WOMP 2006: ALGEBRAIC TOPOLOGY

MEGAN GUICHARD, MICHAEL SHULMAN

One of the main goals of algebraic topology involves the transformation of topological properties into algebraic ones. In large part, this is accomplished by assigning suitable algebraic invariants to topological spaces. In general, this means assigning some sort of algebraic structure (a group, ring, or some other structure) to each topological space in such a way that continuous maps between spaces induce homomorphisms of the algebraic structures. (In the language of category theory, we are looking for functors from the category of topological spaces and continuous maps to a category such as the category of groups.) These groups (or rings, or ...) are algebraic invariants in the sense that spaces which have "the same shape," in some sense which will be made more precise momentarily, are assigned the same groups. Useful invariants will also allow us to distinguish between genuinely different spaces by assigning them different structures.

One very rough invariant which can be assigned to a space is its number of connected components. This gives you a little bit of information, of course, but it leaves a lot to be desired. The first part of today's talk will be devoted to developing another invariant, called the fundamental group.

Note: whenever I say "map," I mean "continuous function."

1. Homotopy and homotopy equivalence

The first question to be dealt with is: when should two spaces be considered the same? Homeomorphic spaces should, of course, be the same, but it turns out that this is a stronger notion of "sameness" than we will want in many applications. For example, for many purposes, a cylindrical strip of paper is really the same thing as a circle; it's just been fattened up a little. Similarly, we'll regard a space which is formed from another space by thickening it up a little, or continuously deforming it without puncturing it, as having the same shape.

To make this more precise, it is helpful to introduce the language of homotopy. In the following, I will denote the closed unit interval $[0, 1] \subset \mathbb{R}$, and id_X will denote the identity map from a space X to itself: $\mathrm{id}_X(x) = x$.

Definition 1. Let X, Y be two spaces, and $f, g : X \to Y$ two continuous maps between them. Then f and g are **homotopic**, denoted $f \simeq g$, if there is a continuous map

$$H : X \times I \to Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)\ \forall x \in X$. Such a map H is called a **homotopy** from f to g. If additionally $H(a, t)$ is independent of t for all a in some $A \subset X$, then H is called a **homotopy rel** A.

Sometimes a homotopy H is written as a family of maps $h_t : X \to Y$, given by $h_t(x) = H(x,t)$. In this context, a homotopy from f to g is a family such that $h_0 = f$ and $h_1 = g$.

With this language, we can express the idea that X is a "fattened up" version of a subspace $A \subset X$. Let $i : A \hookrightarrow X$ be the inclusion of A into X. A retraction $r : X \to A$ is a map such that $r \circ i : A \to A$ is the identity. Then

Definition 2. $A \subset X$ is a **deformation retraction** of X if there is a retraction $r : X \to A$ such that the composition $i \circ r : X \to X$ is homotopic to the identity map $\mathrm{id}_X : X \to X$.

That is, A is a deformation retraction of X if there is a map r such that $r \circ i = \mathrm{id}_A$ $i \circ r \simeq \mathrm{id}_X$. More generally, we define

Definition 3. Two spaces X and Y are **homotopy equivalent** if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. f and g are called **homotopy equivalences**.

It is fairly easy to check that homotopy equivalence of spaces is an equivalence relation. This is the notion of sameness which we will be using.

2. The fundamental group

We now come to the question of finding suitable algebraic invariants of spaces. One of the easiest to define is the fundamental group, which is defined in terms of loops in a space.

Pick a space X, and fix some point $x \in X$. Then we can consider loops on X based at x; that is, maps $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x$. Since we are considering everything up to homotopy, we are actually interested in the equivalence classes of such maps; given a loop α, we will write $[\alpha]$ to mean the set of loops which are homotopic to α rel $\{0,1\}$. Denote the set of such equivalence classes by $\pi_1(X,x)$. Since all loops start and end at the same place, we can define a multiplication on $\pi_1(X,x)$, as follows. If α and β are loops starting and ending at x, then $\alpha \cdot \beta$ is the map given by following β at double speed, followed by α at double speed:

$$(\alpha \cdot \beta)(t) = \begin{cases} \beta(2t) & 0 \leq t \leq \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

We can then define a multiplication on $\pi_1(X,x)$ by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$. It turns out that multiplication is associative: $\alpha \cdot (\beta \cdot \gamma) \simeq (\alpha \cdot \beta) \cdot \gamma$ and so $[\alpha \cdot (\beta \cdot \gamma)] = [(\alpha \cdot \beta) \cdot \gamma]$ for loops α, β, and γ.

