1 Basic Measure Theory

- **Definitions**

  1. E is the cube in $\mathbb{R}^n$: $\{(x_1, x_2, \ldots x_n) : 0 \leq x_i \leq 1 \; \forall \; i = 1, 2, \ldots n\}$
  2. The **upper measure** of a subset A of the cube is
  $$m^*(\mathrm{A}) = \inf \sum_k m(\mathrm{R}_k)$$
  where the inf is taken over all collections $\{\mathrm{R}_k\}$ of rectangles whose union covers A. The **lower measure** of A is
  $$m_*(\mathrm{A}) = 1 - m^*(\mathrm{E} - \mathrm{A}).$$
  3. A subset A of E is said to be **measurable** if $m^*(\mathrm{A}) = m_*(\mathrm{A})$. In this case we say $m(\mathrm{A})$ is this common value.
  4. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called **measurable** if $f^{-1}(\mathrm{S})$ is measurable for all open S in $\mathbb{C}$. *(Note: We can replace "open" by "closed" in this definition, and get exactly the same definition.)*
  5. A function is called **simple** if it is measurable and takes only countably many values.
  6. If f is simple, taking values $y_1, y_2, y_3, \ldots$, then
  $$\int_{\mathrm{A}} f(x)dm = \sum y_i m(\mathrm{A}_i),$$
  where $\mathrm{A}_i = f^{-1}(y_i) \cap \mathrm{A}$.
  7. A measurable function f is **integrable** on A if there exists a uniformly convergent sequence of simple functions $\{f_i\}$ having f as its limit. In this case,
  $$\int_{\mathrm{A}} f(x)dm = \lim_{i\to\infty} \int_{\mathrm{A}} f_i(x)dm$$

- **Exercises**

  1. If a set A is countable, prove from the definitions that it is measurable with measure zero.
  2. The Cantor set C consists of all real numbers in [0,1] whose base 3 representation can be written with no 1's. For instance, 0.00202022 (base 3) is in C, while 0.02201022021 is not. Note that, for example, $1/3 = 0.1 = 0.02222222\ldots$, so 1/3 *is* in C. Find the measure of C. Note that C is uncountable!
  3. Define $f : \mathbb{R} \to \mathbb{C}$ by
  $$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$
  Prove that f is measurable.
  4. Show that the definition of the integral in Definition 7 makes sense; i.e. show that the limit is independent of the choice of the uniformly convergent sequence $\{f_i\}$.

2 Exercises on the Stone-Weierstrass Theorem

NOTATION
$C_c(\mathbb{R})$ = continuous functions on $\mathbb{R}$ with compact support.
$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \,\big|\, \forall\, \epsilon > 0\ \exists\, K \subset \mathbb{R}$ compact such that $|f(x)| \le \epsilon$ for $x \notin K\}$
$C^\infty([-1,1])$ = those functions in $C([-1,1])$ that have a C^∞ extension to an open neighborhood of $[-1,1]$.

1. (a) Are the even polynomials dense in $C([0,1])$?
   (b) Are the even polynomials dense in $C([-1,1])$?

2. The Weierstrass Theorem is not valid for $C(\mathbb{R})$. Prove that if a function $f(x)$ can be uniformly approximated on $\mathbb{R}$ by polynomials, then $f(x)$ is itself a polynomial.

3. Let
$$S = \big\{xp(x) \,\big|\, p(x) \text{ is a polynomial}\big\}$$
$$T = \big\{f(x) \,\big|\, f(x) \text{ is continuous on } [-1,1] \text{ and } f(0) = 0\big\}$$
   Is S dense in T with respect to the norm $\|f\| = \sup_{x\in[-1,1]} |f(x)|$?

4. Let
$$S = \left\{ \sum_{n=0}^{N} f_n(x) g_n(y) \,\middle|\, f, g \in C([0,1]) \right\}$$
   Prove that S is dense in $C([0,1] \times [0,1])$.

5. (a) Let
$$S^1 = \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\}$$
   Prove using the Stone-Weierstrass Theorem that the space of polynomials in z and $\bar{z}$ is dense in $C(S^1)$.
   (b) A trigonometric polynomial is a finite sum of the form
$$f(x) = \sum a_n e^{inx}$$
   Deduce as a consequence of (a) that the trigonometric polynomials are dense in $\{f \in C([-\pi,\pi]) \mid f(-\pi) = f(\pi)\}$.

6. (a) Prove, using the Stone-Weierstrass Theorem, that $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.
   (b) Prove, using the Stone-Weierstrass Theorem, that $C^\infty([-1,1])$ is dense in $C([-1,1])$.
   (c) Prove, using (b), that $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. You may assume $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$.

   In the above exercise, it may be useful to use the function
$$f(x) = \begin{cases} e^{-(x-1)^{-2}} e^{-(x+1)^{-2}} & \text{for } x \in (-1,1) \\ 0 & \text{for } x \in (-1,1) \end{cases}$$
   which is in $C^\infty(\mathbb{R})$, positive on $(-1,1)$ and 0 elsewhere.

