WOMP 2006: FUNCTION SPACES

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1. Convergence of sequences of functions

Definition Let $\{f_n\}$ be a sequence of measurable functions defined on a measure space (X, μ), and let f be another measurable function. Then we say that f_n converges to f

(1) almost everywhere (a.e.) if $f_n(x) \to f(x)$ for all $x \in X$ except on a set of measure zero. In other words it converges pointwise except on a set of measure zero.
(2) uniformly if for all $\epsilon > 0$ there exists N such that for $n \geq N$, and all $x \in X$, $|f_n(x) - f(x)| < \epsilon$.
(3) in L^1 if $f_n, f \in L^1(\mu)$ and

$$\int_X |f_n(x) - f(x)| \, d\mu(x) \to 0.$$

Remark If $\{f_n\}$, defined on a topological space, are continuous and converge uniformly to f then f is also be continuous. Also, if $f_n \to f$ in L^1 then $\int f_n \, d\mu \to \int f \, d\mu$.

Remark There are lots of other types of convergence that come up in various applications. Some examples are convergence in L^p (we'll see that a little bit later), convergence in measure, and convergence in distribution. However, the types above are the ones you'll see most often, and are sufficient to get familiar with the examples and ideas involved.

We'll devote the remainder of the section to exploring what the relations are between the various forms of convergence.

1.1. Almost everywhere convergence and uniform convergence.

It follows from the definitions that if $f_n \to f$ uniformly, then it converges pointwise and therefore a.e.. You've probably seen that the converse does not hold. For example, take $f_n = \chi_{(0,1/n]}$ (where χ_A denotes the characteristic function of A). Then you should verify that $f_n \to f$ a.e. but not uniformly. However, we do have the following positive result:

Theorem 1 (Egorov's Theorem)**.** *Suppose $\mu(X) < \infty$ and $f_n \to f$ a.e.. Then for all $\epsilon > 0$, there exists a measurable set E such that $\mu(E) < \epsilon$, and $f_n \to f$ on $X \setminus E$.*

1.2. Almost everywhere convergence and L^1 convergence.

In general, these two types of convergence are different. For instance, if $f_n = n\chi_{(0,1/n]}$, then $f_n \to 0$ a.e. but not in $L^1(\mathbb{R}, dx)$ (dx denotes Lebesgue measure). For an example of a sequence that converges in L^1 but not a.e., take the same measure space and let $f_1 = \chi_{[0,1/2]}$, $f_2 = \chi_{[1/2,1]}$, $f_3 = \chi_{[0,1/3]}$, $f_4 = \chi_{[1/3,2/3]}$ etc. Then $f_n \to 0$ in L^1 but not a.e..

While the above examples show that a.e. and L^1 convergence are not the same, we do have the following two theorems of which the first is particularly important.

Theorem 2 (Dominated Convergence Theorem)**.** *Suppose that $f_n \to f$ a.e. and there exists a function $g \in L^1(\mu)$ such that $|f_n| \leq g$. Then $f \in L^1(\mu)$, $f_n \to f$ in L^1 and therefore, $\int f_n \, d\mu \to \int f \, d\mu$.*

Theorem 3. *If $f_n \to f$ in L^1, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e..*

1.3. **Uniform convergence and L^1 convergence.** You may recall that if $f_n \to f$ uniformly, then

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx,$$

where the integrals above are Riemann integrals. The same result holds more generally, if we assume that $\mu(X) < \infty$.

Theorem 4. *If $\mu(X) < \infty$ and $f_n \to f$ uniformly, then $f_n \to f$ in L^1.*

The hypothesis $\mu(X) < \infty$ cannot be omitted. For instance, let $f_n = (1/n)\chi_{[0,n]}$. Then $f_n \to 0$ uniformly but not in $L^1(\,dx)$.

2. L^p SPACES

2.1. **Definition.**

Definition Let $1 \leq p < \infty$. We define $L^p(X,\mu)$ to be the set of all measurable functions f on X such that

$$\|f\|_p = \left(\int_X |f|^p \; d\mu\right)^{1/p} < \infty.$$

We define $L^\infty(X,\mu)$ to be the set of all measurable functions f on X that are bounded a.e., i.e. such that

$$\|f\|_\infty = \inf_{\alpha \geq 0}\{|f| \leq \alpha \text{ a.e.}\} < \infty.$$

Definition If $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ are such that $1/p + 1/q = 1$ then p and q are said to be conjugates. Note that 2 is its own conjugate and that 1 and ∞ are conjugates.

