WOMP 2004, BANACH AND HILBERT SPACES

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1. Banach Spaces

Definition A Banach space is a normed linear space (NLS) (i.e. a complex vector space with an associated norm) whose induced metric is complete.

The standard example is $L^p(d\mu)$ where $1 \leq p \leq \infty$ and μ is a measure on a measure space X. Recall that $L^p(d\mu)$ consists of those functions (or, to be exact, those equivalence classes of functions) f such that

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty.$$

One can show that $\|.\|_p$ is a norm on $L^p(d\mu)$, and that its induced metric is complete.

Definition Let X and Y be NLS's and let $T : X \to Y$ be a linear transformation. We define

$$\begin{aligned} \|T\| &= \sup_{x \in X, \|x\| \leq 1} \|Tx\| \\ &= \inf\{\alpha \in \mathbb{R} : \|Tx\| \leq \alpha \|x\|, \forall x \in X\} \end{aligned}$$

If $\|T\| < \infty$ then we say that T is a bounded linear transformation. In particular, if $Y = \mathbb{C}$ and $\lambda : X \to \mathbb{C}$ is bounded, then we say that λ is a bounded linear functional (using λ instead of T for linear functionals makes me happy).

Theorem 1. *T is continuous iff T is bounded.*

Proof. Suppose first that T is continuous. In particular, it's continuous at 0. Therefore, there exists $\delta > 0$ such that if $\|x\| \leq \delta$, $\|Tx\| \leq 1$. Thus, $\|T\| \leq 1/\delta$, and T is bounded.

Conversely, suppose that T is bounded, and let x be some point in X. Let $\epsilon > 0$ be given. Let $\delta = \epsilon / \|T\|$. Then if $\|x - y\| < \delta$,

$$\|Tx - Ty\| = \|T(x - y)\| \leq \|T\| \|x - y\| \leq \epsilon.$$

$\square$

Here is a useful result that is stated without proof:

Theorem 2 (Hahn-Banach)**.** *Let Y be a subspace of a NLS X and let λ be a bounded linear functional on Y. Then there exists a linear functional λ^* on X such that $\lambda^*(x) = \lambda(x)$ for all $x \in Y$ and $\|\lambda^*\| = \|\lambda\|$.*

Let X be a NLS. Consider the set of all bounded linear functionals on X, and associate the norm defined above (as an exercise verify that it is indeed a norm). The resulting space, denoted X^*, is called the dual of X. It is another exercise to verify that X^* is a Banach space (even if X isn't). We can also consider the double dual X^{**}. It is easy to show that X is naturally imbedded in X^{**} by the mapping sending $x \in X$ to the map taking λ to $\lambda(x)$. It is another exercise to verify that this is a norm-preserving imbedding.

In general, it is not true that $X = X^{**}$. If it is true, then X is called reflexive. However, this is true for a large number of nice spaces. You will see in first quarter analysis that for $1 \leq p < \infty$, and for "nice" measure spaces $(X, d\mu)$, that the dual of $L^p(d\mu)$ is $L^q(d\mu)$ where $1/p + 1/q = 1$. It follows that $L^p(d\mu)$ is reflexive for $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ are much trickier...

Definition Let X and Y be NLS's and $T : X \to Y$ a bounded linear map. We define the adjoint $T^* : Y^* \to X^*$ by $T^*(\lambda)(x) = \lambda(Tx)$ (here, $\lambda \in Y^*$, $x \in X$; unravel the definitions to convince yourself that everything here makes sense).

Theorem 3. *If X, Y and T are as above, then $\|T\| = \|T^*\|$.*

Proof. Exercise. □

2. Hilbert Spaces

Definition Let X be a vector space, and let $<,>$ be a map from $X \times X$ to $\mathbb{C}$. We say that $<,>$ is a complex inner product if for all vectors x, y and z, and complex numbers α,

(1) $< y, x >= \overline{< x, y >}$,
(2) $< x + y, z >=< x, z > + < y, z >$,
(3) $< \alpha x, y >= \alpha < x, y >$,
(4) $< x, x >\geq 0$,
(5) $< x, x >= 0$ iff $x = 0$.

Define $\|x\| =< x, x >^{1/2}$. Using the following theorem, one can indeed verify that it is a norm:

Theorem 4 (Cauchy-Schwarz inequality)**.** *If $<,>$ is an inner product on X, then for all $x, y \in X$,*

$$|< x, y >| \leq \|x\|\|y\|.$$

An inner product space X is therefore a normed linear space, and hence a metric space. If the associated metric is complete, then we say that X is a Hilbert space. Therefore, Hilbert spaces are special cases of Banach spaces, and indeed there is a rich theory associated to them that does not hold for general Banach spaces.