Notice that the constant map at x, $c_x(t) = t \, \forall t \in I$, gives a unit for the multiplication in $\pi_1(X,x)$. Also, given a loop α, traversing α backwards gives a loop which we will write α^{-1}; that is, $\alpha^{-1}(t) = \alpha(1-t)$. You can check that $[\alpha \cdot \alpha^{-1}] = [c_x]$. So we have shown (details will be given in the talk)

Proposition 4. $\pi_1(X,x)$, with multiplication defined as above, is a group.

$\pi_1(X, x)$ is called the fundamental group, or first homotopy group, of X at the basepoint x. ($\pi_0(X)$ is often used to denote the set of path-connected components of X; we will talk a little bit about the higher homotopy groups π_n later.)

3. Dependence on the basepoint

It is natural to ask how much the group $\pi_1(X, x)$ depends on the choice of basepoint x; after all, we started out looking for an invariant of spaces.

Let $x, y \in X$, and suppose $\gamma : I \to X$ is a path with $\gamma(0) = x$ and $\gamma(1) = y$. Then we can define a homomorphism $\theta_\gamma : \pi_1(X, x) \to \pi_1(X, y)$ by $\theta_\gamma([\alpha]) = [\gamma \cdot \alpha \cdot \gamma^{-1}]$. This is well-defined, since if h_t is a homotopy from α to some loop α', then the family $\{\gamma \cdot h_t \cdot \gamma^{-1}\}$ gives a homotopy from $\gamma \cdot \alpha \cdot \gamma^{-1}$ to $\gamma \cdot \alpha' \cdot \gamma^{-1}$. Furthermore, since $[\gamma \cdot \alpha \cdot \beta \cdot \gamma^{-1}] = [\gamma \cdot \alpha \cdot \gamma^{-1}] \cdot [\gamma \cdot \beta \cdot \gamma^{-1}]$, θ_γ is a homomorphism of groups. You can check that $\theta_{\gamma^{-1}}$ gives an inverse homomorphism, and so θ_γ is actually an isomorphism of groups.

Therefore, if X is a path-connected space (given any two points x_0, x_1, there exists a path from x_0 to x_1), the isomorphism type of $\pi_1(X, x)$ is independent of the choice of x, and is often written just $\pi_1(X)$. However, it is important to remember that there is a choice of basepoint inherent in the computation of π_1; often you will need to remember what this basepoint is in order to do calculations.

Sometimes it is convenient to work in the category of basepointed spaces, having as objects pairs (X, x); a morphism $(X, x) \to (Y, y)$ is a map $\alpha : X \to Y$ such that $\alpha(x) = y$. In this category, there is a natural choice of basepoint when computing π_1.

Alternately, people sometimes choose to work with the **fundamental groupoid** $\Pi_1(X)$ of a space X. This assigns a *category* to each topological space X. Specifically, the objects of $\Pi_1(X)$ are the points of X, and the morphisms from $x \in X$ to $y \in X$ are the homotopy classes of maps from x to y (homotopy rel endpoints). The notion of composition of paths is the same as before; notice that $\alpha \cdot \beta$ is defined whenever $\beta(1) = \alpha(0)$. For any $x \in X$, the group of endomorphisms in $\Pi_1(X)$ from x to x is just the group $\pi_1(X, x)$. Also, the argument above shows that every morphism in Π_1 is actually an isomorphism, which means that Π_1 is a groupoid.

When the space X is connected, the fundamental groupoid does not really give any more information that the fundamental group, because all the endomorphism groups are isomorphic. However, it can be more useful for dealing with spaces which are not connected, because it gives a way of capturing information about all the different components, rather than just one component.

4. Homotopy invariance

Having developed the fundamental group, an important question is: is it an invariant in the sense above? That is, do homotopy equivalences of spaces induce isomorphisms of fundamental groups?

First, notice that a map $f : X \to Y$ does indeed induce a homomorphism of fundamental groups, usually written $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$, defined by composing loops $I \to X$ with f. That is, f_* takes the equivalence class $[\alpha]$ to $[f \circ \alpha]$.

Proposition 5. If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.

We will prove this in the talk.

