WOMP 2006 Linear Algebra-Rough Outline

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1 References

1. Hoffman and Kunze, Linear Algebra
2. Halmos, Finite Dimensional Vector Spaces
3. Helson, Linear Algebra

2 Outline

I'll try to use Greek letters for scalars $(\alpha, \beta, ...)$ and English letters for vectors $(a, b, v, ...)$.

Definition 2.1. A *vector space* is a set V with an addition operation $+$ and a scalar multiplication over a field k such that

1. $(V, +)$ is a commutative group,
2. $1_k a = a \ \forall a \in V$
3. $(\alpha\beta)a = \alpha(\beta a) \ \forall \alpha, \beta \in k, \forall a \in V$
4. $\alpha(a + b) = \alpha a + \alpha b \ \forall \alpha \in k, \forall a, b \in V$
5. $(\alpha + \beta)a = \alpha a + \beta a \ \forall \alpha, \beta \in k, \forall a \in V$

Definition 2.2. [Briefly]

* *Subspace*
* *Linear combination and Span*
  Note finiteness in definition.
* *Linearly independent set*

* *Direct sum*

* *Basis*
  Theorem: Every vector space has a basis.

* *Dimension*
  Theorem: Dimension is well defined.
  Theorem: For vector spaces over the same field, dimension determines isomorphism.

Definition 2.3. $T : V \to W$ where V and W are vector spaces over k is a *linear transformation* if for all $\alpha, \beta \in k$ and $a, b \in V$,

$$T(\alpha a + \beta b) = \alpha T(a) + \beta T(b).$$

The set of all such T forms the vector space $Hom_k(V, W)$.

If $T : V \to k$ is a linear transformation, T is called a *linear functional*, and $Hom_k(V, k)$ is called the *dual* of V.

Definition 2.4. [More briefly!]

* $ker(T) = \{v \in V | Tv = 0\}$, $nullity(T) = dim(ker(T))$

* $im(T) = \{w \in W | (\exists v) Tv = w\}$, $rank(T) = dim(im(T))$

Theorem (Rank/Nullity) For finite dimensional V,

$$dim(V) = rank(T) + nullity(T).$$

Theorem 2.5 (Change of Basis). *Suppose V is an n-dimensional vector space over k. Let B_1 and B_2 be two bases for V. Then there exists an n by n invertible matrix S (called the* change of basis matrix *such that for all $v \in V$,*

$$[v]_{B_2} = S[v]_{B_1}.$$

Using the same methods,

Theorem 2.6. *Let V be an n-dimensional vector space and W be an m-dimensional one. Then every linear transformation $T : V \to W$ has the form T_A for some m by n dimensional matrix A where T_A is matrix multiplication with respect to given bases for V and W.*

Definition 2.7. n by n matrices A and B (with coefficients in the same field) are *similar* if there exists an invertible matrix S such that $A = SBS^{-1}$.

Theorem 2.8 (Motivation for Similarity). *A and B represent the same linear transformation with respect to different bases for V if and only if A and B are similar.*

We now restrict our attention to $V = \mathcal{C}^n$ over $\mathcal{C}$ and to a linear transformation $T : \mathcal{C}^n \to \mathcal{C}^n$ represented by the matrix A with respect to the standard basis. Note $\mathcal{C}$ is algebraically closed.

Definition 2.9. [Again briefly.]

* *trace(A)*

* *determinant(A)*

Theorem 2.10 (Existence and Uniqueness of Determinants). *$det(A) : (\mathcal{C}^n)^n = Mat_{n,n}(\mathcal{C}) \to \mathcal{C}$ is the only complex function of n variables (the columns) that is multi-linear, skew-symmetric, and normalized so that $det(I_n) = 1$.*

Definition 2.11. [Eigenvalues, Preparation for JCF]

* *Eigenvalue:* λ is an *eigenvalue for T* if $Tv = \lambda v$ for $v \neq 0$. v is called an eigenvector associated to λ.

* *Characteristic polynomial:* $p_A(t) = det(tI - A) = \Pi_{i=1}^{k}(t - \lambda_i)^{m_i}$ for λ_i distinct. We call m_i the *algebraic multiplicity of* λ_i.
  Cayley's Theorem: $p_A(A) = 0$

* *Eigenspace:* $V_\lambda = \{v \in V | Tv = \lambda v\}$, $dim(V_\lambda)$ is the *geometric multiplicity of* λ.
  Note: $dim(V_\lambda) \leq m_i$.

* *diagonalizable:* A is *diagonalizable* if it is similar to a diagonal matrix.
  Theorem: A is diagonalizable if and only if A has n linearly independent eigenvectors.
  Note: Failure to be diagonalizable is a discrepancy between geometric and algebraic multiplicity.

  Theorem: Any A as above is similar to an upper triangular matrix.

* *Generalized eigenspace:* $U_\lambda = \{v \in V | (\exists k > 0)(T - \lambda I)^k v = 0\}$

* *Minimal polynomial:* $m_A(t) = \Pi_{i=1}^{k}(t - \lambda_i)^{j_i}$ is the monic polynomial of least degree which annihilates A.

Theorem 2.12 (Jordan Canonical Form). *Let A be an n by n complex matrix with distinct eigenvalues $\lambda_1, ..., \lambda_k$. Then A is similar to a matrix which is the direct sum of Jordan blocks $J_m(\lambda_i)$ (unique up to a reordering of the blocks) with at least one block for each λ_i where $J_m(\lambda)$ is an m by m matrix of the form:*

$$\begin{pmatrix} \lambda & 1 & 0 & ... & 0 \\ 0 & \lambda & 1 & ... & 0 \\ 0 & 0 & \lambda & 1 & .... \\ 0 & ... & ... & ... & ... \\ 0 & ... & ... & 0 & \lambda \end{pmatrix}$$

Summary of Properties of JCF

- A and B are similar iff they have the "same" JCF.
- Algebraic Multiplicity:
  $m_i = dim U_{\lambda_i} =$ sum of sizes of all Jordan blocks for λ_i.
- Geometric Multiplicity:
  $dim V_{\lambda_i} =$ number of Jordan blocks for λ_i.
- The exponent of $(t - \lambda_i)$ in $m_A(t)$ is the size of largest Jordan block for λ_i (We called this exponent j_i). This is also the index of the nilpotent transformation $(A - \lambda_i I)\restriction_{U_{\lambda_i}}$.