WOMP 2005: HOMOLOGICAL ALGEBRA

Kate Ponto
September 20, 2005

1. Complexes, Homology, and Cohomology

A chain complex is a sequence of homomorphism of Abelian groups

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \to 0$$

where $\partial_n \circ \partial_{n+1} = 0$ for each $n \in \mathbb{N}$. So the image of ∂_{n+1} is contained in the kernel of ∂_n. The maps ∂_n are called differentials. Notation: $C = \{C_n, \partial_n\}$.

Example 1. Some examples of chain complexes. The first nonzero group on the right is C_0.

(1) $\ldots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. Here $C_1 = C_0 = \mathbb{Z}$ and all other groups are zero. All homomorphisms are zero.
(2) $\ldots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0$. These are the same groups as above, but the map $\mathbb{Z} \to \mathbb{Z}$ is the identity map.
(3) $\ldots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$. These are the same groups as above, but the map $\mathbb{Z} \to \mathbb{Z}$ is the multiplication by 2 map.
(4) $\ldots \to 0 \to 0 \to \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/8 \to 0$
(5) $\ldots \to 0 \to 0 \to \mathbb{Z}/4 \xrightarrow{\times 4} \mathbb{Z}/8 \to 0$
(6) $\ldots \to 0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0$ where $f(1) = (1,1)$ and $g(a,b) = a - b$. Then $g \circ f(a) = g(a,a) = a - a = 0$.
(7) $\ldots \to 0 \to \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/4 \to 0$

We define the n^{th} homology group of the chain complex to be the quotient $H_n(C) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$.

Example 2. The chain complexes in Example 1 have the following homology groups.

(1) $H_0(C) = \mathrm{Ker}\partial_0/\mathrm{Im}\partial_1 = \mathbb{Z}/0 = \mathbb{Z}$,
$H_1(C) = \mathrm{Ker}\partial_1/\mathrm{Im}\partial_2 = \mathbb{Z}/0 = \mathbb{Z}$.
(2) $H_0(C) = \mathrm{Ker}\partial_0/\mathrm{Im}\partial_1 = \mathbb{Z}/\mathbb{Z} = 0$,
$H_1(C) = \mathrm{Ker}\partial_1/\mathrm{Im}\partial_2 = 0/0 = 0$.
(3) $H_0(C) = \mathrm{Ker}\partial_0/\mathrm{Im}\partial_1 = \mathbb{Z}/2\mathbb{Z}$,
$H_1(C) = \mathrm{Ker}\partial_1/\mathrm{Im}\partial_2 = 0/0 = 0$.
(4) $H_0(C) = \mathbb{Z}/2\mathbb{Z}$,
$H_1(C) = 0$.
(5) $H_0(C) = \mathbb{Z}/4\mathbb{Z}$,
$H_1(C) = \mathbb{Z}/2\mathbb{Z}$.
(6) $H_0(C) = \mathbb{Z}/\mathbb{Z} = 0$,
$H_1(C) = 0$,
$H_2(C) = 0$.

(7) $H_0(C) = \mathbb{Z}/2,$
$H_1(C) = 0,$
$H_2(C) = \mathbb{Z}/2,$

Some notation: The kernel of ∂_n is often written as $Z_n(C)$ and is called the cycles of C. The image of ∂_{n+1} is written $B_n(C)$ and is called the boundaries of C.

Lemma 1. *If all of the differentials in a chain complex are zero then the homology groups of the complex are isomorphic to the groups of the chain complex.*

Proof. The group $H_n(C) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$. If the differentials are all zero then $\mathrm{Ker}\partial_n = C_n$ and $\mathrm{Im}\partial_{n+1} = 0$ and so $H_n(C) = C_n$. □

A cochain complex is a sequence of homomorphisms of Abelian groups

$$\ldots \leftarrow C^{n+1} \stackrel{\delta^{n+1}}{\leftarrow} C^n \stackrel{\delta^n}{\leftarrow} C^{n-1} \leftarrow \ldots$$

where $\delta^n \circ \delta^{n-1} = 0$ for each n. The difference between chain complexes and cochain complexes is that differentials are in the opposite direction.

