SOME MULTILINEAR ALGEBRA OVER FIELDS WHICH I UNDERSTAND

Most of what is discussed in this handout extends verbatim to all fields with the exception of the description of the Exterior and Symmetric Algebras, which requires more care in non-zero characteristic. These differences can be easily accounted for except for the properties of the Exterior Algebra in characteristic 2, which require a significant amount of work to understand. I will, however, assume that the vector field is $\mathbb{R}$. [1]

1. The Tensor Product

Definition 1. *Let V, U and W be vector spaces. A bilinear map ϕ from $V \times U$ to W is a map which satisfies the following conditions:*

- $\phi(a_1v_1 + a_2v_2, u) = a_1\phi(v_1, u) + a_2\phi(v_2, u)$ *for all* $v_1, v_2 \in V$, $u \in U$, $a_1, a_2 \in \mathbb{R}$.
- $\phi(v, b_1u_1 + b_2u_2) = b_1\phi(v, u_1) + b_2\phi(v, u_2)$ *for all* $v \in V$, $u_1, u_2 \in U$, $b_1, b_2 \in \mathbb{R}$.

Here are some examples of bilinear maps:

(1) If $U = \mathbb{R}$ and V is any vector space, we may define a bilinear map with range $W = V$ using the formula
$$\phi(v, c) = cv.$$
(2) If U is any vector space, and V is the dual vector space $U^\star$, we may define a bilinear map with range $\mathbb{R}$ using the formula
$$\phi(v, \alpha) = \alpha(v).$$
(3) If A is an algebra over $\mathbb{R}$, then the product operation defines a bilinear map $\phi : A \times A \to A$:
$$\phi(a, b) = ab.$$
(4) More generally, if A is an algebra over $\mathbb{R}$, and M is a module over the algebra A, then the action of A on M defines a bilinear map $\phi : A \times M \to M$:
$$\phi(a, m) = am.$$
(5) Going back to reality, we can pick $M = \mathbb{R}^n$, and $A = M(n, n)$, the set of $n \times n$ matrices with coefficients in $\mathbb{R}$. Here, the standard rules for multiplying vectors by matrices make $\mathbb{R}^n$ into a module over $n \times n$ matrices.

Now that you're convinced that bilinear maps are lurking everywhere, let's try to understand all of them at once:

Definition 2. *Let V and U be two vector spaces. The tensor product of V and U is a vector space, denoted $V \otimes U$, for which there exists bilinear map $\alpha : V \times U \to V \otimes U$ which satisfies the following property: Whenever $\phi : V \times U \to W$ is any bilinear map, there exists a unique* **linear map***(linear transformation) $\hat{\phi} : V \otimes U \to W$ such that $\hat{\phi} \circ \alpha = \phi$. In other words, there is a map of vector spaces $\hat{\phi}$ such that the following diagram commutes:*

$$\begin{tikzcd} V \times U \arrow[r, "\alpha"] \arrow[d, "\phi"] & V \otimes U \arrow[dl, dotted, "\hat{\phi}"] \\ W & \end{tikzcd}$$

[1] The opinions expressed therein are solely the responsibility of the author, and then, only partially. Regardless, no other entity bears ANY responsibility for them. In particular, they should not be taken to represent opinions (voiced or otherwise) of the WOMP organizing committee, the University of Chicago Mathematics Department, or (Heaven Forbid!) some supposed consensus of the "Mathematical Community."

