then $\int_0^1 |f_n(x)| \, dx = \frac{1}{n}$ for all $n \geq 1$, and so $f_n \to f$ in $L^1([0, 1], dx)$. However, $f_n \not\to 0$ pointwise anywhere.

So pointwise convergence, uniform convergence, and $L^1$ convergence do not imply each other.

We do, however, have a few positive results:

Theorem 7 If $f_n \to f$ in $L^1$, then there is a subsequence $f_{nk}$ such that $f_{nk} \to f$ pointwise a.e.

Theorem 8 (Dominated Convergence Theorem) Suppose that $f_n \to f$ pointwise a.e., and that there is some function $g$ such that $\int_X |g| \, d\mu < \infty$ and $|f_n(x)| \leq g(x)$ for all $x$ and $n$. Then $f_n \to f$ in $L^1$.

This is proven using Fatou’s Lemma. Note that if $\mu(X) < \infty$, then $g(x) = |f(x)| + 1 \in L^1(X, d\mu)$ whenever $f \in L^1(X, d\mu)$; thus, on a set of finite measure, uniform convergence does imply $L^1$ convergence.

### 3 Subspaces

In a metric space, it is occasionally easy to define or prove things on dense subspaces and then extend them to the entire space; for example, $2^\mathbb{Q}$ can be defined for all rational $x$ using $n$th roots, and can then be extended to all real $x$ by continuity.

$L^p$ spaces have a number of useful dense subspaces.

Theorem 9 If $1 \leq p < \infty$, and $(X, \mu)$ is a measure space, then

- The set of all simple functions whose support has finite measure is dense in $L^p(X, \mu)$. (A function is called simple if its range is a finite set; a function’s support is the closure of the set where it is nonzero.)
- The set of all continuous functions whose support is compact is dense in $L^p(\mathbb{R}, dx)$. (This is true for other spaces as well.)

Neither of these sets are dense in $L^\infty(\mathbb{R}, dx)$. The set of all simple functions with no restrictions on support is dense in $L^\infty(X, \mu)$, but the set of all continuous functions is not.

Since the first set is contained in $L^p$ for all $p$, we have that $L^p \cap L^q$ is dense in $L^p$ for all $1 \leq p < \infty$ and all $1 \leq q \leq \infty$. 


4 Arzela-Ascoli

Arzela and Ascoli proved a very important theorem:

**Theorem 10 (Arzela-Ascoli)** Let \( \{f_n\} \) be a sequence of functions defined on \( \mathbb{R} \). Suppose that the sequence is uniformly bounded, that is, there is a constant \( C \) such that \( |f_n(x)| < C \) for all \( n \) and \( x \). Suppose furthermore that the \( f_n \) are continuous and differentiable, and that the derivatives are also uniformly bounded.

Then there is a subsequence \( f_{n_k} \) which converges uniformly to some continuous function \( f \).

**Proof.** Pick your favorite countable dense subset \( \mathbb{Q} \) of \( \mathbb{R} \), for example, the rationals. Let its elements be \( q_1, q_2, \ldots \).

Then \( \{f_n(q_1)\} \) is a sequence of points in \([-C, C]\), a compact space; thus, there is some increasing sequence \( \{n^1_k\} \) such that the subsequence \( f_{n^1_k}(q_1) \) converges as \( k \to \infty \). Define \( f(q_1) \) to be the number it converges to.

Now, \( \{f_{n^1_k}(q_2)\} \) is also a sequence of points in \([-C, C]\). So we may pick out another subsequence \( n^2_k \) of \( n^1_k \) such that \( f_{n^2_k}(q_2) \to f(q_2) \) for some \( f(q_2) \). Note that since \( n^2_k \) is a subsequence of \( n^1_k \), \( f_{n^2_k}(q_1) \to f(q_1) \).

Repeat this for each \( q_j \). Then let \( n_k = n^j_k \), that is, the \( k \)th element of the subsequence for \( q_k \). Then if \( k \geq j \), there is some \( h \geq k \) with \( n_k = n^j_h \); thus, \( f_{n_k}(q_j) \to f(q_j) \) for all \( j \).

So we have picked out a subsequence that converges on \( \mathbb{Q} \), using nothing more than the fact that the \( f_n \)s are uniformly bounded. We can easily require that \( |f_{n^j_k}(q_k) - f(q_k)| < 2^{-j} \); thus, \( f_{n_k} \to f \) uniformly on \( \mathbb{Q} \).

Fix \( \epsilon > 0 \). Let \( r \in \mathbb{R} \). Then there is some \( q_j \) with \( |r - q_j| < \epsilon/2C \), so that \( |f_n(r) - f_n(q_j)| < \epsilon/4 \) for all \( n \). There is some \( N \) such that \( |f_n(q_j) - f(q_j)| < \epsilon/4 \) for all \( k > N \) and all \( j \). Thus, if \( k, h > N \), then

\[
|f_{n_k}(r) - f_{n_h}(r)| < |f_{n_k}(r) - f_{n_h}(q_j)| + |f_{n_k}(q_j) - f(q_j)| + |f(q_j) - f_{n_h}(q_j)| + |f_{n_h}(q_j) - f_{n_h}(r)| < \epsilon
\]

Thus, \( f_{n_k}(r) \) is a Cauchy sequence for all \( r \), and so it converges to some number \( f(r) \).

So \( \{f_{n_k}\} \) converges uniformly on \( \mathbb{R} \). Any sequence of uniformly convergent continuous functions converges to a function which is itself continuous; thus we are done.