BASIC TOPOLOGY

MOON DUCHIN, ANDREW BLUMBERG

1. Overview

These notes are intended as a slightly idiosyncratic guide to notions of point-set topology and basic algebraic topology which will to some degree be taken as a matter of course during the first year. That is, some professors (but not all) will assume that people have seen much of this material at least once and will thus feel justified in presenting it rapidly or in the most general way possible (e.g., Professor May's notorious discussion of covering spaces in their full categorical glory). This having been said, we certainly allude to more advanced notions here and there; hopefully it will be clear when this is happening, and those remarks can be skipped without loss to the main point.

We don't prove much of anything; the goal is largely to present and review basic definitions of point set topology and introduce ways to think about some basic objects of algebraic topology. For a much more thorough look at all this, look at the books in the references. These notes owe a clear debt to these references.

2. A review of point-set (general) topology

2.1. Preliminaries.

Definition. A *topological space* is a set X with a collection of subsets (referred to as open sets) subject to the following constraints :

1. X itself and the empty set are open sets.
2. The finite intersection of open sets is an open set.
3. The arbitrary union of open sets is an open set.

We refer to the collection of subsets as a *topology* on the set X.

Definition.
1. A *basis* for a topology on a space X is a set of subsets of X such that for every $x \in X$, there is some basis element which contains it and such that if $x \in B_1 \cap B_2$, there exists $B_3 \subset B_1 \cap B_2$ such that $x \in B_3$.
2. The topology generated by a basis is given by the specification that a set U is open if for every point $x \in U$, there exists a basis element which contains x and is contained in U.

Definition. A *continuous* map $f : X \to Y$, where X and Y are topological spaces, is a map such that if $V \subset Y$ is open then $f^{-1}(V) \subset X$ is open.

Example: *Basic topological building blocks*

1. the trivial (or indiscrete) topology; the open sets are X and the empty set.
2. the discrete topology; all subsets of X are open.
3. euclidean topology; given by the standard euclidean metric.
4. subspace topology; a subspace Y of a topological space X inherits a topology where a set U in Y is open if U is of the form $V \cap X$ for V open in X.

Date: November 7, 2002.

Remark. Note that any metric yields a topology (let the basic open sets be the open balls), but there are plenty of non-metrizable topological spaces.

Remark. Sometimes, we study topological structures on objects that possess other sorts of algebraic structure; for example, topological vector spaces, topological groups, etc.

Example: *topologizing matrix groups*
The matrix groups inherit the subspace topology from any of the natural embeddings in $\mathbb{R}^{n^2}$.

We are often interested in topologies obtained by performing transformations on our original space. Sometimes we glue the space together; in this case we obtain the quotient topology.

Definition. The *quotient topology* on A is induced from a surjective map p from X to A as the minimal topology which makes the map continuous; U is open in A if $p^{-1}(U)$ is open in X.

Remark. The most common usage of this is the situation where A is a set of equivalence classes defined on X and p is the map taking x to the class of x.

We also may take the cartesian product of a number of topological spaces. A natural topology to put on the product would specify that the open sets are simply open in each coordinate (this is the "box" topology). Unfortunately, this turns out not to be a good definition. A better topology is the product topology.

Definition. The *product topology* on a cartesian product ΠX_i is defined by specifying that the open sets are those which (i) project to an open set in each coordinate, and (ii) project to the full X_i in all but finitely many coordinates.

This is a more useful definition as it makes a variety of important results true (for example, Tychonoff's theorem is true in the box topology and false in the product topology).

2.2. **Compactness.** Compactness is a finiteness condition for the topology. It is an outrageously powerful property to have in hand; compact spaces are extraordinarily pliable and pleasant to work with.

Definition. A space X is *compact* if every open covering of X admits a finite subcover.

Definition. A space X is *limit-point compact* if every infinite subset of X has a limit point.

Definition. A space X is *sequentially compact* if every sequence has a convergent subsequence.

Remark. In a metric space these definitions are equivalent. In general, the first is stronger than the second which is stronger than the last.

Claim. The image of a compact space under a continuous map is compact.

Remark. We invariably will want to understand the behavior of any new definitions under continuous maps; results such as the previous will proliferate.

Definition. A space X is *paracompact* if every open cover admits a locally finite refinement which is a cover.

Example:
Every compact Hausdorff space is paracompact.

Example: *Stone's theorem*
Every metrizable space is paracompact.

Remark. We care about paracompactness in part because it is enough to demonstrate the existence of partitions of unity, which permit us to do calculus on manifolds and a lot of other wonderful things.