Of course, the above definition may be that of an object which doesn't exist, and which, moreover, may not be unique if it were to exist. The issue of uniqueness does not actually arise, since the requirement that the map $\hat{\phi}$ be unique eliminates any possible ambiguity. This is a common feature of such definitions featuring universal diagrams. As to the problem of existence, we may construct the tensor product by hand in the following exercise:

Exercise 1. *Work out the details of this vague description: Consider the vector space freely generated by all symbols of the form $v \otimes u$ for $v \in V$ and $u \in U$. The tensor product will be the quotient of this vector space by the subspace generated by elements of the form $(cv) \otimes u - c(v \otimes u)$, $(v_1 + v_2) \otimes u - v_1 \otimes u - v_2 \otimes u$, and, since U and V should be treated on equal footing, the corresponding elements if they switch roles. The map α will then map the pair (v, u) to the image of $v \otimes u$ in the quotient; this map will be bilinear because we took the quotient by the correct subspace. Similarly, any map ϕ tell us where the generating set $v \otimes u$ should be mapped to. This map is well defined on the quotient since ϕ is bilinear.*

If we're willing to jump in the mud hole and wrestle with our vector spaces, we get another description of the tensor product.

Exercise 2. *If V has basis $\{e_i\}_{i=1}^n$ and U has basis $\{f_j\}_{j=1}^m$, then $V \otimes U$ has basis $e_i \otimes f_j$.*

To show that this construction works, we may either show that it is isomorphic to the previous construction (in fact, it embeds in the big ugly vector spaces we first constructed, the projection defines a map to the quotient, so it suffices to show that this is an isomorphism), or we may show that it satisfies the universal diagram. For all my skepticism about categorical propaganda, the second option is easier.

As another application of the universal property, we get the following theorem:

Theorem 3. *The tensor product is commutative, associative, and respects duality of vector spaces. In other words, if V_i are vector spaces, then*

$$V_1 \otimes V_2 \cong V_2 \otimes V_1$$
$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$
$$(V_1 \otimes V_2)^\star \cong V_1{}^\star \otimes V_2{}^\star$$

<u>Proof:</u> The direct sum $(V \times W)$ of vector spaces satisfies the same conditions. The universal property allows one to pass from direct sums to tensor products. □

I should remark that the above theorem is true "for all vector spaces at once." In categorical words, the above isomorphisms are all natural. Note further that the second property above means that we can unambiguously write:

$$V_1 \otimes \cdots \otimes V_k$$

for any finite collection of vector spaces without having to worry about parentheses. In fact, this object could have been defined without using induction:

Exercise 3. *Define a k-multilinear map from a product of vector space $V_1 \times \cdots \times V_k$ in analogy with the definition of a bilinear map (For a special case, see Definition 6).*

Use this to define the tensor product of k vector spaces following Definition 2.

From Exercise 2, it is clear that the dimension of $V \otimes U$ is the product of the dimension of the two vector spaces.

Here's another basic result about tensor products:

Exercise 4. *If M and N are $m_1 \times m_2$ and $n_1 \times n_2$ matrices, then they define a map $\mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{m_2} \times \mathbb{R}^{n_2}$ which we may compose with the defining map for $\mathbb{R}^{m_2} \otimes \mathbb{R}^{n_2}$ to get a bilinear map. The definition of the tensor product says that this determines a unique map:*

$$\mathbb{R}^{m_1} \otimes \mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{m_2} \otimes \mathbb{R}^{n_2}.$$

What is the matrix of this map in the basis given above?

You already know another object (Hom) whose dimension is "multiplicative." This is no coincidence:

Theorem 4. *If V and W are two vector spaces, then* $\mathrm{Hom}(V, W) \cong W \otimes V^\star$,

<u>Proof:</u> We define the map in one direction by mapping $w \otimes \phi$ to the map which takes v to $\phi(v)w$. Once it is shown that this map is injective, a dimension count proves the required isomorphism. $\square$

2. The Tensor Algebra

We now consider a single vector space V. Since the tensor product is commutative and associative, we may define:

$$V^{\otimes i} \quad = \quad \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}}$$

and using the fact we have isomorphisms $V^{\otimes i} \otimes V^{\otimes j} \cong V^{\otimes i+j}$, these vector spaces fit together in a (graded) algebra:

$$T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i},$$

where $V^{\otimes 0} = \mathbb{R}$. As usual, there is a categorical definition of the Tensor Algebra (stated here as a theorem):