The cohomology groups of a cochain complex are defined to be $H^n(C) = \mathrm{Ker}\delta^{n+1}/\mathrm{Im}\delta^n$.

Exercise 1. Compute the homology of the following complex.

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \stackrel{\partial_2}{\to} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \stackrel{\partial_1}{\to} \mathbb{Z} \to 0$$

C_2 is generated by U and L, C_1 is generated by a, b, c. Define $\partial_1(a) = \partial_1(b) = \partial_1(c) = 0$ and $\partial_2(U) = \partial_2(L) = a + b - c$.

Exercise 2. Compute the homology of the following complex.

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \stackrel{\partial_2}{\to} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \stackrel{\partial_1}{\to} \mathbb{Z} \oplus \mathbb{Z} \to 0$$

C_2 is generated by U and L, C_1 is generated by a, b, c, C_0 is generated by v and w. Define $\partial_1(a) = \partial_1(b) = \partial_1(c) = w - v$, $\partial_2(U) = -a + b + c$, and $\partial_2(L) = a - b + c$.

2. Exact Sequences

Let α and β be homomorphisms of Abelian groups. We say that

$$A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C$$

is exact at B if $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$. A sequence of homomorphisms of Abelian groups

$$A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} A_{n-2} \longrightarrow \ldots \longrightarrow A_1 \xrightarrow{\alpha_1} A_0$$

is exact if it is exact at A_i for $i = 1, \ldots n-1$.

An exact sequence $0 \to A \to B \to C \to 0$ is called a short exact sequence. A sequence of homomorphism of Abelian groups

$$\ldots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots$$

that is exact for each A_n is called a long exact sequence.

Example 3. Of the chain complexes in Example 1 the following are exact.

(1) Not exact, $\mathrm{Ker}(\mathbb{Z} \stackrel{0}{\to} \mathbb{Z}) = \mathbb{Z} \neq \mathrm{Im}(0 \to \mathbb{Z}) = 0$
(2) Exact, $\mathrm{Im}(0 \to \mathbb{Z}) = 0 = \mathrm{Ker}(\mathbb{Z} \stackrel{id}{\to} \mathbb{Z})$ and $\mathrm{Im}(\mathbb{Z} \stackrel{id}{\to} \mathbb{Z}) = \mathrm{Ker}(\mathbb{Z} \to 0) = \mathbb{Z}$.
(3) Not exact, $\mathrm{Im}(\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}) = 2\mathbb{Z} \neq \mathrm{Ker}(\mathbb{Z} \to 0) = \mathbb{Z}$.
(4) Not exact, $\mathrm{Im}(\mathbb{Z}/4 \stackrel{\times 2}{\to} \mathbb{Z}/8) \neq \mathrm{Ker}(\mathbb{Z}/8 \to 0)$.

(5) Not exact, $\mathrm{Ker}(\mathbb{Z}/4 \xrightarrow{\times 4} \mathbb{Z}/8) \neq \mathrm{Im}(0 \to \mathbb{Z}/4)$.
(6) Exact, $\mathrm{Ker}(f) = \mathrm{Im}(0 \to \mathbb{Z})$, $\mathrm{Im}(f) = \mathrm{Ker}(g)$ and $\mathrm{Im}(g) = \mathrm{Ker}(\mathbb{Z} \to 0)$.
(7) Not exact, $\mathrm{Ker}\partial_0 = \mathbb{Z}/4 \neq \mathrm{Im}\partial_1 = 2\mathbb{Z}/4\mathbb{Z}$, and $\mathrm{Ker}\partial_2 = 2\mathbb{Z}/4\mathbb{Z} \neq \mathrm{Im}\partial_3 = 0$.

Lemma 2. *A chain complex of Abelian groups C is exact if and only if its homology is trivial, that is $H_n(C) = 0$ for all n.*

Proof. The homology of a chain complex is defined to be $\mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$. These are zero if and only if $\mathrm{Ker}(\partial) = \mathrm{Im}(\partial)$, or the sequence is exact. □

There are several properties that can be described by exact sequences.