3 Hilbert and Banach Space Exercises

1. **When does a norm come from an inner product?**

   (a) Suppose E is a vector space with inner product $<,>$, and let $\|f\|^2 =< f, f >$. Prove the Parallelogram Law:

   $$\|f + g\|^2 + \|f - g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$$

   (b) Using the Parallelogram Law, prove that there does not exist an inner product $<,>$ such that $\|f\|^2 =< f, f >$ for the norm

   $$\|f\| = \sup_{x \in [-1,1]} |f(x)|$$

   on $C([-1, 1])$.

   (c) Using the Parallelogram Law, prove that there does not exist an inner product $<,>$ such that $\|f\|^2 =< f, f >$ for the norm

   $$\|f\| = \left(\int |f|^p\right)^{1/p}$$

   on $L^p(\mathbb{R})$ for $p \neq 2$ and $1 \leq p \leq \infty$.

2. **The importance of the completeness axiom for Hilbert and Banach spaces**. Completeness is needed to prove a key lemma in Hilbert space theory: If C is a nonempty closed convex subset of a Hilbert space E and $x \in E$ then there is a unique point $c_0 \in C$ such that

   $$\inf_{c \in C} \|x - c\| = \|x - c_0\|$$

   From this lemma, the projection operator is constructed and the Riesz Representation Theorem is proved. Completeness is also necessary for two of the three major theorems in Banach space theory: the Open Mapping Theorem and the Principle of Uniform Boundedness. Below is an elementary and frequently used application of completeness.

   Suppose E is a normed linear space and F is a Banach space. Let $L' : E' \to F$ be a bounded linear operator defined on a dense subspace E' of E. Prove that L' has a unique extension L to a bounded linear operator $L : E \to F$.

3. **An application of orthogonality**

   (a) Let F be an inner product space and let $E = \{e_1, \ldots, e_n\}$ be a finite orthonormal set in F. For $f \in F$, let

   $$Pf = \sum_{k=1}^{n} < f, e_k > e_k$$

   Then show $(f - Pf) \perp E$.

(b) Let $f \in L^2([0,1])$. Prove, using (a), that for each positive integer n, there is a unique polynomial p_n of degree $\leq n$ such that $\|f - p\| \geq \|f - p_n\|$ for each polynomial p of degree $\leq n$.

4. **Banach or not?**

(a) Is $\{f \in C[-1,1] \mid f(0) = 0\}$ with $\|f\| = \sup_{x\in[-1,1]} |f(x)|$ a Banach space?

(b) Is $\{f \in C[-1,1] \mid f(x) = 0 \text{ in a neighborhood of } 0\}$ with $\|f\| = \sup_{x\in[-1,1]} |f(x)|$ a Banach space?

(c) Is $C([-1,1])$ with $\|f\| = f(0)$ a Banach space?

(d) Is $C_c(\mathbb{R})$ with $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ a Banach space?

(e) Is $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \bigm| \forall \epsilon > 0\ \exists K \subset \mathbb{R} \text{ compact such that } |f(x)| \leq \epsilon \text{ for } x \notin K\}$ with $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ a Banach space?

L^p Space Exercises

1. Show that for a measure space X such that $\mu(X) < \infty$ and $p_1 \leq p_2$, then $L^{p_2}(X) \subset L^{p_1}(X)$, but this is not true if $X = \mathbb{R}$.

2. Show that if $1 \leq p_1 \leq p \leq p_2 \leq \infty$ then $L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R}) \subset L^p(\mathbb{R})$.

3. By Exercise 7, for a given function f, the set of all p, $1 \leq p \leq \infty$ for which $f \in L^p(\mathbb{R})$ must be of one of the following types: (a) empty (b) finite or infinite open, closed, or half-open interval (c) a single point. Each of these cases does arise, and we verify this for case (c) now.

Let

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \dfrac{1}{\sqrt{x}|\ln x|} & \text{if } 0 < x < \frac{1}{2} \text{ or } x > 2 \\ 0 & \text{if } \frac{1}{2} \leq x \leq 2 \end{cases}$$

Show that $f \in L^2(\mathbb{R})$ but $f \notin L^p(\mathbb{R})$ for $p \neq 2$ and $1 \leq p \leq \infty$.

4. Let $0 < p < 1$.

(a) Define

$$\|f\| = \left(\int |f|^p \right)^{1/p}$$

Show that $\|\cdot\|$ is NOT a norm on $L^p(\mathbb{R}) = \{f \,|\, \int |f|^p < \infty\}$.

(b) Define

$$\|f\| = \int |f|^p$$

Show that $\|\cdot\|$ is also NOT a norm on $L^p(\mathbb{R})$, but that $d(f,g) = \|f - g\|$ does define a metric on $L^p(\mathbb{R})$.