Theorem 5. *There is a linear map $i : V \to T(V)$ such that whenever $\phi : V \to A$ is a linear map from V to an algebra A, there exists a unique map of algebras $\hat{\phi} : T(V) \to A$ such that $\phi = \hat{\phi} \circ i$. In other words, there exists a unique $\hat{\phi}$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\phi"'] & T(V) \arrow[dl, dotted, "\hat{\phi}"] \\ A & \end{tikzcd} .$$

<u>Proof:</u> i is the inclusion of $V = V^{\otimes 1}$ into $T(V)$. Since $T(V)$ is generated as an algebra by $V^{\otimes 1}$, the rest follows. $\square$

Exercise 5. *Construct $T(V)$ as the quotient of a "huge vector space" in analogy with the construction of the tensor product.*

One advantage of the categorical definition, is that it give us a lot of information about maps from the Tensor Algebra. For example, if $f : V \to W$ is a map of vector spaces, we can use the universal property of $T(V)$ to conclude that there is a unique map of algebras $T(f)$ which makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"] & T(V) \arrow[d, dotted, "T(f)"] \\ W \arrow[r, "i"] & T(W). \end{tikzcd}$$

Further, if $g : W \to U$ is another map, then the maps $T(g) \circ T(f)$ and $T(g \circ f)$ agree. This is a consequence of the universal property applied to the following diagram:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"] & T(V) \arrow[d, dotted, "T(f)"] \\ W \arrow[r, "i"] \arrow[d, "g"] & T(W) \arrow[d, dotted, "T(g)"] \\ U \arrow[r, "i"] & T(U). \end{tikzcd}$$

This property is what allows us to think of the tensor algebra as a functor. It takes a vector space and returns a (non-commutative) algebra, but it does so in some coherent sense, taking maps of vector spaces to maps of algebras.

There is an alternative way of looking at the tensor product:

Definition 6. *Let W be a vector space. A map $f : V^k \to \mathbb{R}$ is a k-linear form if it satisfies the following conditions:*

- $f(v_1, \ldots, cv_i, \ldots, v_k) = cf(v_1, \ldots, v_i, \ldots, v_k)$ *for all* $c \in \mathbb{R}$
- $f(v_1, \ldots, v_i + v_i', \ldots, v_k) = f(v_1, \ldots, v_i, \ldots, v_k) + f(v_1, \ldots, v_i', \ldots, v_k)$

We will denote the space of k-linear forms on V by $\mathcal{L}^k(V)$. In Differential Geometry the term "multilinear form" is reserved for what is called here "skew-symmetric multilinear form" (See Definition 10).

Multiplication defines a map

$$\mathcal{L}^k(V) \oplus \mathcal{L}^m(V) \to \mathcal{L}^{k+m}(V)$$

$$(\phi, \psi) \to (\phi \cdot \psi)(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}) = \phi(x_1, \ldots, x_k)\psi(x_{k+1}, \ldots, x_{k+m})$$

This allows us to turn the space of all multilinear maps into an algebra:

$$\mathcal{L}(V) = \bigoplus_{k=0}^{\infty} \mathcal{L}^k.$$

Theorem 7.

$$\mathcal{L}(V) \cong T(V^\star)$$

<u>Proof:</u> We will do this at the level of graded vector spaces, and leave checking the compatibility of the algebra structures to the diligent reader. First, we observe that by Theorem 3, we have an isomorphism $(V^{\otimes r})^\star \cong (V^\star)^{\otimes r}$, so it suffices to show:

$$(V^{\otimes r})^\star \cong \mathcal{L}^r(V).$$

This is a good example of the use of the universal property of the tensor product. Keep the following diagram in mind:

$$\begin{tikzcd} V \times \cdots \times V \arrow[r] \arrow[d] & V^{\otimes r} \arrow[dl, dotted] \\ \mathbb{R} & \end{tikzcd}$$