(1) $0 \to A \xrightarrow{\alpha} B$ is exact if and only if α is injective.
(2) $A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is surjective.
(3) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is an isomorphism.
(4) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if α is injective, β is surjective and $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

Lemma 3. *For a short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of Abelian groups the following are equivalent.

(1) *There exists a homomorphism $p : B \to A$ such that $p\alpha = id : A \to A$.*
(2) *There exists a homomorphism $s : C \to B$ such that $\beta s = id : C \to C$.*
(3) *There exists an isomorphism $B \to A \oplus C$ such that the following diagram commutes.*

$$\begin{tikzcd} & & B \arrow[dr, "\beta"] \arrow[d] & & \\ 0 \arrow[r] & A \arrow[ur, "\alpha"] \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

The map $A \to A \oplus C$ is $a \mapsto (a, 0)$ and $A \oplus C \to C$ is given by $(a, c) \mapsto c$.

A short exact sequence is said to be split exact if it satisfies any of these equivalent conditions.

Example 4. Suppose that in the following diagram the row is exact and $h \circ f = 0$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] & 0 \\ & D & & \end{tikzcd}$$

Prove that there exists a unique homomorphism $k : C \to D$ such that $k \circ g = h$.

Let $c \in C$. Since the row is exact g is surjective and there exists $b \in B$ such that $g(b) = c$. Then let $k(c) = h(b)$. To show that this does not depend on the choice of b, let $b' \in B$ such that $g(b') = c$. Then $g(b - b') = g(b) - g(b') = c - c = 0$ so there exists $a \in A$ with $f(a) = b - b'$. $h(b) = h(b' + f(a)) = h(b') + h(f(a)) = h(b')$ since $h(f(a)) = 0$.

Similar techniques can be used to show this is a homomorphism.

Lemma 4 (Five lemma)**.** *In the commutative diagram of Abelian groups below, if the two rows are exact and α, β, δ, and ϵ are isomorphisms then γ is also an isomorphism.*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

Proof. First show that γ is injective. Let $c \in C$ be such that $\gamma(c) = 0$. Then $k' \circ \gamma(c) = 0 = \delta \circ k(c)$. The map δ is injective so $k(c) = 0$. Since the top row is exact there is an element $b \in B$ so that $j(b) = c$. Since $j' \circ \beta(b) = \gamma(c) = 0$, there exists $a' \in A'$ so that $i'(a') = \beta(b)$. The map α is surjective so there exists $a \in A$ such that $\alpha(a) = a'$. Then $\beta(i(a) - b) = \beta \circ i(a) - \beta(b) = i' \circ \alpha(a) - \beta(b) = \beta(b) - \beta(b) = 0$, and since β is injective $b = i(a)$. Since the top row is exact, $c = j(b) = j \circ i(a) = 0$.

To show that γ is surjective, let $c' \in C'$. Since δ is surjective there exists $d \in D$ so that $\delta(d) = k'(c')$. $\epsilon \circ l(d) = l' \circ \delta(d) = l' \circ k'(c') = 0$ and since ϵ is injective, $l(d) = 0$. The top row is exact so there exists $c \in C$ so that $k(c) = d$. $k'(c' - \gamma(c)) = k'(c') - k' \circ \gamma(c) = 0$ so there exists $b' \in B'$ such that $j'(b') = c' - \gamma(c)$. The map β is surjective so there exists $b \in B$ so that $\beta(b) = b'$. Then $\gamma(c + j(b)) = \gamma(c) + \gamma \circ j(b) = \gamma(c) + j' \circ \beta(b) = \gamma(c) + j'(b') = c'$. $\square$

Definition 1. *Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \partial'_n\}$ be chain complexes. A map of chain complexes is maps $f_n : C_n \to D_n$ for each n such that the squares below commute.*

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} & D_{n-1} \end{array}$$

Proposition 5. *A map of chain complexes $f : C \to D$ induces a map $H_n(f) : H_n(C) \to H_n(D)$ for each n.*

Proof. Let $c \in \mathrm{Ker}\partial_n$. Note that $\partial'_n(f(c)) = f(\partial_n(c)) = 0$ so f induces a map $f|_{\mathrm{Ker}(\partial_n)} : \mathrm{Ker}(\partial_n) \to \mathrm{Ker}(\partial'_n)$

Let $p : \mathrm{Ker}(\partial'_n) \to \mathrm{Ker}\partial'_n/\mathrm{Im}\partial'_{n+1}$ be the quotient map. If $c \in \mathrm{Im}\partial_n$, then there exists $b \in C_{n+1}$ so that $\partial_{n+1}(b) = c$. Then $p(f(c)) = p(f(\partial_{n+1}(b))) = p(\partial'_{n+1}(f(b))) = 0$, and so $p \circ f$ induces a map $H_n(C) \to H_n(D)$ for all n.