Exercise 1. Consider the (uncountable) open cover of $\mathbb{R}^2$ by all unit squares. Verify that $\mathbb{R}^2$ is paracompact by finding a locally finite refinement of this cover.

For suitable combination operations on spaces, compactness behaves nicely. Clearly for finite products a product of compact spaces is compact. A very useful theorem is that under the product topology on an infinite product, an infinite product of compact spaces is compact (Tychonoff's theorem).

2.3. **Compactification.** It is sometimes useful to embed a space in a larger compact space. For example, in functional analysis one can often utilize properties which apply to compact spaces and extend to their (not necessarily compact) subspaces.

Definition. A space X is *locally compact* if for each point x there is a compact subset C which contains a neighorhood of X.

Definition. Let X be locally compact. The *one-point compactification* of X is the space $Y = X \cup \{\infty\}$ topologized by letting the open sets of Y be the open subsets of X as well as all sets of the form $Y - C$ for C a compact subset of X.

Exercise 2. Show that the one-point compactification of X is compact.

Remark. We need local compactness to ensure that Y is Hausdorff.

The one-point compactification of a space X is in some sense the minimal embedding of X into a compact space. There are in fact a wide variety of compactifications.

Remark. The maximal compactification is the Stone-Cech compactification which has an important extension property for continuous functions to $\mathbb{R}$.

2.4. **Connectedness.**

Definition. A space X is *connected* if there does not exist a *separation* (disjoint open sets U,V such that $X = U \cup V$ and U and V are nonempty).

Claim. The image of a connected set under a continuous map is connected.

Claim. The cartesian product of connected sets is connected (with the product topology; false for the box topology).

Definition. A space is path connected if every pair of points can be joined by a path in X.

There are spaces which are connected but not path connected; the "topologist's sine curve" is the canonical textbook example.

When we speak of *connected components* (or *path components*), we mean the division of the space into maximal connected (or path connected) subspaces.

2.5. **Separation axioms.** It is often the case that we want to rule out spaces which are obtained by perverted gluings or ensure that the topology we've chosen is fine enough to do useful work. As such, there is a family of separation axioms which specify the degree to which points can be distinguished within our topology.

Definition. A space X is said to be *Hausdorff* (or T2) if for any pair of points there exist disjoint open neighborhoods.

Definition. A space X is said to be T1 if for any pair of points each has a neighborhood which does not contain the other (point).

Exercise 3. The diagonal of a space X is defined as the elements of $X \times X$ of the form (x, x). Show that X is Hausdorff if and only if the diagonal of X is closed.

Remark. In certain situations, one actually wants to take the this diagonal criterion in place of the usual definition of Hausdorffness. For example, in algebraic geometry.

The Hausdorff property is well-behaved under many composition operators. However, quotients (gluing) can cause trouble.

Exercise 4. (*) Give an example of a quotient of a Hausdorff space which is not Hausdorff.

Exercise 5. Show constructively that matrix groups are Hausdorff.

2.6. **Countability axioms.**

Definition. A space X is *first-countable* if it has a countable basis at every point (this means that at each point there is a countable collection of open sets such that an arbitrary open set about the point contains at least one of this distinguished collection).

Remark. All metric spaces are first-countable.

Remark. First-countability is useful in that it permits us to talk about sequences rather than nets for convergence. (*Nets* are sequences indexed over arbitrary (possibly uncountable) sets).

Definition. A space X is *second-countable* if it has a countable basis.

Remark. Second-countability is useful because it is sufficient for the Urysohn lemma and partitions of unity. It also, usefully, rules out some famous perverse examples, like the "long line" (which you'll hear about in the context of manifolds).

2.7. **Topology on function spaces.** Spaces whose points are functions (function spaces) are the source of a great deal of deep mathematics; topologizing these spaces leads to the whole field of functional analysis. We will briefly discuss some topologies one works with on $C(X, Y)$ (the set of continuous functions from X to Y).

Definition. The *compact-open topology* on the space $C(X, Y)$ is given by defining the basic open sets as follows: for each pair (K, U), where $K \subset X$ is compact and $U \subset Y$ is open, we take $\{f \in C(X, Y) : f(K) \subset U\}$.

Exercise 6. Show that the compact-open topology is strictly finer than the product topology on $C(X, Y)$.

Definition. The *topology of compact convergence* on a metric space Y is given by defininining the basic open set associated to a compact set K to be functions f such that $|f(x) - f(y)| < \delta$ for $x, y \in K$.

Claim. The compact-open topology and compact convergence topologies are equivalent.

Remark. The utility of the compact-open topology is that it can be defined even in the absence of a metric.