5. Some basic fundamental group computations

In order for an invariant to be of practical use, it should be relatively easy to compute; it turns out that, with a few tools, the fundamental groups of a very large number of spaces can be computed.

Proposition 6. Let $*$ denote the space consisting of a single point. Then $\pi_1(*) = \mathbf{1}$, the one-element group.

Proof. There is only one map $I \to *$. □

Corollary 7. If X is contractible (homotopy equivalent to a point), then $\pi_1(X)$ is the trivial group. In particular, $\pi_1(\mathbb{R}) = \mathbf{1}$.

Mathematicians often want to talk about spaces with trivial π_1, and so they have their own name:

Definition 8. A path-connected space X having $\pi_1(X) = \mathbf{1}$ is a **simply connected** space.

Theorem 9. Let S^1 denote the circle. Then $\pi_1(S^1) \cong \mathbb{Z}$.

The isomorphism is given by taking an equivalence class of paths to the number of times they wind around the circle; a careful proof of this requires the language of covering spaces, which will be covered in the second half of the talk.

Let X and Y be spaces, and $X \times Y$ the Cartesian product (with the product topology). Then, by definition, a map $I \to X \times Y$ is the same thing as a pair of maps $I \to X$ and $I \to Y$; that is, every $\alpha : I \to X \times Y$ is given by $\alpha(t) = (\alpha_1(t), \alpha_2(t))$. The same is true of homotopies of such maps, and so it follows that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. In particular,

Example 10. Let T^2 be the torus $S^1 \times S^1$. Then $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.

It is sometimes also possible to calculate the fundamental group of quotient spaces. For example,

Example 11. $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, the two-element group.

More generally, as we will see in the second half of this talk, if X is a simply connected space, and G is a sufficiently nice group acting freely on X, then the quotient space X/G has $\pi_1(X/G) = G$.

6. The van Kampen theorem

The most useful method of calculating fundamental groups, however, involves the van Kampen theorem, which allows you to compute $\pi_1(X)$ from knowledge about π_1 of some subsets of X, together with knowledge about how these subsets fit together.

The van Kampen theorem, in its full generality, can be rather confusing at first sight. One special case is

Theorem 12. *Let X be a space, and let $U, V \subset X$ be open subsets such that $U \cup V = X$ and such that U and V are path-connected, and $U \cap V$ is simply connected. Then $\pi_1(X)$ is isomorphic to the free product $\pi_1(U) * \pi_1(V)$.*

Recall that, if G and H are groups, then the **free product** $G*H$ has as elements finite strings $\beta_1 h_1 \beta_2 h_2 \dots \beta_n h_n$, with $\beta_i \in G$, $h_i \in H$; the group operation is concatenation of strings, modulo multiplication in G and H.

Example 13. $\pi_1(S^1 \vee S^1)$, the figure 8, is $\mathbb{Z} * \mathbb{Z}$. More generally, a wedge of k circles has fundamental group the free product of k copies of $\mathbb{Z}$ (also known as the free group on k generators).

Slightly more generally, if we drop the requirement that $U \cap V$ be simply connected, we can still describe $\pi_1(X)$. In this case, the inclusions $i : U \cap V \to U$ and $j : U \cap V \to V$ induce homomorphisms $i_* : \pi_1(U \cap V) \to \pi_1(U)$ and $j_* : \pi_1(U \cap V) \to \pi_1(V)$; it turns out that $\pi_1(X)$ is the quotient of $\pi_1(U) * \pi_1(V)$ given by identifying $i_*([\alpha])$ and $j_*([\alpha])$ for $[\alpha] \in \pi_1(U \cap V)$.

Theorem 14. *Let $U, V \subset X$ be subspaces with $U \cup V = X$, and suppose $U, V, U \cap V$ are all path-connected. Let $R < \pi_1(U) * \pi_1(V)$ be the normal closure of the group generated by elements of the form $(i_*([\alpha]))^{-1}(j_*([\alpha]))$, for $[\alpha] \in \pi_1(U \cap V)$. Then*

$$\pi_1(X) = \pi_1(U) * \pi_1(V)/R = \pi_1(U) *_R \pi_1(V).$$

Example 15. $\pi_1(S^2) = \mathbf{1}$; to see this, write S^2 as the union of its two hemispheres. More generally, $\pi_1(S^n) = \mathbf{1}$ for $n \geq 2$.

