Lie groups and Lie algebras
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1 Examples of Lie groups

Lie groups often appear as groups of symmetries of geometric objects or as groups acting on a vector space preserving a tensor such as an inner product or a volume form.

Example. The group $SO_2(\mathbf{R})$ of (orientation-preserving) linear isometries of $\mathbf{R}^2$, also known as the group of 2×2 orthogonal matrices with positive determinant.

Exercise. Show that every such matrix can be written

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

The map

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \mapsto \qquad a + ib$$

is an isomorphism from $SO_2(\mathbf{R})$ to the multiplicative group $S^1 = \{e^{it} : t \in \mathbf{R}\}$. In fact, this map has nice topological properties, as well ....

In general, $SO_n(\mathbf{R})$ is the subgroup of $GL_n(\mathbf{R})$ preserving the standard inner product on $\mathbf{R}^n$.

Example. The group $SL_2(\mathbf{R})$ of 2×2 matrices with determinant 1. This is also the subgroup of $GL_2(\mathbf{R})$ preserving the standard volume form and preserving orientation on $\mathbf{R}^2$.

The quotient $PSL_2(\mathbf{R}) = SL_2(\mathbf{R})/\{\pm I\}$ has a very important description as the group of orientation-preserving isometries of hyperbolic space $\mathbf{H}^2$. The group $PSL_2(\mathbf{C})$ acts by fractional linear transformations on the complex plane $\mathbf{C}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}$$

and such a transformation leaves the upper half-plane, $\{z \in \mathbf{C} : Imz > 0\}$, invariant if and only if $a, b, c, d \in \mathbf{R}$. Note that $-I$ defines the same fractional linear transformation as I.

The action of $PSL_2(\mathbf{R})$ on the upper half-plane is transitive. The action is not, however, simply transitive—that is, the stabilizer of a point is a non-trivial subgroup. In general the stabilizers of points p and $g.p$ are related by

$$\mathrm{Stab}_{g.p} = g\mathrm{Stab}_p g^{-1}$$

The image $\overline{SO_2(\mathbf{R})}$ of $SO_2(\mathbf{R})$ in $PSL_2(\mathbf{R})$ can be identified with the stabilizer of the point i, so for the hyperbolic plane $\mathbf{H}^2$, point stabilizers are all conjugate to $\overline{SO_2(\mathbf{R})}$.

There is a $PSL_2(\mathbf{R})$-equivariant bijection

$$PSL_2(\mathbf{R})/\overline{SO_2(\mathbf{R})} \to \mathbf{H}^2$$

mapping the coset of $\overline{SO_2(\mathbf{R})}$ to i. The quotient $PSL_2(\mathbf{R})/\overline{SO_2(\mathbf{R})}$ has a natural topology—in fact, a smooth structure—that is respected by this identification, as well ....

Exercise. Spaces with a transitive G action for a Lie group G are called *homogeneous spaces*. Write $S^2 = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| = 1\}$ as a homogeneous space—in this case, a quotient of matrix groups. Write $\mathbf{P}^n(\mathbf{R})$, the set of lines in $\mathbf{R}^{n+1}$, as a quotient of matrix groups.

2 Lie groups via definitions

In the above examples, groups act on spaces and are, themselves, geometric objects.

Definition. A *Lie group* is a group that is also a smooth manifold such that the multiplicaton map

$$\begin{aligned} \mu &: G \times G \to G \\ \mu &: (g,h) \mapsto gh \end{aligned}$$

and the inversion map

$$\begin{aligned} \iota &: G \to G \\ \iota &: g \mapsto g^{-1} \end{aligned}$$

are smooth.

The examples above are subgroups of $\mathrm{GL}_n\mathbf{R}$ and have a natural topology as closed subsets of $\mathbf{R}^{n^2}$.

We now turn to the differentiable structure of Lie groups. Consider the set of *vector fields* on a Lie group G. A vector field on G is a smooth choice of $\mathbf{v}(g)$ in T_gG for each $g \in G$, or a smooth section of the tangent bundle TG. One can add and scale vector fields by adding or scaling in the tangent space at each point. Thus the set of vector fields on G forms a vector space $\mathbf{v}(G)$. The group G acts on its tangent bundle by left multiplication

$$L_g : (h, v) \mapsto (hg, D(L_g)_h(v)) \qquad \text{for} \qquad g, h \in G, v \in T_hG$$

Given a vector field $X \in \mathbf{v}(G)$, left translation gives another vector field $L_{g*}X$ with

$$(L_{g*}X)(h) = D(L_g)_{g^{-1}h}(X(h))$$

A vector field X on a Lie group is *left-invariant* if

$$(L_{g*}X)(h) = X(h) \qquad \text{for all} \qquad g, h \in G$$

The left-invariant vector fields on G form a subspace $\mathbf{v}_G(G)$ of $\mathbf{v}(G)$. For each $v \in T_eG$, there is a unique vector field satisfying $X(g) = D(L_g)_e(v)$ for all $g \in G$, i.e. there is a unique left-invariant vector field with $X(e) = v$.

Exercise. This correspondence is a vector space isomorphism of T_eG with $\mathbf{v}_G(G)$.

Example. $T_e(\mathrm{SO}_n\mathbf{R})$

Example. $T_e(\mathrm{SL}_n(\mathbf{R}))$

Exercise. Find T_eG for G preserving a general positive definite form.