3. Covering spaces

The basic situation is as follows : you have a topological space X. Over it lie a bunch of potential *covers*, which are spaces E which somehow "fold" or "coil" over X; that is, there are maps $E \to X$ with nice properties– locally just homeomorphisms, and globally consistently k-to-one.

A simple way to visualize this is to think about a "stack of pancakes"; multiple copies of a space projecting down. Here, $E = X \times F$ for some F which is either a finite set or $\mathbb{Z}$. This is called the trivial cover. The wrinkle is of course that in most interesting cases these copies are glued together in various twisted ways.

Definition. A *k-fold cover* of B consists of a space E and a map $p : E \to B$ and a number k (possibly infinity) such that p is continuous, $|p^{-1}(x)| = k$ for all $x \in B$, and for each $x \in B$ there is a neighborhood U such that $p^{-1}(U)$ is a disjoint union of k neighborhoods in E, each homeomorphic to U (via p).

This is not the most economical definition, but it addresses all of the salient features. Note that often one requires a covering space to be connected, so as to rule out trivial covering spaces.

Remark. The set $p^{-1}(x)$ for a point x in B is often referred to as the *fiber* of the covering and B as the *base space*. This terminology alludes to the more general notion of a fibration.

Remark. The requirement that there exists a neighborhood U such that $p^{-1}(U)$ is a disjoint union of sets V_i such that V_i is homeomorphic to U via p is called *local triviality.* Local triviality just says that the covering space is locally homeomorphic to $U \times F$. Recall that a simple example of a covering space is simply $U \times F$; this is globally trivial. More general covering spaces are locally trivial but may be twisted around so as to prohibit global triviality.

Example:
Any space is its own covering space under the identity map.

Example: $\mathbb{R}$ *over the circle.*
A covering map is $t \mapsto e^{it}$, which gives an infinite-degree cover. An intuitive way to draw the picture is having $\mathbb{R}$ coil to form a spiral over the circle.

Example: $\mathbb{R}^n$ *over the n-torus.*
(Note this specializes to the previous example for $n = 1$.) This works for general n by taking products of the map in the previous example. In the case $n = 2$, this is the familiar picture of the square folding into a torus (à la "pac-man").

Example: *Projective space.*
Here is an object which comes up again & again in the first year and onwards. First we'll look at $\mathbb{R}P^n$, the real projective space. There are several equivalent ways of thinking about $\mathbb{R}P^n$, including these that follow:

- The space of lines in $\mathbb{R}^{n+1}$.
- The sphere S^n with antipodes associated: $x \sim -x$.
- The collection of $(n+1)$-tuples of real numbers, not all zero, up to scaling by nonzero reals:

$$x \sim \lambda x, \quad x \in \mathbb{R}^{n+1} \setminus \mathbf{0},\ \lambda \in \mathbb{R} \setminus 0.$$

(Note the equivalences between these.)

In particular, $\mathbb{R}P^1$ ("the projective line") and $\mathbb{R}P^2$ ("the projective plane") come up a lot and can be understood very concretely.

Exercise 7. S^n 2-covers $\mathbb{R}P^n$ via the association of antipodes.

Exercise 8. Consider the sets $A_i := \{(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n) : y_j \in \mathbb{R}\}$ for $1 \leq i \leq n$. Note that each of these is a copy of $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$. Show that they provide an open cover of $\mathbb{R}P^n$. (In the language of manifolds, the embeddings are charts which comprise a complete atlas.)

Exercise 9. Show that real projective space is compact and Hausdorff.

Remark. Understanding what covers are possible for a given space carries useful geometric information. For instance, any manifold which can be covered by Euclidean space inherits its flat metric via the covering map. In particular, in the case of two-dimensional surfaces, each can be covered by exactly one of the flat plane, the hyperbolic plane, and the sphere. This correlates to the total curvature of the surface: zero, negative, or positive.

3.1. **Covering maps and liftings.** It's often very important to think about lifting things (maps, paths, etc.) from the base space to a covering space.

Definition. A *lift* of a path $f : I \to X$ to E (given a specified member of the fiber over the basepoint of the path) is a map g such that $p \circ g = f$.

Claim. Any path in the base space has a unique lift to the covering space.

Definition. A *covering map* between covering spaces of B is a continuous map which is compatible with the projections.

Definition. A *deck transformation* is an automorphism of a covering space.

Remark. In order to respect the projection map, note that deck transformations must permute the fibers. Note that not all permutations of the fibers produce permissible deck transformations, though. It turns out that the deck transformations (which form a group) are intimately related to an another algebraic invariant of the base space (which leads us to our next section).