From a categorical perspective, this version of the theorem is saying that the diagram

$$\begin{array}{ccc} \pi_1(X) & \longleftarrow & \pi_1(V) \\ \uparrow & & \uparrow \\ \pi_1(U) & \longleftarrow & \pi_1(U \cap V) \end{array}$$

is a pushout. Notice that

$$\begin{array}{ccc} X & \longleftarrow & V \\ \uparrow & & \uparrow \\ U & \longleftarrow & U \cap V \end{array}$$

is also a pushout in the category of topological spaces; so van Kampen's theorem is saying that in this situation π_1 takes pushouts to pushouts.

A pushout is a special type of colimit, which gives rise to a more general version of the theorem: this calculates $\pi_1(X)$ given a cover $\{U_i\}$ of path-connected open sets, each containing the basepoint of X, whose pairwise intersections are again path-connected. In this case, X can be written as the colimit of a diagram given by the U_i and inclusions of their intersections, and van Kampen's theorem says that $\pi_1(X)$ is the corresponding colimit of the induced diagram of $\pi_1(U_i)$. This translates into saying that $\pi_1(X)$ is a quotient of the free product of the groups $\pi_1(U_i)$. For more details, see Hatcher's book for a statement which spells out $\pi_1(X)$ as a quotient of a free product, or May's book for a more categorical treatment. May's book also has a statement of the theorem for fundamental groupoids, for the very categorically-minded; this can be used to calculate $\pi_1(X)$ given a cover $\{U_i\}$ whose pairwise intersections are not path-connected. (For example, it can be used to calculate π_1 of the circle.)

7. Other invariants

Recall that the very first invariant we mentioned was π_0, the set of path components of a space. π_1 does a slightly better job of distinguishing spaces from one another; it tells us, for example, that $\mathbb{R}$, S^1, and T^2 are not homotopy equivalent. However, it fails to distinguish between the higher dimensional spheres, and so other invariants are needed.

The easiest to define are the higher homotopy groups π_n. $\pi_1(X)$ can be thought of as the group of homotopy classes of basepointed maps $(S^1, s) \to (X, x)$; $\pi_n(X)$ is defined to be the group of homotopy classes of basepointed maps $(S^n, s) \to (X, x)$ (the group operation can be defined in a way similar to the operation on π_1). Unfortunately, the higher homotopy groups are very hard to compute in general. Calculating the higher homotopy groups of spheres shows every sign of being a permanently open question in algebraic topology.

There are also invariants, called the homology and cohomology groups of a space, which are easier to compute but do not distinguish between spaces as well, and various "generalized (co)homology theories," such as K-theory. Some of these will probably be covered in the algebraic topology course.

8. Group actions and quotients

Consider a torus. You may be familiar with the fact that we can represent it as a square with opposite sides identified. If we glue one pair of opposite sides, we get a cylinder, and then if we glue the remaining pair of sides, we get the torus.

On the other hand, consider the point of view of a 2-dimensional being living inside the torus. If she looks in any of the four directions, then the light coming to her eyes has passed all the way around the torus, so she sees the back of her own head. So from her perspective, it looks as if the square has an extra copy of itself on each side, and so on, giving an infinite grid of squares, all of which are 'really' the same square.

From a mathematical point of view, this means that we can obtain a torus T^2 by starting with the plane $\mathbb{R}^2$ and quotienting by the equivalence relation

$$(x, y) \sim (x + m, y + n) \qquad (n, m) \in \mathbb{Z} \times \mathbb{Z}.$$

Observe that $\mathbb{Z} \times \mathbb{Z}$ is also known as $\pi_1(T^2)$; this is clearly not a coincidence!

A similar thing happens with the circle: we can obtain S^1 by starting with $\mathbb{R}$ and quotienting by

$$x \sim y + n \qquad n \in \mathbb{Z}.$$

Once again, $\mathbb{Z}$ is also known as $\pi_1(S^1)$.

These quotients are actually a special sort of quotient: they are generated by an 'action' of the group in question.

8.1. **Group actions.** Recall the following definition from abstract algebra:

Definition 16. Let G be a group and X be a set. An **action** of G on X is a function $G \times X \to X$, written $g \cdot x$, such that

(1) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$
(2) $e \cdot x = x$

The **quotient** (or **orbit space**) of X by the action of G is the set X/G of equivalence classes of X under the equivalence relation

$$x \sim g \cdot x \qquad g \in G.$$

A set with a G-action is sometimes called a G**-set**. What we need for our purposes is a topological version of this, as follows.

