Covering Spaces

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1 Introduction

Given a topological space X, we're interested in spaces which "cover" X in a nice way. Roughly speaking, a space Y is called a covering space of X if Y maps onto X in a locally homeomorphic way, so that the pre-image of every point in X has the same cardinality. It turns out that the covering spaces of X have a lot to do with the fundamental group of X. The subgroups of $\pi_1(X)$ correspond exactly to the connected covering spaces of X. Also, for nice enough spaces X, there's a special covering space called the universal cover, on which $\pi_1(X)$ acts. Covering spaces are important not just for algebraic topology but also for differential geometry, Lie groups, Riemann surfaces, geometric group theory ...

2 Definition and basic examples

Throughout, all spaces are topological spaces and all maps are continuous.

Definition 1. *A* covering space *or* cover *of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the following condition: every point $x \in X$ has an open neighborhood $U_x \subseteq X$ such that $p^{-1}(U_x)$ is a disjoint union of open sets, each of which is mapped by p homeomorphically onto U_x.*

You can visualize the pre-image of the neighborhood U_x as a "stack of pancakes", each pancake being homeomorphic to U_x. Some more terminology: sometimes the space X is called the *base space*, the map p is called the *covering map* or *projection*, and the pre-image $p^{-1}(x)$ of some point x in the base space is called the *fiber over x*.

Examples.

1. There's always the trivial cover: a space covers itself, with the covering map being the identity map.

2. The map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{it}$ is a covering map, wrapping the real line round and round the circle. The pre-image of a little open arc in the circle is a collection of open intervals in the real line, offset by multiples of 2π.

3. Another cover of the circle is the map $p : S^1 \to S^1$ given by $p(z) = z^n$, where n is a positive integer. This wraps the circle around itself n times.

4. Consider the equivalence relation on $\mathbb{R}^2$ given by $(x, y) \sim (x + m, y + n)$, where m and n are any integers. Let $p : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ be the quotient map. Then the image of p is the torus obtained by identifying opposite sides of a square, and p is a covering map.

5. The real projective plane $\mathbb{R}P^2$ can be thought of in several equivalent ways: as the set of lines through the origin in $\mathbb{R}^3$, as S^2 with the equivalence relation $x \sim -x$, and as the set of non-zero points of $\mathbb{R}^3$ with the equivalence relation $x \sim \lambda x$, where λ is a non-zero scalar. If we select the second way of thinking about $\mathbb{R}P^2$, then S^2 is a covering space for $\mathbb{R}P^2$, with the covering map being the quotient map.

6. The figure-of-eight graph has lots of covering spaces, and I'll draw some of them on the board.

Given a neighborhood U_x in the base space, the fiber over each point in U_x must have the same cardinality. So, if the base space is connected, this cardinality is constant over the whole space. The cardinality of each fiber is then called the *number of sheets of the covering.* The cover of S^1 in Example 3 has n sheets, while the cover of $\mathbb{R}P^2$ by S^2 is a two-sheeted covering.

3 Liftings

In this section $p : \widetilde{X} \to X$ is always a covering space.

A *lift* of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$. There are several key results about existence and uniqueness of liftings, and these have important applications.

For instance, since a covering space is a topological space, it has a fundamental group. The following proposition relates the fundamental group of a covering space to the fundamental group of the base space, and is proved using liftings of homotopies.

Proposition 2. *Fix basepoints $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. Then the homomorphism*

$$p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$$

is injective.

So, we may identify $\pi_1(\widetilde{X}, \widetilde{x}_0)$ with the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ of $\pi_1(X, x_0)$. The choice of basepoint does matter here: different choices of $\widetilde{x}_0$ in the fiber over x_0 will yield conjugate subgroups of $\pi_1(X, x_0)$.

Examples.

1. Let $p : \mathbb{R} \to S^1$ be the covering map $p(t) = e^{it}$ and, for each positive integer n, let $p_n : S^1 \to S^1$ be the covering map $p_n(z) = z^n$. Then $\pi_1(\mathbb{R}, 0)$ is trivial, so its image under p_* is the trivial subgroup of $\pi(S^1, 1) = \mathbb{Z}$. The image of $(p_n)_*$ is the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$.
2. Let $p : S^2 \to \mathbb{R}P^2$ be the covering map which identifies antipodes. Since S^2 has trivial fundamental group, the image under p_* is also trivial.
3. The fundamental group of a covering space which is a graph can be calculated using the Seifert–Van Kampen Theorem. You can then use Proposition 2 to show that, for instance, the free group on two generators has subgroups which are free on three generators, and on countably many generators.

