REVIEW OF POINT-SET TOPOLOGY I
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The foundational material contained in this talk is meant to be exclusively review. To supplement the ideas covered here, a great reference is *Topology: A First Course* by James R. Munkres.

1. A Topological Space

Definition 1.1. A **topology** on a set X is a collection of subsets $\mathcal{T}_X$ called *open sets* such that
(1) X and $\emptyset$ are open,
(2) arbitrary unions of open sets are open, and
(3) finite unions of open sets are open.

We refer to the pair $(X, \mathcal{T}_X)$ as a **topological space**.

Definition 1.2. A subset $Y \subseteq X$ is **closed** if its complement in X is open.

Example 1.3. Trivial and discrete topologies.

2. Metric Spaces

Metric spaces are our first examples of topological spaces.

Definition 2.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that
(1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(3) $d(x, z) \geq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The pair (X, d) is called a **metric space**.

Example 2.2. The standard examples of metric spaces are:
(1) $(\mathbb{R}^n, d)$, where $d(x, y) := ||x - y|| := \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$ is the **Euclidean norm**.
(2) $(\mathbb{R}^n, d_p)$, where $d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$ is the p**-norm**.
(3) $(\mathbb{R}^n, \rho)$, where $\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ is the **square metric**.

If (X, d) is a metric space, we can define a topology on X as follows. We say that $U \subseteq X$ is open if for all $x \in U$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\} \subset U$. In this case we say that d *induces* the topology on X.

Fact 2.3. The metrics d, d_p, and ρ all induce the same topology on $\mathbb{R}^n$. We call this the *standard topology* on $\mathbb{R}^n$ and sometimes denote it by $\mathcal{T}_d$.

Definition 2.4. A topological space $(X, \mathcal{T}_X)$ is **metrizable** if there exists a metric d that induces the topology $\mathcal{T}_X$.

From now on we will omit the notation $\mathcal{T}_X$ and refer to the topological space X.

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Definition 2.5. We say that $x \in A \subseteq X$ is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Definition 2.6. The union of a subset $A \subseteq X$ and its limit points is called the **closure** of A, denoted $\overline{A}$. Equivalently, $\overline{A}$ is the smallest closed set in X that contains A.

Fact 2.7 (Sequence Lemma)**.** Let X be metrizable and $A \subset X$. A sequence $\{x_n\} \subset A$ converges to $x \iff x \in \overline{A}$.

Definition 2.8. A subset $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

Example 2.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

3. More Topological Spaces

Example 3.1. We can build other topological spaces out of the ones that we know as follows:

(1) Given $(X, \mathcal{T}_X)$ and $Y \subset X$, define $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$. We call $(Y, \mathcal{T}_Y)$ the **induced** or **subspace topology**.
(2) Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we can define a topology on $X \times Y$ by forming the topology $\mathcal{T}_p$ *generated* by all sets of the form $U \times V$, where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Here, $W \in \mathcal{T}_p$ if for all $x \in W$ there exists a set $U \times V$ such that $x \in U \times V \subset W$, $U \in \mathcal{T}_X$, and $V \in \mathcal{T}_Y$. The resulting topology is called the **product topology** on $X \times Y$.
(3) $(\mathbb{R}^\omega, \mathcal{T}_p)$ where $\prod_{i=1}^{\infty} U_i \in \mathcal{T}_p$ if $U_i \in \mathcal{T}_d$ and only finitely many of the U_i are not all of $\mathbb{R}$, defines the product topology on countably many copies of $\mathbb{R}$.
(4) $(\mathbb{R}^\omega, \mathcal{T}_b)$ where $\prod_{i=1}^{\infty} U_i \in \mathcal{T}_b$ if $U_i \in \mathcal{T}_d$, defines the **box topology** on $\mathbb{R}^\omega$.

Definition 3.2. Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if for each open subset $V \subset Y$, the set $f^{-1}(V) \subset X$ is an open subset.

Fact 3.3. If $f : X \to \prod_{i \in I} Y_i$ is given by $f(x) = (f_i(x))$ coordinatewise where $\prod Y_i$ has the product topology, then f is continuous $\iff$ each f_i is continuous.

This fact illustrates one of the many advantages of using the product topology over the (at first, seemingly more natural) box topology.

Definition 3.4. Given a topological space X, a set $\mathcal{B}$ of open sets is called a **basis** for $\mathcal{T}_X$ if every set in $\mathcal{T}_X$ is a union of elements in $\mathcal{B}$.

