Topology - I

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1 Topological Spaces

There are many different ways to define a topological space; the most common one is as follows:

Definition 1.1 *A* topological space *(often just "space") consists of a set X together with a* topology *on X. The latter is a collection of subsets of X, called* open *sets, such that:*

(1) The union of any collection of open sets is open, and

(2) The intersection of finitely many open sets is open.

(3) X and the empty set are open.

The same notion of a topology can also be defined by giving any one of the following things, satisfying appropriate axioms (which I won't describe here):

- The *closed* sets (those whose complement is open)
- The "closure" operation on subsets of X (the *closure* of a set is the smallest closed set containing it, also the intersection of all such closed sets)
- The *interior* operation on subsets of X (dually, the largest open set contained in a subset)
- The neighborhood filter of each point (a *neighborhood* of x is a set containing an open set containing x; see section 3 for filters)
- A *local basis* for the neighborhood filter of each point (a set that generates the filter by taking supersets)
- A *basis* of open sets (such that every open set is a union of basis sets)
- A *subbasis* of open sets (such that every open set is a union of finite intersectios of subbasis open sets). No axioms are required here: any collection of subsets of X forms a subbasis for some topology.

In practice, we usually use one of the last three methods. Topologies generated by "small" bases are nicer in many ways (as we will see). A topology generated by a countable local basis at each point is called *first countable*; one generated by a countable basis is *second countable*. A related notion is separability: a space is *separable* if it has a countable dense subset (a subset is *dense* if its closure is the whole space).

Example 1.2 *Let X be a totally ordered set, i.e. equipped with a transitive, antisymmetric, total binary relation $<$. The* order topology *on X is defined by the subbasis of "rays" $\{y \mid x < y\}$ and $\{y \mid y < z\}$ for x, z varying over X.*

The order topology on the real numbers coincides with the usual topology, as does the metric topology (see below). A more exotic example is the following:

Example 1.3 *Let ω_1 be the first uncountable ordinal, and let $L = \omega_1 \times [0, 1)$ with the lexicographic ordering, i.e. $(\alpha, x) < (\beta, y)$ if $\alpha < \beta$, or if $\alpha = \beta$ and $x < y$. L with the order topology is called the* long line*. If we include the greatest element ω_1 in the space, we get the* extended long line *L^*.*

The long line is a fruitful source of counterexamples. Most of them depend on the fact that ω_1 is *regular*; in particular, any sequence of ordinals less than ω_1 has an upper bound which is still less than ω_1.

Exercise 1 *Show that a set is open if and only if it is a neighborhood of each of the points in it.*

Exercise 2 *Prove that a collection of filters $\{\mathcal{N}_x : x \in X\}$ are the neighborhood filters of a unique topology if and only if for every $x \in X$ and every $U \in \mathcal{N}_x$,*

(1) $x \in U$, and also

(2) the set $\{y \in U \mid U \in \mathcal{N}_y\}$ is in $\mathcal{N}_x$.

Such a coherent collection of filters is often called a fundamental system of neighborhoods *for the topological space X.*

2 Metric Spaces and Separation

Definition 2.1 *A* distance function *on a set X is a function $d : X \times X \to \mathbb{R}$ such that:*

(1) $d(x, y) = d(y, x)$

(2) $d(x, z) \geq d(x, y) + d(y, z)$ (the triangle inequality*)*

(3) $d(x, y) \geq 0$ with equality if and only if $x = y$

A set equipped with a distance function is called a metric space*.*

Any distance function on a set gives rise to a topology by taking as a local basis around x the *open balls* $B(x, r) = \{y : d(y, x) < r\}$ for $r \in \mathbb{R}$. However, multiple distance functions on the same set often give rise to the same topology:

Exercise 3 *Show that the metrics*

$$\begin{aligned} d_1(x, y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ d_2(x, y) &= \max(|x_1 - y_1|, |x_2 - y_2|) \\ d_3(x, y) &= |x_1 - y_1| + |x_2 - y_2| \end{aligned}$$

define the same topology on $\mathbb{R}^2$.