The standard example of a Hilbert space is $L^2(d\mu)$ with inner product

$$< f, g >= \int_X f\overline{g}\, d\mu.$$

Definition Let X be an inner product space and $x, y \in X$. We say that x and y are orthogonal if $< x, y >= 0$. If M is a subspace of X we define

$$M^\perp = \{y \in X : \; < x, y >= 0 \text{ for all } x \in M\}.$$

Analogous to the familiar situation in $\mathbb{R}^n$, there exists orthogonal projection onto subspaces in any Hilbert space (well, closed subspaces at least):

Theorem 5. *Let M be a closed subspace of a Hilbert space X. There exist linear mappings $P : X \to M$ and $Q : X \to M^\perp$ with the following properties:*

(1) *For all $x \in X$, $x = Px + Qx$.*
(2) *If $x \in M$, then $Px = x$ and $Qx = 0$; if $x \in M^\perp$, then $Px = 0$, $Qx = x$.*
(3) *For all $x \in X$, $\|x - Px\| = \inf\{\|x - y\| : \; y \in M\}$*

Corollary 1. *If $M \neq X$ is a proper closed subspace of X then $M^\perp \neq \{0\}$.*

Proof. Given $x \in X \backslash M$, write $x = Px + Qx$. Qx is a nonzero vector in $M^\perp$. □

We will use the above decomposition to study the dual of a Hilbert space. Given a vector $z \in X$, we can define a linear functional L_z by $L_z(x) =< x, z >$. Using the Cauchy-Schwarz inequality, one verifies that L_z is indeed bounded, and that $\|L_z\| = \|z\|$. The interesting part is that actually, all linear functionals on X occur in this way:

Theorem 6. *If L is a bounded linear functional on a Hilbert space X, then there exists a unique $z \in X$ such that $L = L_z$. Furthermore, $\|L\| = \|z\|$.*

It is easy to check that the map sending z to L_z is linear. Therefore, X and X^* are isometrically isomorphic as Banach spaces, and so for all practical purposes are the same: **if X is a Hilbert space, $X = X^*$.**

Proof. First, let's settle uniqueness. If $L_z = L_y$, then $< x, y > = < x, z >$ for all $x \in X$, i.e. $< x, y - z > = 0$ for all $x \in X$. Taking $x = y - z$, we see that $y - z = 0$, or $y = z$.

Now for existence. If $Lx = 0$, for all x, take $z = 0$. If not, then $M = \ker L$ is a proper closed subspace of M. By the previous corollary, there exists $y \in M^{\perp}$, $y \neq 0$. By considering, $y/\|y\|$, we may assume that $\|y\| = 1$.

Let $z = \overline{Ly}y$. I claim that $L = L_z$. Given $x \in X$, let $u = L(x)y - L(y)x$. Since $Lu = 0$, we have that

$$0 = < Lu, y > = Lx. < y, y > - Ly. < x, y > = Lx - Ly. < x, y > = Lx - < x, z >,$$

giving us the desired result. □

Consider a map $T : X \to X$. Recall the definition of the adjoint $T^* : X^* \to X^*$. Identifying X^* with X, we notice that $T^* : X \to X$. What is T^*? Well, given $z \in X$, we identify z with L_z. Then

$$T^*(L_z)(x) = L_z(Tx) = < Tx, z > .$$

Therefore, by the above theorem, T^*z is the unique vector in x satisfying $< x, T^*z > = < Tx, z >$ for all $x \in X$. We say that T is self-adjoint if $T = T^*$ or, equivalently, if $< x, Tz > = < Tx, z >$ for all $x, z \in X$.

3. Orthonormal Bases

Definition Let X be a Hilbert space. A collection of vectors $\{v_\alpha\} \subset X$ is orthonormal if for all pairs of vectors, $< v_\alpha, v_\beta > = \delta_{\alpha\beta}$. An orthonormal set is an orthonormal basis for X if it's a maximal orthonormal set (i.e. if another vector is added to the set, it is no longer orthonormal).

Theorem 7. *Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in X. Let $M = \operatorname{span}\{v_1, \ldots, v_n\}$. It is an exercise to verify that M is closed. Let P be as in Theorem 5. Then for $x \in X$,*

$$Px = \sum_{k=1}^{n} < x, v_k > v_k.$$

Proof. Given $x \in X$, let $y = \sum_{k=1}^{n} < x, v_k > v_k$. For $1 \leq j \leq n$, $< y, v_j > = < x, v_j >$ by orthonormality. Thus, $< x - y, v_j > = 0$ for $1 \leq j \leq n$, and thus $x - y \in M^{\perp}$. Since, $y \in M$, it follows by the uniqueness assertion in Theorem 5 that $y = Px$. □

Theorem 8 (Bessel's Inequality)**.** *If $\{v_\alpha : \alpha \in A\}$ is an orthonormal set, then for $x \in X$,*

$$\sum_{\alpha \in A} |< x, v_\alpha >|^2 \leq \|x\|^2.$$

Let's clarify what we mean by the sum of possibly uncountably many positive numbers. If A is any index set, and $x_\alpha \geq 0$, we let

$$\sum_{\alpha \in A} x_\alpha = \sup_{B \subset A,\ B \text{ finite}} \sum_{\beta \in B} x_\beta.$$