Another important result on liftings concerns liftings of paths in the base space.

Proposition 3. *Let $f : I \to X$ be a path with starting point $f(0) = x_0$. Then for each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \widetilde{X}$ so that $\tilde{f}(0) = \widetilde{x}_0$.*

In particular, once we fix a basepoint x_0 in X, then for each $\widetilde{x}_0 \in p^{-1}(x_0)$, every loop in X based at x_0 has a unique lift to a path in the covering space $\widetilde{X}$ starting at $\widetilde{x}_0$. This is used to prove the following result.

Proposition 4. *When X and $\widetilde{X}$ are path-connected, the number of sheets of the covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ equals the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$.*

4 Maps between covering spaces

Suppose $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are two covering spaces. A *homomorphism* of covering spaces is a map $f : \widetilde{X}_1 \to \widetilde{X}_2$ so that $p_1 = p_2 \circ f$. An *isomorphism* of covering spaces is an invertible map (that is, homeomorphism) $f : \widetilde{X}_1 \to \widetilde{X}_2$ so that $p_1 = p_2 \circ f$.

An isomorphism from a covering space to itself is sometimes called a *deck transformation* or *covering transformation* (think of shuffling a deck of cards). Deck transformations permute fibers. The set of deck transformations of a covering space forms a group under composition. By a unique lifting property, a deck transformation is completely determined by where it sends a single point.

Examples.

1. Each translation of the real line by an integer multiple of 2π is a deck transformation of the covering space $p : \mathbb{R} \to S^1$, where $p(t) = e^{it}$. The group of deck transformations is isomorphic to $\mathbb{Z}$.

2. Rotating the circle S^1 by an integer multiple of $2\pi/n$ is a deck transformation of the covering space $z \mapsto z^n$. The group of deck transformations is cyclic of order n.

3. Each translation of R^2 by a vector (m, n), where m and n are integers, is a deck transformation of the covering space of the torus. The group of deck transformations is isomorphic to $\mathbb{Z}^2$.

5 The universal cover and subgroups of the fundamental group

We saw that the induced homomorphism from the fundamental group of a covering space to the fundamental group of the base space is injective. This leads to the question: can every subgroup of $\pi_1(X, x_0)$ be realized as $p_*(\pi_1(\widetilde{X}_0, \widetilde{x}_0))$ for some covering space $p : \widetilde{X} \to X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$? It turns out that the answer is yes if X is a reasonably nice space (path-connected, locally path-connected and semilocally simply connected, to be precise).

To prove this, you first construct a *universal cover*: that is, a covering space $\widetilde{X}$ of X which is simply connected. The universal cover is unique up to isomorphism.

Examples. The universal cover of the circle is the real line, of the torus is $\mathbb{R}^2$, of $\mathbb{R}P^2$ is the sphere S^2, and of the figure-of-eight graph is the infinite 4-valent tree.

Since the universal cover $\widetilde{X}$ is simply connected, $\pi_1(\widetilde{X}, \widetilde{x}_0)$ is trivial, so its image under p_* is the trivial subgroup of $\pi_1(X, x_0)$. To realize all the other subgroups of $\pi_1(X, x_0)$, you take quotients of the universal cover.

Another important feature of the universal cover is that the fundamental group of the base space acts on the universal cover by deck transformations. The action is determined as follows. Take a basepoint $x_0 \in X$ and a preimage $\widetilde{x}_0$ of x_0 in the universal cover $\widetilde{X}$. Then each element of $\pi_1(X, x_0)$ is represented by a loop $f : I \to X$ based at x_0. There is a unique lift $\tilde{f} : I \to \widetilde{X}$ starting at $\widetilde{x}_0$. Then we define the action of the homotopy class $[f]$ on $\widetilde{x}_0$ by

$$[f] \cdot \widetilde{x}_0 = \tilde{f}(1).$$

The quotient under this group action is the base space.

6 References

For the more geometrically minded, Allen Hatcher's *Algebraic Topology* and William Massey's *A Basic Course in Algebraic Topology* are recommended. Peter May's *A Concise Course in Algebraic Topology* shows the approach of algebraic topologists today.