Metric spaces are nicer than arbitrary spaces in many ways. For example, every metric space is first countable, since it suffices to take the sets $B(x, \frac{1}{n})$ for $n \in \mathbb{N}^+$ as a local basis. Moreover, in a metric space, it is also always possible to "separate" any two distinct points by open sets. More generally:

Definition 2.2 *A topological space is...*

- *...T_0 if for any two points, there is an open set containing one but not the other.*
- *...T_1 if for any two points, there is an open set containing the first one but not the second.*
- *...T_2 (or* Hausdorff*) if for any two points, there are two disjoint open sets each containing one of the points.*

Then every metric space is Hausdorff. Note that $T_2 \Rightarrow T_1 \Rightarrow T_0$. (There are also many other "separation" properties of this flavor, with names like T_3, T_4, $T_{3\frac{1}{2}}$, normal, regular, completely regular, etc.)

Exercise 4 *The* Zariski topology *on the spectrum of a (commutative) ring has as points the prime ideals of the ring, and as closed sets the sets of prime ideals containing an arbitrary given ideal. Show that this topology is T_0 but not necessarily T_1 (and hence not Hausdorff).*

3 Convergence, Nets, and Filters

Definition 3.1 *A sequence $\{x_n\}$ of points of a space X* converges *to a point x if for any neighborhood U of x, there is an $N > 0$ such that whenever $n > N$, $x_n \in U$.*

Note that in general, a sequence can converge to more than one point, but this is not possible in a Hausdorff space such as a metric space.

Exercise 5 *Show that in a first countable space, a point x is in the closure of some set A if and only if there exists a sequence of points in A converging to x.*

Thus in first countable spaces, the topology can be characterized by the convergent sequences; however, in general, we need a more general notion to capture all the topological information.

Definition 3.2 *A* directed set *is a partially ordered set D (that is, a set with a reflexive, transitive, antisymmetric binary relation $\leq$) such that for any elements $d, d' \in D$ there is an element $d'' \in D$ with $d \leq d''$ and $d' \leq d''$. A* net *in a set X is a function from a directed set D into X.*

Thus every sequence is a net, with $D = \mathbb{N}$. We can easily generalize convergence to nets:

Definition 3.3 *A net $\phi : D \to X$ in a topological space X* converges *to a point x if for any neigborhood U of x, there exists $d_0 \in D$ such that whenever $d_0 \leq d$, $\phi(d) \in U$.*

Example 3.4 *In the extended long line L^*, the point ω_1 is not the limit of any sequence (except a sequence that is eventually constant), but it is the limit of interesting nets. (Note that any ordinal is a directed set.)*

There is another way to generalize convergence, which is perhaps less intuitive, but often more useful. Instead of nets, we use filters.

Definition 3.5 *A* filter *on a set X is a nonempty collection $\mathcal{F}$ of nonempty subsets of X such that:*

(1) Finite intersections of elements of $\mathcal{F}$ are in $\mathcal{F}$, and

(2) Any superset of an element of $\mathcal{F}$ is in $\mathcal{F}$.

Example 3.6 *Every point $x \in X$ defines a* principal filter *consisting of all sets containing x. If X is a topological space, then every point x defines a* neighborhood filter *consisting of all neighborhoods of x. Moreover, any net (and in particular, any sequence) $\{x_d\}$ for $d \in D$ defines an* elementary filter *consisting of the sets A such that there exists $d_A \in D$ with $x_d \in A$ for all $d \geq d_A$.*

Definition 3.7 *A filter $\mathcal{F}$ on a topological space X* converges *to a point $x \in X$ if for any neighborhood U of x, there is a set $A \in \mathcal{F}$ with $A \subseteq U$.*

Clearly a net converges to a point if and only if its associated elementary filter converges to that point. Conversely, we can:

Exercise 6 *Given a filter on a space X, define a net from it which converges to a point if and only if the given filter converges to that point.*

Thus the concepts of net and filter can be used "equivalently" to generalize sequences. We will generally stick to filters.