Proof. Let $\{v_{\alpha_1}, \ldots, v_{\alpha_n}\} \subset \{v_\alpha\}$ be a finite subset. Let $y = x - \sum_{k=1}^{n} < x, v_{\alpha_k} > v_{\alpha_k}$. Then for $1 \leq j \leq n$, $< y, v_{\alpha_j} > = 0$ by orthonormality. Thus,

$$\begin{aligned}
\|x\|^2 &= < y + \sum_{k=1}^{n} < x, v_{\alpha_k} > v_{\alpha_k}, y + \sum_{k=1}^{n} < x, v_{\alpha_k} > v_{\alpha_k} > \\
&= \|y\|^2 + \| \sum_{k=1}^{n} < x, v_{\alpha_k} v_{\alpha_k} > \|^2 \\
&= \|y\|^2 + \sum_{k=1}^{n} | < x, v_{\alpha_k} > |^2 \\
&\geq \sum_{k=1}^{n} | < x, v_{\alpha_k} > |^2.
\end{aligned}$$

Since the finite subset was arbitrary, the result is proved. □

Theorem 9. *If $\{v_\alpha : \alpha \in A\}$ is an orthonormal set, then the following are equivalent:*

1. *$\{v_\alpha\}$ is an orthonormal basis (i.e. a maximal orthonormal set).*
2. *$\operatorname{span}\{v_\alpha\}$ is dense in X.*
3. *For every $x \in X$, $\|x\|^2 = \sum_{\alpha \in A} |< x, v_\alpha >|^2$.*
4. *For all $x, y \in X$, $< x, y > = \sum_{\alpha \in A} < x, v_\alpha > \overline{< y, v_\alpha >}$.*

Proof. (1) ⇒ (2): Let M be the closure of $\operatorname{span}\{v_\alpha\}$. If $\operatorname{span}\{v_\alpha\}$ is not dense, then M is a proper closed subspace. By the corollary to Theorem 5, $M^\perp \neq \{0\}$ so that there exists a nonzero vector orthogonal to every v_α. Therefore, $\{v_\alpha : \alpha \in A\}$ is not maximal, a contradiction.

(2) ⇒ (3): Fix $x \in X$, and $\epsilon > 0$. Since $\operatorname{span}\{v_\alpha\}$ is dense in X, there exist $v_{\alpha_1}, \ldots, v_{\alpha_n}$ such that some finite linear combination of these vectors is within ϵ distance of X. Let $y = \sum_{k=1}^{n} < x, v_{\alpha_k} > v_{\alpha_k}$. By Theorems 5 and 7, it follows that

$$\|x - y\| = \|x - Px\| \leq \epsilon.$$

Thus, $\|x\| \leq \|y\| + \epsilon$, and so

$$(\|x\| - \epsilon)^2 \leq \|y\|^2 = \sum_{k=1}^{n} |< x, v_{\alpha_k} >|^2 \leq \sum_{\alpha \in A} |< x, v_\alpha >|^2 \leq \|x\|^2,$$

by Bessel's inequality. Since ϵ was arbitrary, the result holds.

(3) ⇒ (4): Exercise.

(4) ⇒ (1): If $\{v_\alpha : \alpha \in A\}$ is not maximal, then there exists $u \neq 0$ in X so that $< u, v_\alpha > = 0$ for all $\alpha \in A$. Taking $x = y = u$ in (4) gives us the required contradiction.

□

I will mention, but not prove, that every Hilbert space contains an orthonormal basis. The proof is a standard Zorn's Lemma type proof. Those of you who actually like that kind of stuff, can go ahead and prove it on your own. It is also possible to prove that any two orthonormal bases have the same cardinality.

4. Problems

Problem 1. *Prove that indeed*

$$\sup_{x \in X, \|x\| \leq 1} \|Tx\| = \inf\{\alpha \in \mathbb{R} : \|Tx\| \leq \alpha \|x\|, \forall x \in X\}$$

in the definition of $\|T\|$.

Problem 2. *Given X and Y NLS's, let Λ denote the set of bounded linear transformations from X to Y. Prove that the operator norm is indeed a norm on Λ. Furthermore, if Y is a Banach space, prove that Λ is also a Banach space.*

Problem 3. *Fix $x \in X$. Prove that the map taking $\lambda \in X^*$ to $\lambda(x)$ is a linear functional on X^* and hence is in X^{**}. In addition show that this map has norm $\|x\|$. Hint: Use the Hahn-Banach theorem.*

Problem 4. *Let X and Y be NLS's and $T : X \to Y$ be a bounded linear map. Prove that $\|T\| = \|T^*\|$. Hint: You will want to use the Hahn-Banach theorem at one point.*

Problem 5. *Let M be a subspace of a NLS X. Prove that M is dense in X iff there does not exist a non-zero bounded linear functional λ on X such that $\lambda(x) = 0$ for all $x \in M$.*

Problem 6. *If M is a subspace of a Hilbert space X, prove that $M^\perp$ is closed (even if M isn't). Also, prove that $(M^\perp)^\perp = \overline{M}$, where $\overline{M}$ denotes the closure of M in X.*

Problem 7. *Prove that (3) ⇒ (4) in Theorem 9.*

Problem 8. *Prove that a Hilbert space X is separable (contains a dense subset that's countable) iff X contains an orthonormal basis that's at most countable.*