Theorem 5.
(1) *If $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ are conjugate, $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

(2) *If $1 \leq p \leq \infty$, and $f, g \in L^p(\mu)$, then $f + g \in L^p(\mu)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The first inequality is called Hölder's Inequality, and the second Minkowski's Inequality. The case $p = q = 2$ in (1) is usually referred to as Schwarz's Inequality.

Theorem 6. *For $1 \leq p \leq \infty$, $(L^p(\mu), \|.\|_p)$ is a complete, normed vector space (almost).*

The reason that I wrote "almost" is that, as defined above, $\|.\|_p$ on $L^p(\mu)$ is only a seminorm; i.e. it has all the properties of a norm, except that it is not necessarily true that $\|f\|_p = 0$ implies $f = 0$. Indeed, if $f = 0$ a.e., then $\|f\|_p = 0$. To circumvent this problem, we notice that being equal a.e. is an equivalence relation and we mod out by it. In other words, we consider the elements of L^p to be equivalence classes of functions, instead of functions themselves. Then $(L^p(\mu), \|.\|_p)$ becomes a complete normed vector space. In practice, one can usually just think of elements of L^p as functions.

Note that for $1 \leq p < \infty$, $f_n \to f$ in L^p if

$$\|f_n - f\|_p^p = \int_X |f_n(x) - f(x)|^p \; d\mu(x) \to 0.$$

For L^∞, $f_n \to f$ in L^∞ if $f_n \to f$ uniformly except on a set of measure zero.

2.2. **Examples and subspaces.**

Examples (1) Assume that $\mu(X) = 1$. One can show using Jensen's Inequality (see Rudin 3.3 for example), that for all measurable functions f, $\|f\|_p \leq \|f\|_q$ whenever $1 \leq p \leq q \leq \infty$, and therefore, $L^p \supset L^q$ for $1 \leq p \leq q \leq \infty$. The last fact is also true for any finite measure space, although it is not necessarily true that $\|f\|_p \leq \|f\|_q$.

(2) Consider the measure space $(\mathbb{N}, \mu)$ where μ is counting measure, i.e. $\mu(A)$ is the cardinality of A if A is finite, and ∞ if it is infinite. Note that functions from $\mathbb{N}$ to $\mathbb{C}$ are simply complex-valued sequences, and given $\mathbf{x} = (x_1, x_2, \ldots)$,

$$\|\mathbf{x}\|_p^p = \sum_{n=1}^{\infty} |x_n|^p \,,$$

$$\|\mathbf{x}\|_\infty = \sup_n |x_n| \,.$$

$L^p(\mathbb{N}, \mu)$ is usually denoted by ℓ^p. You should verify that in this case, we actually have that $\ell^p \subset \ell^q$ for $1 \leq p < q \leq \infty$ (the opposite of (1)).

(3) In general, one does not have such nice inclusions. For instance, you should be able to come up with examples in $L^p(\mathbb{R}, dx)$ to show that no $L^p(\mathbb{R}, dx)$ space is contained in any other. One result which is true for any measure space, is that if $p < r < q$ then $L^p \cap L^q \subset L^r$.

The last topic I want to touch on is dense subspaces of L^p. In analysis, it is often useful to know that a subset is dense, because oftentimes properties can be easily proved for elements of the subset and then extended to the entire space. You will see many examples of this, especially when you study bounded operators on L^p.

Theorem 7. (1) *Let S be the set of all simple functions whose support has finite measure. Then S is dense in $L^p(\mu)$, $1 \leq p < \infty$. Note that this implies that $L^p \cap L^q$ is dense in L^p for any $1 \leq p, q < \infty$.*

(2) *Consider the case where $(X, \mu) = (\mathbb{R}, dx)$ (the result actually holds more generally for any locally compact Hausdorff space with a Borel measure). Let $C_c(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ with compact support. Then $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R}, dx)$, $1 \leq p < \infty$.*

Remark (1) holds for L^∞, with bounded simple functions (their support may have infinite measure). (2) is not at all true in the L^∞ case. Note that convergence in L^∞ is the same as uniform convergence (except possibly on a set of measure zero). However, we know that if a sequence of continuous functions converges uniformly, the limit function must be continuous. It follows therefore, that the closure of $C_c(\mathbb{R})$ in the L^∞ norm cannot be all of L^∞. In fact, its closure is $C_0(\mathbb{R})$, the set of continuous functions that vanish at infinity (see Rudin 3.16 for more details).

References

[1] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 3rd edition, 1987.