Exercise 7 *Prove that in a Hausdorff space, any filter, and any net, converges to at most one point. Give examples of a filter and a net (in a non-Hausdorff space) that converge to multiple points.*

4 Continuous Functions

Definition 4.1 *A function $f : X \to Y$ between topological spaces is* continuous *if for every open set $U \subseteq Y$, the preimage $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open in X.*

Equivalent definitions include:

- For every *closed* set $A \subseteq Y$, the preimage $f^{-1}(A)$ is closed in X.

- For *any* set $A \subseteq X$, the *direct* image $f(A) = \{f(x) \mid x \in A\}$ satisfies $f(\overline{A}) \subseteq \overline{f(A)}$ (where $\overline{B}$ is the closure of B).
- For any $x \in X$ and any neighborhood N of $f(x)$ in Y, $f^{-1}(N)$ is a neighborhood of x in X.
- For any filter $\mathcal{F}$ which converges to x in X, the *direct image filter* $f_*(\mathcal{F})$ (which is the filter generated by the direct images of the elements of $\mathcal{F}$) converges to $f(x)$ in Y.

The last condition says that f preserves the notion of closeness, as captured by convergent filters. Easier to check, in many situations, is the following:

Exercise 8 *Show that a function $f : X \to Y$ between* metric *spaces is continuous if and only if it maps all convergent* sequences *in X to convergent sequences in Y.*

But as before, remember that sequences only suffice for sufficiently nice spaces. It is also easy to check that for metric spaces, this definition of continuity coincides with the traditional "epsilon-delta" definition.

Finally, continuous functions give us a convenient way to say when two topological spaces are "the same" topologically.

Definition 4.2 *A* homeomorphism *between topological spaces X and Y is a continuous function from X to Y which has an inverse function that is also continuous.*

Exercise 9 *Show that a function is a homeomorphism if and only if it is bijective and for any set $A \subseteq X$, A is open in X if and only if $f(A)$ is open in Y.*

A *topological property* is one which is invariant under homeomorphism. The separation (T_i) and countability properties defined above are topological properties; so are connectedness and compactness defined below.

In general, a continuous bijection need not be a homeomorphism; however between nice enough spaces this is true:

Proposition 4.3 *A continuous bijection from a compact space (section 6) to a Hausdorff space is a homemorphism.*

5 Connectedness

Definition 5.1 *A topological space (more generally, a subset of a topological space) is* connected *if it is not the union of two disjoint nonempty open sets.*

Since the union of connected sets with nonempty intersection is connected, every connected set (in particular, every point) is contained in a maximal connected set, called its *connected component.*

Definition 5.2 *A space X is* path-connected *if for any two points $x, y \in X$ there is a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

As with connectedness, every space decomposes into *path-connected components*. Path-connectedness implies connectedness (and path components live inside components), but not the reverse except in nice spaces, such as *locally path-connected* ones, which are those having a basis consisting of path-connected open sets.

Example 5.3 *The* (closed) topologist's sine curve *is the following subset of the plane, with the induced topology:*

$$S = \{(x, \sin(x)) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}.$$

Then S is connected but not path-connected.

Example 5.4 *The extended long line L^* is also connected, but not path-connected.*

It is easy to see that not only are connectedness and path-connectedness topological properties (preserved by homeomorphism), they are preserved by all continuous functions.

6 Compactness

What is the appropriate notion of "finiteness" or "finite in extent" for topological spaces? We take motivation from the following nontraditional characterization of when a *set* is finite.