Definition 17. Let X be a topological space and G a group. An **action** of G on X is a function $G \times X \to X$, written $g \cdot x$, such that

(1) For each $g \in G$, the function $(g \cdot -) : X \to X$ is continuous[1];
(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$
(3) $e \cdot x = x$

The **quotient** (or **orbit space**) of X by the action of G is a space X/G defined as the quotient space of X by the equivalence relation

$$x \sim g \cdot x \qquad g \in G.$$

A space with a G-action is sometimes called a G**-space**.

Example 18. $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$, and $\mathbb{R}/\mathbb{Z} \cong S^1$.

[1]In fact, since G is a group, each function $(g \cdot -)$ has a continuous inverse $(g^{-1} \cdot -)$, so it is actually a homeomorphism.

Example 19. Let $\mathbb{Z}/2$, the group with two elements, act on the n-sphere S^n by reflection in the origin. In other words, if

$$S^n = \{(x_0, \dots, x_n) | \sum x_i^2 = 1\}$$

and a is the non-identity element of $\mathbb{Z}/2$, then

$$a \cdot (x_0, \dots, x_n) = (-x_0, \dots, -x_n).$$

Then $S^n/(\mathbb{Z}/2) \cong \mathbb{RP}^n$.

There are a couple of more abstract ways to rephrase the definition of a group action. Firstly, let $\mathrm{Aut}(X)$ denote the group of homeomorphisms from X to itself, with the group structure given by composition. Then an action of G on X is the same as a group homomorphism from G to $\mathrm{Aut}(X)$.

Secondly, let BG denote the category with a single object $*$ and morphisms $BG(*, *) = G$, with composition given by the group structure of G. Then an action of G on X is the same as a functor from BG to the category **Top** of topological spaces and continuous maps which sends the object $*$ to X.

This last description is useful because it allows us to give a universal characterization of the quotient. Namely, the quotient space X/G is simply the colimit, in **Top**, of the above functor. It also makes the generalization to group actions on other objects straightforward. For example, a *set* with a G-action, as defined above, is simply a functor from BG to **Set**. Similarly we have G-manifolds, G-schemes, and so on, simply by changing the target category.

8.2. **Fundamental groups of quotients.** In the examples of S^1, T^2, and $\mathbb{RP}^2$, the fundamental group of the quotient was the same as the group involved in the action. This is true in general, although we require a few conditions. First of all, the space we start with ($\mathbb{R}$, $\mathbb{R}^2$, or S^n) should be simply connected; otherwise its fundamental group will get in the way.

The second requirement is more subtle. Consider $\mathbb{Z}/2$ acting on $\mathbb{R}^2$ by a 180 degree rotation. The quotient is a 'cone' which is in fact homeomorphic to $\mathbb{R}^2$ again, and hence still simply connected. What's gone wrong here is that the action has a *nontrivial fixed point*: the nonidentity element a sends the origin to itself. To get around this, we require our actions to be 'free'.

Definition 20. An action of a group G on a space X is **free** if whenever $g \in G$ is not the identity, then $g \cdot x \neq x$ for all $x \in X$.

Actually, we need a slightly stronger condition, but 'morally' it is still a 'freeness' condition.

Definition 21. We will call an action of a group G on a space X **continuously free**[2] if any $x \in X$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for all nonidentity $g \in G$.

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[2]This terminology is nonstandard. Hatcher calls such an action a **covering space action**, in light of the results in §10. Other authors may call it something like 'free and properly discontinuous'.

Theorem 22. If X is path-connected, simply-connected, and locally path-connected[3], and G acts continuously freely on X, then $\pi_1(X/G) \cong G$.

Sketch of Proof. Let $p\colon X \to X/G$ be the quotient map, let x_0 be a basepoint in X, and let $y_0 = p(x_0)$ its image in X/G. We define maps in either direction, and leave it to you to check that they are inverse isomorphisms.

Given $g \in G$, choose a path in X from x_0 to $g \cdot x_0$; this gives a loop in X/G based at y_0, so an element of $\pi_1(X/G, y_0)$. Such paths exist since X is path-connected, and the choice of path doesn't matter since X is simply-connected.