An element of $(V^{\otimes r})^\star$ is just a linear map $V^{\otimes r} \to \mathbb{R}$. Composing with the multilinear map

$$V \times \cdots \times V \to V^{\otimes r}$$

we obtain an r-linear form on V. To go in the other direction, we use Exercise 3, to see that every r-linear form on V induces a unique linear map from $V^{\otimes r}$ to $\mathbb{R}$. $\square$

3. The Symmetric Algebra

If we were interested only in commutative algebras, we would be studying the Symmetric Algebra over V:

Definition 8. *The Symmetric Algebra* $\mathrm{Sym}(V)$ *is defined by the following universal property: There exists a linear map* $j : V \to \mathrm{Sym}(V)$*, such that whenever* $\phi : V \to A$ *is a linear map into a commutative algebra* A*, there exists a unique map of algebras* $\hat{\phi} : \mathrm{Sym}(V) \to A$ *such that* $\phi = \hat{\phi} \circ i$*. In other words, the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\phi"] & \mathrm{Sym}(V) \arrow[dl, dotted, "\hat{\phi}"] \\ A & \end{tikzcd} .$$

Exercise 6. $\mathrm{Sym}(V)$ *is generated as an algebra by* V*. (Hint: Consider the subalgebra generated by* V*; apply the universal property).*

Theorem 5 gives a map $T(V) \to \mathrm{Sym}(V)$. By the above exercise, this map is surjective (since the generating set is in the image).

Exercise 7. *Write* $\mathrm{Sym}(V)$ *explicitly (?) as a quotient of the "huge vector space" you used in Exercise 5.*

In fact, the grading on $T(V)$ descends to a grading of $\mathrm{Sym}(V)$, so we may write:

$$\mathrm{Sym}(V) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V).$$

Just as finding an explicit description of the tensor product allows us to give a basis for it in the case of finite-dimensional vector spaces, we can use the image of the explicit basis produced in Exercise 2 to prove the following result:

Exercise 8. *If V has basis $\{e_i\}_{i=1}^n$, prove that* $\mathrm{Sym}^j(V)$ *has basis consisting of all elements of the form:*

$$e_1^{a_1} e_2^{a_2} \dots e_n^{a_n}, \Sigma_{i=1}^n a_i = j.$$

Use this to compute the dimension of $\mathrm{Sym}^j(V)$.

If this reminds you of polynomials in several variables, you're on the right track. The analogue of polynomials for abstract vector spaces are symmetric multilinear forms:

Definition 9. *A map $f : V^r \to \mathbb{R}$ is a symmetric r-linear form if it is r-linear and is invariant under the action of the symmetric group.*

Here, the symmetric group S_r acts on V^r by permuting components. Symmetric multilinear forms are an algebra if we define $\phi \cdot \psi$ as follows:

$$(\phi \cdot \psi)(v_1, \dots, v_{r+s}) = \frac{1}{(r+s)!} \Sigma_{\sigma \in S_{r+s}} \phi(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \psi(v_{\sigma(r+1)}, \dots, v_{\sigma(r+s)})$$

In other words, we force the product of two multilinear forms to be symmetric by averaging over all possible permutations. This is precisely the point that doesn't work in the same way for vector spaces of non-zero characteristic, since we are dividing by a number that may not be invertible in the field. The more general thing to do requires one to not take the average, but simply take the sum over all permutations. We can now prove the analogue of Theorem 7

Exercise 9. *Prove that the algebra of symmetric multilinear forms on V is isomorphic to* $\mathrm{Sym}(V^\star)$.

This shows that if $V = \mathbb{R}^n$, then $\mathrm{Sym}(V^\star)$ is, indeed, the space of polynomials in n variables.