Exercise 10. Verify that the deck transformations of a covering space form a group.

4. The fundamental group

4.1. **Some religion.** The construction of the fundamental group is an excellent example of the basic enterprise of algebraic topology; for each topological space, we give a procedure for building an algebraic object and a procedure for converting maps of topological spaces to maps of algebraic objects. In this way, we reduce problems about topological spaces, which are hard, to problems about algebraic objects, which are easy.

Remark. In other words, algebraic topology is about constructing functors from the category of topological spaces (with morphisms continuous maps) to various categories of algebraic objects (say groups with morphisms homomorphisms).

4.2. **Definition.** The fundamental group of X, denoted $G = \pi_1(X)$, is perhaps the most basic algebraic invariant of a topological space. G is the collection of loops in X (from a particular basepoint, say) where two loops are regarded as being equivalent if you can "slide one around" to make it identical to the other.

Remark. By "sliding around" we're referring to the notion of homotopy. We'll discuss this in more detail in the section on homotopies. However, briefly, a homotopy between two curves γ_1 and γ_2 is a continuous map f from $I \times I$ to X where $f(x, 0)$ is $\gamma_1(x)$ and $f(x, 1)$ is $\gamma_2(x)$.

Definition. $\pi_1(X, x) = \{f \colon I \to X \text{ s.t. } f(0) = f(1) = x\}/\sim$ where two loops are equivalent if one can be continuously deformed into the other (are homotopic).

Remark. Maps as above can also be regarded as $f : S^1 \to X$ s.t. $f(1) = x$. This is the version which generalizes readily to the "higher homotopy groups" $\pi_n(X)$ of which π_1 is the first.

There is a group structure on π_1; composition works by performing the loops in sequence, with the "speed" doubled.

Exercise 11. Prove that this in fact imposes the structure of a group on π_1; demonstrate the existence of a unit, verify associativity, and verify the existence of inverses.

Remark. In fact, the basepoint is irrelevant most of the time: if x and y are two points from X and $h : x \to y$ is any smooth path, then an isomorphism between $\pi_1(X, x)$ and $\pi_1(X, y)$ is given by $[f] \mapsto [h.f.h^{-1}]$. This is why the usual notation, simply $\pi_1(X)$, is justified whenever X is path-connected. Note however that there is no canonical isomorphism; it depends on the path taken.

Example: $\mathbb{R}^n$*,* S^2*.*
Here, there are no nontrivial loops: any loop can be continuously deformed to a path which stays for the whole time at the basepoint. We say that a space X with $\pi_1(X) = \{e\}$ is *simply connected.*

Example: *The circle.*
$\pi_1(S^1) = \mathbb{Z}$ because the only topologically relevant information about a loop is how many times it goes around (forward or back).

Remark. It's actually moderately involved to prove that the fundamental group of a circle is $\mathbb{Z}$, although it's intuitively fairly clear. It's worth looking at a standard proof of this fact.

Example: *The torus.*
$\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$. To see this, consider the two fundamental loops. Any loop on the torus can be continuously deformed to look like a sequence of these two loops (forward and back), so we only need to know how many times each loop appears in the sequence.

Remark. The torus is $T = S^1 \times S^1$. In fact, it's true in general, as in this case, that fundamental group commutes with Cartesian products. See Exercise 14 below.

Remark. These examples are a bit deceptive, we should note. In general it's very hard to calculate homotopy groups, even for relatively simple spaces. The complete calculation of the homotopy groups of spheres is not complete at this time. This having been said, there are more powerful tools for computing π_1, notably the Siefert-Van Kampen theorem.

Continuous maps of spaces induce homomorphisms of fundamental groups. Understanding these induced maps is often very important. In particular, covering maps induce homomorphisms of fundamental groups.

Remark. A few notes about the relation between covering spaces and fundamental groups. The fundamental group of the covering space, $H = \pi_1(E)$, is a subgroup of the fundamental group of the base space, $G = \pi_1(X)$. Covering spaces can be thought of as fiber bundles with discrete fibers; the fundamental group G acts naturally on the fiber with stabilizers isomorphic to H. In this way, we can identify the fiber, a set of size k, with the quotient G/H. In particular, when the covering space is simply connected ($H = \{e\}$), the degree of the cover matches the order of the fundamental group of the base space. Simply connected covering spaces are called *universal covers*, often denoted $\tilde{X}$, and play an important role.

5. Homotopy

Here we discuss in more detail the equivalence relation used to define the fundamental group.