$\square$

A short exact sequence of chain complexes is a commutative diagram like that below where the rows are chain complexes and the columns are short exact sequences.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 6. *A short exact sequence of chain complexes*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

induces a long exact sequence in homology.

$$\ldots \to H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(f)} H_{n-1}(B) \to \ldots$$

The proof of this theorem is not particularly difficult, but it is long and has many parts. Instead of including a complete proof here, Proposition 5 and the following propositions are the major steps in the proof. The complete proof can be found in [1, p 114] and [2, p 46].

Proposition 7. *For each n the sequence*

$$H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C)$$

is exact.

Proposition 8. *For each n there exists a homomorphism $\partial : H_n(C) \to H_{n-1}(A)$.*

Proof. First define a map $\phi : Z_n(C) \to H_{n-1}(A)$. Let $c \in Z_n(C)$, since $g_n : B_n \to C_n$ is surjective there exists $b \in B_n$ such that $g_n(b) = c$. Consider $\partial(b) \in B_{n-1}$, then

$$g_{n-1}(\partial(b)) = \partial(g_n(b)) = \partial(c) = 0$$

so $\partial(b) \in \mathrm{Ker}(g_{n-1}) = \mathrm{Im}(f_{n-1})$. Since f_{n-1} is a monomorphism, there exists a unique $a \in A_{n-1}$ such that $f_{n-1}(a) = \partial(b)$. Then

$$f_{n-2}(\partial(a)) = \partial(f_{n-1}(a)) = \partial(\partial(b)) = 0,$$

and f_{n-2} is injective so $\partial(a) = 0$. Therefore $a \in Z_{n-1}(A)$ and the map $Z_n(C) \to Z_{n-1}(A)$ given by $c \mapsto a$ followed by the projection map $Z_{n-1}(A) \to Z_{n-1}(A)/B_{n-1}(A)$ defines the map ϕ. There are three more steps to this proof:

(1) Prove this map is independent of the choices of b.
(2) Prove this is a homomorphism.
(3) Prove the kernel of this map contains $B_n(C)$= boundaries of C.

□

Proposition 9. *The sequence*

$$H_n(B) \xrightarrow{H_n(g)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{H_{n-1}(f)} H_{n-1}(B)$$

is exact for each n.

The following exercises use the definition of an exact sequence. Solutions can be found in [2].

Exercise 3. In an arbitrary exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

of homomorphisms of Abelian groups, the following are equivalent

(1) f is surjective.
(2) g is the trivial homomorphism.
(3) h is injective.

Exercise 4. In an arbitrary exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$$

of homomorphisms of Abelian groups $C = 0$ if and only if f is surjective and k is injective.

Exercise 5. If a sequence $0 \to C \to 0$ of Abelian groups is exact then $C = 0$.

Exercise 6. In an arbitrary exact sequence

$$A \xrightarrow{d} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E \xrightarrow{k} F$$

of homomorphisms of Abelian groups the following are equivalent.

(1) g is an isomorphism.
(2) f and h are trivial homomorphisms.
(3) d is surjective and k is injective.

References

[1] Hatcher, Allen. Algebraic topology. Cambridge University Press, Cambridge, 2002. Really an Algebraic Topology book, but includes basic definitions as they are needed.

[2] Hu, Sze-Tsen. Introduction to Homological Algebra. Holden-Day, Inc., San Francisco, 1968. Older reference, but starts at the beginning and includes lots of details.

[3] Mac Lane, Saunders. Homology. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Also an older reference, less approachable than Hu, but more approachable than Weibel.

[4] Weibel, Charles A. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. A useful book but probably not the place to start.