4. The Exterior Algebra

One of the fundamental tools in Differential Geometry is the space of k-linear forms. If we eliminate the manifold, we are left with the Exterior Algebra of a vector space. As for the Tensor and Symmetric Algebras, there are different ways of approaching this object (Warning: This is also called the Alternating Algebra, and, in older and physics texts, the Grassmann Algebra. Using the theory of graded vector spaces, one can treat the Exterior and Symmetric Algebras as manifestations of the same things: Free Algebras on a graded vector space).

Definition 10. *A skew-symmetric k-linear form is a k-linear form ϕ which satisfies the following additional property:*

$$\phi(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = -\phi(v_1, \dots, v_{i+1}, v_i, \dots, v_n)$$

This is where the trouble begins in characteristic 2. Since $-1 = 1$ in that situation, skew-symmetry and symmetry are equivalent. The stronger condition (which is equivalent to skew-symmetry away from the prime 2.) is that of a k-alternating map:

$$\phi(v_1, \dots, v_n) = 0$$

if there exists $i \neq j$ such that $v_i = v_j$. The reason most people prefer to use the skew-symmetric definition is that it is nicely compatible with the sign homomorphism of the symmetric group:

Exercise 10. *Use the fact that transpositions generate the symmetric group to prove that a k-linear map ϕ is skew-symmetric if and only if:*

$$\phi(v_1, \cdots, v_n) = -\text{sign}(\sigma)\phi(v_{\sigma(1)}, \cdots, v_{\sigma(n)})$$

for all permutations σ.

Use this to conclude the equivalence of the alternating and skew-symmetry conditions away from 2 (i.e: whenever the only number that satisfies $-x = x$ is 0).

One advantage of the alternating point of view is that it makes the following result clearer:

Exercise 11. *Prove that the zero map is the only k-linear map on a one dimensional vector space if $k > 1$.*

Again, skew-symmetric k-linear maps form an algebra under an appropriately chosen average:

$$(\phi \wedge \psi)(v_1, \ldots, v_{r+s}) = \frac{1}{r! + s!}\Sigma_{\sigma \in S_n} sign(\sigma)\phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}).$$

This is, once again, one of those situations where one has to make choices. This choice works only in characteristic 0. But even in characteristic 0, authors often have different conventions in order to make some formula or the other look nice (Another reasonable choice is to replace $r! + s!$ by $(r + s)!$). Just be aware of these differences, but don't worry too much about them, since they rarely lead to anything worse than being a factorial sign away from the right answer. I have chosen this convention because it eliminates multiplicative constants in the determinant.

As should be clear by now, one may proceed from here in different directions in order to define the Exterior Algebra. We will opt for the one encountered in most differential geometry textbooks:

Definition 11. *The Exterior Algebra of $V^\star$, denoted $\Lambda(V^\star)$ is the algebra of skew-symmetric k-linear maps from V^r to $\mathbb{R}$. We write:*

$$\Lambda(V^\star) = \bigoplus_{i=0}^{\infty} \Lambda^i(V^\star)$$

Exercise 12. *Define the Exterior Algebra of a vector space without passing through its dual, taking as a model Definition 8.*

Explicitly, one finds that if V has basis $\{e_i\}_{i=1}^n$, then $\Lambda^k(V)$ has basis:

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad i_1 < \cdots < i_k.$$

Note that this implies that the dimension of $\Lambda^k V$ is $\binom{n}{k}$. In particular, the 0'th and the n'th exterior powers are 1 dimensional, and $\Lambda^k V$ is 0 if $k > n$. Another way of proving this is explained in the following exercise:

Exercise 13. *Prove that:*

$$\Lambda^k(V \oplus W) = \bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W.$$

Use induction and Exercise 10 to prove the dimension formula above.

So far, we have more or less described what our constructions do to vector spaces, ignoring maps. If we are to go any further, we have to address this problem. In fact, the Symmetric and Exterior Algebras are also functors (See the discussion after Exercise 5).

