Hyperbolic Space Isometries

**Theorem 1** \( \text{Isom}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R}) \)

Consider the upper half-space model of the hyperbolic plane. The metric is \( ds = 1/(\text{Im}z) \, dz \).

1. Translations of the form \( z \mapsto z + b \), dilations of the form \( z \mapsto az \), and inversions of the form \( z \mapsto -1/z \) are all isometries.

2. The group of Möbius transformations \( z \mapsto \frac{az+b}{cz+d} \) with \( ad - bc \neq 0 \) is the subgroup of \( \text{Isom}^+(\mathbb{H}^2) \) generated by translations, dilations, and inversions.

3. \( PSL_2(\mathbb{R}) \) is isomorphic to the group of the above Möbius transformations.

4. The group of Möbius transformations is transitive on the set of triples of points in \( \mathbb{H}^2 \).

5. An isometry is determined by its action on any three non-collinear points.

**Theorem 2** The set of geodesics of \( \mathbb{H}^2 \) is the set of lines/circles perpendicular to the real-axis.

1. Vertical lines are geodesics.

2. The group of Möbius transformations acts transitively on the set of lines/circles perpendicular to the real axis.