Conversely, given an element $[\alpha] \in \pi_1(X/G, y_0)$, choose a neighborhood U of x_0 which is disjoint from all its images under the G-action. Then p maps U homeomorphically to a neighborhood of y_0, so the first part of the path α can be lifted from $p(U)$ to U. Repeat this all the way along the loop α to lift it to a loop $\widetilde{\alpha}$ in X. (Hey, I said this was a sketch, right?) The second endpoint of $\widetilde{\alpha}$ is in $p^{-1}(y_0)$, hence is equal to $g \cdot x$ for some $g \in G$, which is unique since the action is free. $\square$

There are actually ways to make a similar result true *without* needing the action to be free, but they require either introducing a more general kind of object, called an 'orbifold' or 'stack', to be the quotient, or using a more complicated construction called the 'homotopy orbit space'.

There are also more general versions of this theorem not requiring X to be connected (in which case you need to use the fundamental group*oid* instead of the fundamental group) or simply connected (in which case you get involved in group extensions).

9. Universal covers

Now we ask the natural question: can we go the other way? Given a space Y with $\pi_1(Y) = G$, can we construct a simply connected space X on which G acts such that $X/G \cong Y$? After all, this is what we did for T^2 and S^1. The answer is yes, for nice enough Y, and the procedure is straightforward.

Think about the torus. The way we got a plane was by saying that someone living in the torus would see one different copy of an object for each way in which light could travel from that object to her eyes; in other words, for each *path* from the location of that object to her eyes (the basepoint). So really what we're doing is taking one copy of each point x for each path from the basepoint to x. This motivates the following construction.

Let Y be a space with basepoint y_0 and let $\widetilde{Y}$ denote the set of endpoint-preserving-homotopy classes of paths in Y starting at y_0. Let $p\colon \widetilde{Y} \to Y$ take each equivalence class of paths to its second endpoint. In order for this map to be surjective, we clearly need Y to be path-connected; no surprises there. Now let's try to topologize $\widetilde{Y}$.

Let $[f]$ be an element of $\widetilde{Y}$, with $p([f]) = f(1) = y$, and let U be some neighborhood of y in Y. We'd like to lift U to a neighborhood of $[f]$ in $\widetilde{Y}$ which maps homeomorphically onto it. To do this, we need to be able to extend f essentially

[3]This means that the topology on X has a basis of path-connected open sets.

uniquely to a path from y_0 to y' for each $y' \in U$. Certainly this is possible if U is contractible, since in this case, we can connect y to y' by an essentially unique path and join it onto the end of f. Thus, it suffices to assume that Y is **locally contractible** (has a basis of contractible opens)[4]. We can then define the set $\widetilde{U}$, consisting of the equivalence classes of all these paths, to be open in $\widetilde{Y}$, and as U ranges over a basis of contractible opens, this will generate a topology on $\widetilde{Y}$.

Theorem 23. If Y is path-connected, locally path-connected, and semi-locally simply-connected, then $\widetilde{Y}$, as defined above, is path-connected, simply-connected, and locally path-connected, and $G = \pi_1(Y, y_0)$ acts continuously freely on $\widetilde{Y}$ with quotient Y.

Sketch of Proof. The action of G on $\widetilde{Y}$ is straightforward: just precompose a path with a loop at y_0. Since π_1 and $\widetilde{Y}$ are both defined using homotopy classes of maps, this action is free, and the conditions on Y make it continuously free (insert waving of hands). Finally, since the action preserves the endpoint of paths, and any two paths with the same endpoint differ by a loop, it is clear that the quotient is Y. □

The space $\widetilde{Y}$ is called the **universal cover** of Y, for reasons we'll see in a moment.

Example 24. We've already seen that $\widetilde{S^1} \cong \mathbb{R}$, $\widetilde{T^2} \cong \mathbb{R}^2$, and we can check too that $\widetilde{\mathbb{RP}^n} \cong S^n$.

Example 25. Let Y be the wedge of two circles (the 'figure-eight' space). Its fundamental group is the free group F_2 on two generators. Its universal cover is a graph whose vertices are the elements of F_2: it is a '4-valent tree'. This is also called the **Cayley graph** of F_2; see Hatcher for more details.

10. General covering spaces

The maps $p\colon \widetilde{Y} \to Y$ we have been considering are actually a very special kind of quotient map; in all cases the space downstairs is covered by open sets which lift to a collection of disjoint homeomorphic images upstairs. This type of map is so important that it has a name.