Exercise 14. *If $f : V \to W$ is a map of vector spaces, then the Universal Property of the Exterior Algebra defines maps of (skew-commutative) algebras*

$$\Lambda(f) : \Lambda(V) \to \Lambda(W).$$

Show that this makes the exterior algebra into a functor from the category of vector spaces to that of (skew-commutative) algebras.

Tracing all the diagrams in the above exercise should yield the following formula:

$$\Lambda(\phi)(v_1 \wedge \cdots \wedge v_k) = \phi(v_1) \wedge \cdots \wedge \phi(v_k).$$

In particular, $\Lambda(\phi)$ decomposes as $\Lambda^k(\phi)$ for $0 \leq k \leq n$. Here's a (non-standard) definition that shows some of the use of this machinery:

Definition 12. *Let A be a linear transformation on a vector space V of dimension n. The determinant of A is the eigenvalue of the map induced by $\Lambda^n(A)$ on $\Lambda^n V$.*

In order for this definition to make sense, one has to remember that $\Lambda^n V$ is 1-dimensional. To see that this is equivalent to the elementary (but unenlightening) definition of the determinant, we pick a basis $\{e_i\}_{i=1}^n$, write A as a matrix $(a_{ij})_{1\leq i,j\leq n}$ in that basis, and compute:

$$\begin{aligned}
\Lambda(A)(e_1 \wedge \cdots \wedge e_n) &= A(e_1) \wedge \cdots \wedge A(e_n) \\
&= (\Sigma_{j_1=1}^n a_{1j_1} e_{j_1}) \wedge \cdots \wedge (\Sigma_{j_n=1}^n a_{nj_n} e_{j_n}) \\
&= \Sigma_{1\leq j_1,\ldots,j_n\leq n} (a_{1j_1} \cdots a_{nj_n}) e_{j_1} \wedge \cdots \wedge e_{j_n}
\end{aligned}$$

At this point, one observe that whenever some $j_k = j_l$, the corresponding term does not contribute to the sum, since $e_{j_1} \wedge \cdots \wedge e_{j_n} = 0$. So the only terms that matter are those in which none of the j_k are equal. We can then think of $\{j_k\}_{k=1}^n$ as simply a reordering of the indexing set $\{1, \ldots, n\}$, so we write $j_k = \sigma(k)$ for $\sigma \in S_n$. We are now in firm, familiar ground, and we can continue the computation:

$$\begin{aligned}
\Lambda(A)(e_1 \wedge \cdots \wedge e_n) &= \Sigma_{\sigma\in S_n} (a_{1\sigma(1)} \cdots a_{n\sigma(n)}) e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} \\
&= \Sigma_{\sigma\in S_n} (a_{1\sigma(1)} \cdots a_{n\sigma(n)}) \mathrm{sign}(\sigma) e_1 \wedge \cdots \wedge e_n
\end{aligned}$$

In particular, the eigenvalue of $\Lambda(A)$ is

$$\Sigma_{\sigma\in S_n} \mathrm{sign}(\sigma)(a_{1\sigma(1)} \cdots a_{n\sigma(n)}),$$

which is hopefully the right answer.

Exercise 15. *Use the functoriality of the exterior algebra to prove that*

$$\det(AB) = \det(A)\det(B).$$

References

[1] Spivak, Michael. A Comprehensive Introduction to Differential Geometry, Vol 1. Publish Or Perish, Houston, TX, 1999. Despite the scary title and the fascinating cover, this book includes most of what one needs to know about basic differential geometry (including all the multilinear algebra), with long-winded, sometimes quirky explanations, and lots of exercises. Check out the yellow pig.

[2] Hoffman, Kenneth, and Kunze, Ray. Linear Algebra. Prentice-Hall, Saddle River, NJ, 1971. A standard introductory Linear Algebra Text.

[3] Harris, Joe. Algebraic Geometry, A First Course. Springer-Verlag, New York, 1992. See Lecture 6 for how to think geometrically about $\Lambda^i V$.