Example 3.5. Here are some examples of bases for some familiar topological spaces:

(1) $\{B_q(x) : q \in \mathbb{Q}^n\}$ is a basis for $\mathbb{R}^n$ with the standard topology.
(2) $\mathcal{B}_Y := \{U \cap Y : U \in \mathcal{B}_X\}$ is a basis for the subspace topology on Y if $\mathcal{B}_X$ is a basis for X.
(3) $\mathcal{B}_{X \times Y} := \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ is a basis for the product topology on $X \times Y$.

Definition 3.6. Given a topological space $(X, \mathcal{T})$, a set $\mathcal{S} \subset \mathcal{T}$ is called a **subbasis** for the topology $\mathcal{T}$ if every open set is a union of finite intersections of sets in $\mathcal{S}$.

Fact 3.7. Given a collection of subsets $\mathcal{S}$ of X, there exists a unique topology on X such that $\mathcal{S}$ is a subbasis, namely the topology generated by $\mathcal{S}$.

Example 3.8. Define the projection maps $\pi_j : \prod_{i \in I} X_i \to X_j$ by $\pi_j((x_i)_{i \in I}) = x_j$. Then the set $\bigcup_{j \in I} \{\pi_j^{-1}(U_j) : U_j \in \mathcal{T}_{X_j}\}$ defines a subbasis that generates the product topology.

4. Compactness

The notion of a compact topological space is meant to generalize the desirable properties of a closed interval $[a, b] \subset \mathbb{R}$ to arbitrary topological spaces.

Definition 4.1. A topological space X is **compact** if every open cover has a finite subcover.

Example 4.2. Let's consider $\mathbb{R}$ with the standard topology.

(1) $[a, b]$ is compact.
(2) $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.
(3) $\mathbb{R}$ is not compact.
(4) $(0, 1]$ is not compact.

Proposition 4.3. The image of a compact set under a continuous map is compact.

Definition 4.4. A topological space X is called **Hausdorff** if for every $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Proposition 4.5. All closed subsets of a compact space are compact. The converse holds if the space is also Hausdorff.

Proposition 4.6. Let $f : X \to Y$ be a continuous bijection, and suppose X is compact and Y is Hausdorff. Then f is a *homeomorphism*; *i.e.*, f^{-1} is also continuous.

Definition 4.7. In the context of metric spaces, there are several more tangible characterizations of compactness.

(1) A topological space X is **sequentially compact** if every sequence has a convergent subsequence.
(2) A topological space X is **limit point compact** if every infinite subset has a limit point.

Theorem 4.8. Let X be metrizable. The following are equivalent:

(1) X is compact.
(2) X is sequentially compact.
(3) X is limit point compact.

Theorem 4.9. X and Y are compact $\iff$ $X \times Y$ is compact.

Corollary 4.10 (Heine-Borel)**.** A subset $X \subset \mathbb{R}^n$ is compact $\iff$ X is closed and bounded.

Theorem 4.11 (Tychanoff)**.** Arbitrary products of compact sets are compact in the product topology.

Tychanoff's Theorem is deep. The proof uses the ideas of filters, together with yet another alternative characterization of compactness.

Definition 4.12. A collection $\mathcal{C}$ of subsets of X satisfies the **finite intersection property** if every finite subcollection of sets in $\mathcal{C}$ has a nonempty intersection.

Theorem 4.13. X is compact $\iff$ for every collection $\mathcal{C}$ of closed sets satisfying the finite intersection property, $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

When dealing with function spaces, we can use the notion of compactness to define yet another useful topology.

Definition 4.14. Let X and Y be topological spaces, and let $K \subset X$ be compact and $U \subset Y$ be open. The sets $\mathcal{C}(K, U) := \{f \in \mathcal{C}(X, Y) : f(K) \subset U\}$ form a subbasis for the **compact-open topology**. Here, $\mathcal{C}(X, Y)$ denotes the set of continuous functions from X to Y.

5. Local Compactness

Definition 5.1. A topological space X is **locally compact at** x if there exists a compact set $K \subset X$ that contains a neighborhood of x. X is said to be **locally compact** if it is locally compact at each of its points.

Example 5.2. Here are several examples illustrating local compactness:

(1) $\mathbb{R}^n$ is locally compact.
(2) $\mathbb{R}^\omega_p$ is not locally compact.

A very nice, often studied class of topological spaces are the locally compact Hausdorff spaces.

Theorem 5.3. *X is locally compact Hausdorff $\iff$ there exists a set $Y \supset X$ such that*

(1) *$Y - X$ is a single point, and*
(2) *Y is compact Hausdorff.*

Moreover, Y is unique up to homeomorphism and is called the **one-point compactification** *of X.*

Example 5.4. The one-point compactification of $\mathbb{R}$ is the circle S^1. The one-point compactification of R^2 is $S^2 \simeq \mathbb{C} \cup \{\infty\}$, the *Riemann sphere*.

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