Exercises on Vector Bundles

Exercise 1. Show that for a manifold M, the tangent bundle TM also has the structure of a manifold. If M is an n-manifold, what is the dimension of TM?

Exercise 2. Show that, for odd n, the sphere S^n admits a nowhere-vanishing vector field.

Exercise 3. Show that an n-dimensional vector bundle $E \to M$ is trivial if and only if thre are n sections $s_1, \ldots, s_n$ which, in each fiber, are linearly independent. Show that all bundles have local systems of n linearly independent sections.

Exercise 4. Show that the normal bundle is trivial for all spheres $S^n \subset \mathbb{R}^{n+1}$.

Exercise 5. Show that the Möbius strip is actually a nontrivial line bundle over S^1.

Exercise 6. Show that the twice-twisted Möbius strip is actually the trivial line bundle over S^1 (the cylinder).

Remark. Think about the twice-twisted Möbius strip as a physical object. It seems obvious that it is not the same as the cylinder, yet the bundles are equivalent. This is because the "twisting" is dependent on the bundle's embedding in $\mathbb{R}^3$, an extrinsic property, rather than on anything intrinsic to the bundle itself.

Exercise 7. Now show that all line bundles over S^1 are equivalent to either the trivial bundle or the standard Möbius strip.

Exercise 8. Suppose you have a multiplication law in $\mathbb{R}^n$ making it into a (non-associative) division algebra. Then show that

(1) For each point $p \in S^{n-1}$, there is a unique $a \in \mathbb{R}^n$ such that $p = a \cdot e_1$. (Existence of inverses.)
(2) If $a \in \mathbb{R}^n$ is nonzero, then $a \cdot e_1, \ldots, a \cdot e_n$ are linearly independent.
(3) If $p = a \cdot e_1$, then the projections of $a \cdot e_2, \ldots, a \cdot e_n$ on $T(S^{n-1})_p$ are linearly independent.
(4) Multiplication by any fixed element a is continuous.
(5) TS^{n-1} is trivial.
(6) $T(\mathbb{R}P^{n-1})$ is trivial.

And note that regular multiplication, complex multiplication, and that given by the quaternions and octonians ensure that the above results hold for $n = 1, 2, 4, 8$.

Exercise 9. If there are vector bundles $E_1 \subset E_2$, define the quotient bundle E_2/E_1 (you must show that your definition satisfies local triviality).

Exercise 10. If you have a Euclidean metric on E_2, show that $E_2/E_1 \simeq E_1^{\perp}$.

Exercise 11. What is the dimension of $Sym^k(E)$ if E is an n-plane bundle?

Exercise 12. (*) Show that if M is a contractible manifold, any bundle over M is equivalent to the trivial bundle.

Selected References

- Milnor and Stasheff, *Characteristic Classes*, Ch 1-3
- Spivak Vol 1, Ch 3
- Atiyah, *K-theory*, early chapters