Definition 26. A **covering map** is a continuous map $p\colon X \to Y$ which is surjective, and such that each $y \in Y$ has an open neighborhood $U \subset Y$ such that $p^{-1}(U)$ is a disjoint union of components which are open in X and mapped homeomorphically to U by p. X is called the **covering space** and Y is called the **base space**.

Example 27. All our examples so far are covering maps. In fact, for any Y satisfying the conditions of Theorem 23, the space $\widetilde{Y}$ is a covering space.

Example 28. The map $z \mapsto z^2$ from $S^1 \to S^1$, which wraps the circle around itself twice, is a covering map. More generally, $z \mapsto z^n$ is a covering map.

[4]Actually, the slightly weaker condition of being locally path-connected and **semi-locally simply connected** suffices. See May or Hatcher for the definition.

Example 29. The map from an infinite cylinder to a torus, which wraps it around infinitely many times in one direction, is a covering map. Similarly, the map from a torus to itself which wraps it around itself n times in one direction is a covering map.

Example 30. On page 58, Hatcher has a lot of examples of covering spaces of the figure-eight space.

We observe that, in general, the covering space need not be simply-connected. However, its fundamental group $\pi_1(X)$ maps into the fundamental group $\pi_1(Y)$ of the base as a subgroup which depends on the covering map chosen.

For example, the map $z \mapsto z^2$ corresponds to the subgroup $2\mathbb{Z} \subset \mathbb{Z}$, and $z \mapsto z^n$ corresponds to $n\mathbb{Z}$. The cylinder mapping onto the torus corresponds to

$$\mathbb{Z} \times \{0\} \subset \mathbb{Z} \times \mathbb{Z}$$

and the double cover of the torus by itself corresponds to

$$\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}.$$

Theorem 31. Let Y satisfy the hypotheses of Theorem 23. Then for every subgroup $H \subset \pi_1(Y)$, there is a unique (up to isomorphism) path-connected covering space $p\colon Y_H \to Y$ such that $p_*(\pi_1(Y_H)) = H$.

Remark 32. I've stated this kind of sloppily; to be more precise we should equip both spaces X with compatible basepoints and specify that the isomorphisms of covering spaces preserve basepoints. Two pointed covering spaces which are isomorphic in a way *not* preserving basepoints correspond to subgroups of $\pi_1(Y)$ which are not equal but *conjugate*. See May or Hatcher for details.

Sketch of Proof. To construct Y_H, start with the universal cover $\widetilde{Y}$ and define two points $[\alpha]$ and $[\beta]$ in $\widetilde{Y}$ (which, recall, are equivalence classes of paths in Y) to be equivalent when they have the same endpoint, and their composite $[\beta\alpha]$ lies in $H \subset \pi_1(Y)$. Let Y_H be the quotient of $\widetilde{Y}$ by this equivalence relation. The rest is easy to check. □

11. Path-lifting and non-connected spaces

Theorem 31 is great, but it only deals with *connected* spaces. However, not only are there oodles of non-connected spaces, there are oodles of non-connected coverings of connected spaces. For instance, we can map two circles to a circle, or two copies of any space to itself. In order to deal with these, we need to reformulate the classification theorem a bit. Here our treatment will get even sketchier than before; the reader is encouraged to work out the details.

11.1. **Path Lifting.** Let $p\colon X \to Y$ be a covering map, y_0 a basepoint in Y, and let $F = p^{-1}(y_0)$. Way back in Theorem 22, we constructed a group element from an element of $\pi_1(Y)$ by lifting the path downstairs to a path upstairs. We can do this sort of lifting for any covering map in the same way: lift the path progressively in each fundamental neighborhood, and patch them together; thus we have:

Proposition 33. Given a path α in Y starting at y_0, and a point $x \in F$, there is a unique path $\widetilde{\alpha}$ in X starting at x such that $p \circ \widetilde{\alpha} = \alpha$. Moreover, homotopic paths lift to homotopic paths.

11.2. **Non-connected covers.** Assume for now that the base space Y is still path-connected (we will lift this restriction next). We can use path lifting to construct an action of $\pi_1(Y, y_0)$ on the fiber F, for any covering space X. Given $[\alpha] \in \pi_1(Y, y_0)$ and $x \in F$, lift α to a path starting at x in X and let $[\alpha] \cdot x$ be its second endpoint.