3 Lie algebras

Definition. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear product

$$[,] : \mathfrak{g} \times \mathfrak{g} :\to \mathfrak{g}$$

satisfying

$$\begin{array}{llll} (i) & [X,Y] & = -[Y,X] & \text{(skew-symmetry)} \\ (ii) & [X,[Y,Z]] & +[Z,[X,Y]] + [Y,[Z,X]] = 0 & \text{(Jacobi identity)} \end{array}$$

for all $X, Y, Z \in \mathfrak{g}$.

Example. The vector space $\mathbf{R}^3$ equipped with the cross-product

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$$

Example. Given any finite-dimensional vector space V, the space of endomorphisms $End(V) = \{T : V \to V : T \text{is linear}\}$ equipped with the bracket

$$[T,S] = T \circ S - S \circ T$$

Exercise. Find a basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ for the Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ such that $\mathbf{e}_i \mapsto \mathbf{b}_i$ gives an isomorphism from $(\mathbf{R}^3, \times)$ to $(\mathfrak{sl}_2(\mathbf{R}), [,])$.

4 The Lie algebra-Lie group correspondence

Definition. The *exponential map* takes an $n \times n$ matrix X to the infinite series

$$\exp(X) = \sum_{k=0}^{\infty} X^k/k!$$

where $X^0 = I$.

Let $N = \max\{|X_{ij}| : 1 \leq i,j \leq n\}$. Then $|(X^k)_{ij}| \leq (nN)^k$. So $|\exp(X)_{ij}| \leq \exp(nN)$ and the series $\exp X$ converges. The matrix $\exp X$ is invertible, with inverse $\exp(-X)$, so exp maps $\mathrm{M}_n\mathbf{R}$ into $\mathrm{GL}_n\mathbf{R}$.

Exercise. Show that if $XY = YX$, then $\exp(X+Y) = \exp X \exp Y$.

In particular, $\exp(s+t)X = \exp(sX)\exp(tX)$, so $t \mapsto \exp(tX)$ is a homomorphism from $\mathbf{R}$ to $\mathrm{GL}_n\mathbf{R}$. The curve $\gamma_X(t) = \exp(tX)$ is a *one-parameter subgroup* of G. Note that on $\mathrm{M}_1\mathbf{R} = \mathbf{R}$, exp is the familiar function into R^*. The relation $\frac{\mathrm{d}}{\mathrm{dt}}\exp(tX) = X\exp(tX)$ holds. For the curve $\gamma(t) = \exp(tX)$, one has $\gamma'(0) = X$. Conversely, a curve γ in a Lie group G with $\gamma(0) = I$ satisfying $\gamma(s+t) = \gamma(s)\gamma(t)$ is determined by its derivative at 0. (There is a definition of the exponential map $\mathfrak{g} \to G$ for a general Lie group G that agrees with this definition when G is a matrix group.)

Exercise. (a) If α is an automorphism of a Lie algebra $\mathfrak{g}$, then

$$\exp(\alpha X \alpha^{-1}) = \alpha(\exp X)\alpha^{-1}$$

(b) If $\lambda_1, \dots, \lambda_n$ are the characteristic roots of X, with multiplicities, then $\exp\lambda_1, \dots, \exp\lambda_n$ are the characteristic roots of $\exp X$.

A nifty corollary is $\det(\exp X) = \exp(\mathrm{tr} X)$, giving another derivaton of the Lie algebra of SL_n.

For a general Lie group G, one can realize the Lie algebra $\mathfrak{g}$ as the algebra $\mathbf{v}_G(G)$ under the bracket of vector fields $[X, Y] = XY - YX$. Alternatively, one can define the bracket on T_eG via the one-parameter subgroups:

$$[X, Y] = \lim_{s \to 0, t \to 0} \frac{1}{st}(\gamma_X(t)\gamma_Y(s)\gamma_X(t)^{-1})$$

where γ_X and γ_Y are the one-parameter subgroups corresponding to X and Y, respectively. One can now check that the vector space isomorphism $\mathbf{v}_G(G) \to T_eG$ defined above is in fact a Lie algebra isomorphism.

The exponential map is *natural* in the following sense: the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{D}\rho} & \mathfrak{h} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\rho} & H \end{array}$$

commutes.

A consequence is

Theorem. A Lie group homomorphism $\rho : G \to H$, with G connected, is determined by the Lie algebra homomorphism $\mathrm{D}\rho : \mathfrak{g} \to \mathfrak{h}$.

With some more work, one can prove

Theorem. For Lie groups G, H with G connected and simply connected, a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is the derivative of a homomorphism $\rho : G \to H$ if and only if ϕ is a Lie algebra homomorphism.

5 Recommended Reading

- Chevalley, *Theory of Lie Groups*, Princeton Landmarks in Math. [explicit treatment of the classical matrix groups]
- Katok, *Fuchsian Groups*, Chicago Lectures in Math. [the group of isometries of the hyperbolic plane, PSL_2, and its discrete subgroups; elementary treatment with incredible parallels to general structure theory]
- Fulton and Harris, *Representation Theory: A First Course*, Springer Readings in Math. [case-by-case treatment of the classical linear groups and their representations interspersed with general theory; recommended for those interested in algebraic geometry]
- Knapp, *Lie Groups: Beyon an Introduction*, Birkhäuser [recommended for those interested in geometry]