Proposition 6.1 *A set X is finite if and only if it is* inaccessible by directed unions *in the following sense: Whenever $\mathcal{A}$ is a collection of subsets of X which is directed (i.e. for all $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ with $A \subseteq C$ and $B \subseteq C$) and such that $X = \bigcup_{A \in \mathcal{A}} A$, then in fact $X \in \mathcal{A}$.*

A straightforward modification of this definition yields

Definition 6.2 *A topological space is* compact *if it is inaccessible by directed unions of* open *sets. More generally, a subset of a space is* compact *if whenever it is contained in a directed union, it is contained in some element of the union. (This is equivalent to being compact in the subspace topology.)*

There are, as usualy, several other equivalent characterizations of this notion:

- Every cover by open sets has a finite subcover (the "usual" definition).
- For every collection of closed sets such that the intersection of any finite subcollection is nonempty, the intersection of the whole collection is nonempty.
- For any other space T, the projection $X \times T \to T$ is a closed function.

- Every net has a convergent subnet (though one has to be careful with the definition of "subnet").
- Every filter is contained in a convergent filter.
- Every *ultrafilter* (a filter not properly contained in any other filter) converges to some point.

Intuitively, a compact space is one in which there is nowhere to go "off to infinity." We might try to formalize that notion is as follows:

Definition 6.3 *A space is* sequentially compact *if every sequence has a convergent subsequence.*

In general, however, sequential compactness neither implies, or is implied by, compactness! (Note that in a compact space, any sequence, considered as a net, has a convergent subnet, but a subnet of a sequence need not be a subsequence.)

Example 6.4 *The set of functions from the interval* $[0, 1]$ *to itself, with the topology of pointwise convergence, is compact (by Tychonoff's theorem), but not sequentially compact: the sequence of functions defined by* $g_n(x) =$ *the* n^{th} *digit in the binary expansion of* x *has no convergent subsequence.*

Example 6.5 *The long line* L *is sequentially compact (essentially because* ω_1 *is regular) but not compact.*

However, as usual, for nice spaces they coincide.

Proposition 6.6 *In a metric space, or in a second countable Hausdorff space, compactness is equivalent to sequential compactness.*

There are many other important properties related to compactness, such as *local compactness* (every point has a compact neighborhood) and *paracompactness* (every open cover has a locally finite refinement). Paracompactness is important for the existence of partitions of unity on manifolds, while local compactness ensures that a space can be nicely "made compact."

More precisely, the *one-point compactification* of a space X is the space $X \sqcup \{\infty\}$, with all the open sets of X together with the complements of closed, compact sets. This space is compact, and it is Hausdorff if X is both Hausdorff and *locally* compact. Intuitively, we have prevented "shooting off to infinity" by including infinity in our space. There are many other compactifications which have different uses.

Example 6.7 *The compact subsets of the real line are those which are closed and bounded.* $\mathbb{R}$ *is locally compact and Hausdorff, and its one-point compactification is homeomorphic to a circle; we have "joined up the ends" by adding an extra point.*

Finally, we ask how continuous maps interact with compactness. It is easy to show:

Proposition 6.8 *The image of a compact space under a continuous function is compact. That is, if $f : X \to Y$ is continuous and X is compact, then $f(X)$ is a compact subset of Y.*

That is, not only is compactness a topological property (preserved under homeomorphism), it is preserved by all continuous functions, just like connectedness. In particular, continuous functions map convergent things to convergent things; however, they sometimes map divergent things (which "shoot off to infinity") to convergent things as well (think of a constant function). To speak of functions which "preserve divergence" we introduce the following:

Definition 6.9 *A function between topological spaces is* proper *if the inverse image of every compact set is compact.*

That this captures the notion of "preserving divergence" is shown by:

Proposition 6.10 *Say that a sequence* escapes to infinity *if only finitely many terms of it lie in any compact subset. Consider the following properties of a continuous function $f : X \to Y$:*

(1) f is proper

(2) For any sequence $\{x_n\}$ which escapes to infinity in X, the sequence $\{f(x_n)\}$ also escapes to infinity in Y.

Then the first implies the second always, and if X is such that compactness coincides with sequential compactness (such as a metric space or a second countable Hausdorff space), then the second implies the first.