Note that since p is a covering map, F is a discrete space, hence really just a set. We now observe that X is path-connected precisely when the action of $\pi_1(Y)$ on F is **transitive** (every point is mapped to every other point by some group element), and make use of the following result:

Proposition 34. For any group G, there is a bijection between subgroups of G and isomorphism classes of transitive G-sets. The subgroup H corresponds to the coset space G/H with the obvious action.

Thus, using the action of $\pi_1(Y)$ on F, we have recovered the correspondence between subgroups of $\pi_1(Y)$ and connected covering spaces. However, now it is clear that the not-necessarily-connected covering spaces should be classified by sets with a not-necessarily-transitive action. This is, in fact, the case.

Moreover, it can be extended to classify maps between covering spaces. Formally speaking, we have:

Theorem 35. Let Y be connected, locally path-connected, and semi-locally simply-connected. Then there is an equivalence of categories between (1) the category of sets with a $\pi_1(Y)$-action and maps that preserve the action, and (2) covering spaces of Y and maps between them over Y.

11.3. **Non-connected bases.** Now, what if the base space Y is not connected? We've already seen that for non-connected spaces, instead of the fundamental group $\pi_1(Y)$, it is more appropriate to consider the fundamental *groupoid* $\Pi_1(Y)$. Moreover, if we recall that a G-set is the same as a functor from BG to **Set**, it is clear that the correct generalization to groupoids is just to consider functors from the groupoid to **Set**. The result is:

Theorem 36. Let Y be locally path-connected and semi-locally simply-connected. Then there is an equivalence of categories between (1) the category of functors from $\Pi_1(Y)$ to **Set** and natural transformations, and (2) covering spaces of Y and maps between them over Y.

If you've followed me thus far, then perhaps, you may find this version of the theorem *easier* to visualize than the simpler versions for connected spaces, as I do. Recall that the objects of $\Pi_1(Y)$ are the points of Y and the morphisms are homotopy classes of paths. Then the functor corresponding to a given covering space $p\colon X \to Y$ simply sends each point $y \in Y$ to the fiber $p^{-1}(y)$ over y, and each class of paths $[\alpha]$ to the function obtained by lifting the path α and looking at the endpoint. (It is a little less straightforward, however, to see that *any* such functor can be realized by a covering space.)

12. Vistas

What we've done, finally, is to classify something *over* Y (namely, covering spaces) in terms of maps *from* something related to Y (namely, its fundamental groupoid) *to* the collection of fibers (namely, sets). This is a fundamental theme running through topology and geometry: the use of *classifying maps* and *classifying spaces*. Here are a few more examples:

(1) Covering spaces over Y can also be classified by homotopy classes of maps from Y itself to $F_n(\mathbb{R}^\infty)/\Sigma_n$, the 'configuration space of n-element sets.' See [1] for a development of this idea.
(2) Real vector bundles over a nice space Y can be classified by homotopy classes of maps from Y to a *Grassmannian*, the 'configuration space of hyperplanes'.
(3) Principal bundles with fiber G over Y can be classified by homotopy classes of maps from Y into a space BG constructed from G, called the 'classifying space' of G.

These classifying maps are one starting point for *cohomology*, another algebraic invariant of a space, which also (in this way) tells us about what different sorts of gadgets can live over it. See reference [2] for a lot more about this idea.

References

[1] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. A different and very modern take on the subject. Many readers may find their treatment of homology perverse! However, they introduce notions like fibrations and cofibrations, which are essential to more advanced homotopy theory, and discuss covering spaces and vector bundles from the point of view of more general classifying spaces.

[2] John C. Baez and Michael Shulman. Lectures on n-categories and cohomology. Available on the arXiv at http://arxiv.org/abs/math.CT/0608420. No doubt many people will laugh at me for including this in an introductory talk on algebraic topology, but the first section ("The Basic Principle of Galois Theory") is actually a good and pretty accessible introduction to the relationship between covering spaces, groupoids, and classifying spaces. In the interests of full disclosure, I should say that I had nothing to do with the writing of that particular section.

[3] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. A pretty comprehensive introduction to algebraic topology, and a standard reference and textbook. Can be difficult to learn from because he frequently fails to emphasize key points and separate the essentials from the details.

[4] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Very concise, but worth having. Excels at explaining new and better ways to look at things you already understand. Written for teachers of algebraic topology, not as an introductory textbook.