Definition. A *homotopy* is a family of maps $f_t : X \to Y$, $t \in I$ such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. Two functions $g, h : X \to Y$ are homotopic if they can be realized as the endpoints f_0, f_1 of a homotopy.

Remark. In the previous definition of homotopy we were concerned with paths; maps from I to X, so the map F was from $I \times I$ to X. The definition generalizes to all maps between spaces.

Example: *Straight-line homotopy.*
Here, between two maps to $f_0, f_1 : X \to \mathbb{R}^n$, you simply slide from point to point linearly. This is simply given by $f_t(x) = (1-t)f_0(x) + tf_1(x)$. That is, if you drew a straight line between where your initial function and final function would land you for a given input, the time parameter moves linearly along that trajectory. For instance, any two loops in $\mathbb{R}^n$ are homotopic via straight-line homotopy. (This is why $\pi_1(\mathbb{R}^n) = \{e\}$.)

Example: *Deformation retraction.*
A deformation retraction of a space X onto a subspace A is a homotopy such that $f_0 =$id., $f_t(X) \subset A$, and $f_t|_A =$id. for all t. Think of it as sucking a space down into a subspace by smooth sliding. If X deformation retracts to a point, it is called *contractible.*

- $\mathbb{R}^2$ deformation retracts onto an open disk, a line, or a point. (In fact, these can all be achieved by straight-line homotopies.)
- The twice-punctured disk deformation retracts onto a figure eight.
- The Möbius strip deformation retracts onto its central circle.
- The printed letters c,f,h,k,l,m,n,r,s,t,u,v,w,x,y,z are all contractible. The letters a,b,d,e,g,o,p,q, all retract to a circle, and the remaining letters i and j to two points.

Remark. Fundamental group is invariant under deformation retraction. [why?] In fact, deformation retraction is one example of a relation called *homotopy equivalence* under which π_1 is invariant. Homotopy equivalence includes homeomorphism also as a special case, but allows for collapsing balls to points, making it much more general.

Exercise 12. Show that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$. Use this result to show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$, and that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

Exercise 13. Generalize the last part of the previous exercise; show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$.

Exercise 14. Show that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$ in general.

Exercise 15. Show that $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$ is contractible.

6. Collected exercises

Below are all the exercises in the body of the document, plus a few others which are interesting.

Exercise 1. Consider the (uncountable) open cover of $\mathbb{R}^2$ by all unit squares. Verify that $\mathbb{R}^2$ is paracompact by finding a locally finite refinement of this cover.

Exercise 2. Show that the one-point compactification of X is compact.

Exercise 3. The diagonal of a space X is defined as the elements of $X \times X$ of the form (x, x). Show that X is Hausdorff if and only if the diagonal of X is closed.

Exercise 4. (*) Give an example of a quotient of a Hausdorff space which is not Hausdorff.

Exercise 5. Show constructively that matrix groups are Hausdorff.

Exercise 6. Show that the compact-open topology is strictly finer than the product topology on $C(X, Y)$.

Exercise 7. S^n 2-covers $\mathbb{R}P^n$ via the association of antipodes.

Exercise 8. Consider the sets $A_i := \{(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n) : y_j \in \mathbb{R}\}$ for $1 \leq i \leq n$. Note that each of these is a copy of $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$. Show that they provide an open cover of $\mathbb{R}P^n$. (In the language of manifolds, the embeddings are charts which comprise a complete atlas.)

Exercise 9. Show that real projective space is compact and Hausdorff.

Exercise 10. Verify that the deck transformations of a covering space form a group.

Exercise 11. Prove that this in fact imposes the structure of a group on π_1; demonstrate the existence of a unit, verify associativity, and verify the existence of inverses.

Exercise 12. Show that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$. Use this result to show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$, and that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

Exercise 13. Generalize the last part of the previous exercise; show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$.

Exercise 14. Show that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$ in general.

Exercise 15. Show that $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$ is contractible.

Exercise 16. (*) Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Exercise 17. (*) Show that there is no retraction of the Mobius band onto its boundary circle.

Exercise 18. (*) If $X \subset \mathbb{R}^3$ is the union of the spheres of radius $1/n$ tangent to the yz-plane at the origin, show that $\pi_1(X) = \{e\}$.

7. References for further reading

- Hatcher, *Algebraic Topology I*
  - a truly great, readable book, available at `http://www.math.cornell.edu/~hatcher`
- Massey, *A Basic Course in Algebraic Topology*
  - not as great, but a reference
- Munkres, *Topology, a first course*
  - the classic accessible general (point-set) topology reference.
- May, *A Concise Course in Algebraic Topology*
  - you'll see this up close in the 3^{rd} quarter