WHEN AN INFINITELY-RENORMALIZABLE ENDOMORPHISM OF THE INTERVAL CAN BE SMOOTHED$^{(\heartsuit)}$

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Abstract

Let K be a closed subset of a smooth manifold M, and let $f : K \to K$ be a continuous self-map of K. We say that f is smoothable if it is conjugate to the restriction of a smooth map by a homeomorphism of the ambient space M. We give a necessary condition for the smoothability of the faithfully infinitely interval-renormalizable homeomorphisms of Cantor sets in the unit interval. This class contains, in particular, all minimal homeomorphisms of Cantor sets in the line which extend to continuous maps of an interval with zero topological entropy.

I. Introduction

We consider the faithfully infinitely interval-renormalizable homeomorphisms of Cantor sets embedded in the unit interval. We begin with some definitions (a detailed topological description of these maps is given in [BORT]). If K is a nonempty topological space and $f : K \to K$ is continuous, we say that (K, f) is a **dynamical system**. A **morphism** from (K, f) to (K', f') is defined by a continuous map $\phi : K \to K'$ such that $f' \circ \phi = \phi \circ f$. The dynamical system (K, f) is n**-renormalizable** if there is a morphism from (K, f) to $(\mathbb{Z}/n\mathbb{Z}, m \mapsto m+1)$, where $\mathbb{Z}$ denotes the group of integers, and n**-interval renormalizable** if K is a compact subset of the real line and the fibers of ϕ belong to disjoint closed intervals. Let $IRen(f)$ be the set of $n \in \{1, 2, \ldots\}$ for which (K, f) is n-interval renormalizable. We say that (K, f) is **infinitely interval-renormalizable** if $IRen(f)$ is an infinite set. (K, f) is **faithfully infinitely interval-renormalizable** if it is infinitely

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$^{(\heartsuit)}$ Dedicated to Benoit Mandelbrot on his seventieth birthday: the present paper is one small step toward understanding "universality at the transition to turbulence", as first conjectured by P. Coullet and one of the authors (see [CT] and [TC]: these researches were deeply influenced by Benoit Mandelbrot's writings.)

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interval-renormalizable, K is a Cantor set and f is a homeomorphism of K that is conjugate to a generalized adding machine. [2] For the sake of brevity, we shall use the term **infinitely renormalizable** for "faithfully infinitely interval-renormalizable." If (K, f) is infinitely renormalizable, then the set $IRen(f) = \{q_0 < q_1 < \ldots\}$ has the properties:

(a) $\qquad q_0 = 1\,,$

(b) $\qquad q_{n+1} = q_n \cdot a_n,$ where $a_n \in I\!N,\ a_n \geq 2\,.$

(For the proof, see [BORT]). We call a sequence of natural numbers $\{q_n\}_0^\infty$ a **renormalization sequence** if it satisfies properties (a) and (b).

We consider the question of whether the regularity of an infinitely-renormalizable map can be altered by changing the geometry, but not the order structure, of the embedded Cantor set. All infinitely-renormalizable homeomorphisms are conjugate by a homeomorphism of the ambient interval to a map with a continuous extension to the interval that has bounded variation on the interval. There are examples of infinitely-renormalizable homeomorphisms that are not conjugate in this sense to any Hölder continuous endomorphism of the interval. In Section III, we state a necessary condition for an infinitely-renormalizable homeomorphism to be C^1-smoothable and a necessary condition for such a map to be $C^{1+\alpha}$-smoothable, for some $0 < \alpha \leq 1$. In Section IV we compare our results to what is known about the smoothability of infinitely-renormalizable embeddings of the two-disk into itself.

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II. Preliminaries

Let $I = [0, 1]$ be the unit interval. Let $\{q_n\}_0^\infty$ be a renormalization sequence and set $a_n = \frac{q_{n+1}}{q_n}$. We say that a Cantor set $K \subset I$ is **$\{q_n\}$-presented** if

$$K = \bigcap_{n=0}^{\infty}\Big(\bigcup_{i=0}^{q_n-1} K_{n,i}\Big), \text{ where } K_{n,i} = [l_{n,i}, r_{n,i}],$$

for an indexed family of real numbers $r_{n,i}, l_{n,i} \in I$ satisfying:

$$l_{0,0} = 0, \quad r_{0,0} = 1, \quad r_{n,i} < l_{n,i+1}, \quad l_{n,i} = l_{n+1,i\cdot a_n}, \quad r_{n,i} = r_{n+1,(i+1)\cdot a_n - 1},$$

[2] A **generalized adding machine** is a quasi-periodic motion obtained by adding 1 on the compact abelian group $\hat{Z\!\!Z}_Q = \lim_{\leftarrow} Z\!\!Z/q_i Z\!\!Z$, where $q_0 < q_1 < \ldots$ is a sequence of natural numbers ordered by divisibility.

and

$$\lim_{n\to\infty} \max_{i\in\{0,\ldots,q_n-1\}} |r_{n,i} - l_{n,i}| = 0.$$

These conditions imply that $K_{n+1,j} \subset K_{n,i}$ for $j \in \{ia_n, \ldots, (i+1)a_n - 1\}$.
In the presentation above, we refer to the set $K_{n,i}$ as an **interval at level** n. An open interval of the form $G_{n,i} = (r_{n,i}, l_{n,i+1})$ is called a **gap** in K at level n. Let $\tilde{K}_{n,i} = K \cap K_{n,i}$ be the piece of the Cantor set K corresponding to the interval $K_{i,j}$.

The following condition is equivalent to the definition of infinitely-renormalizable stated in the Introduction (see [BORT]). Given a Cantor set $K \subset I$, a homeomorphism $f : K \to K$ is infinitely renormalizable if and only if there exists a renormalization sequence $\{q_n\}$ and a $\{q_n\}$-presentation of K, such that for every n and for every $0 \le i \le q_n - 1$,

$$f(\tilde{K}_{n,i}) = \tilde{K}_{n,\sigma_n(i)},$$

where σ_n is a cyclic permutation of the set $\{0, \ldots, q_n - 1\}$.

To each infinitely-renormalizable homeomorphism we associate a renormalization sequence $\{q_n\}$ (or, equivalently, the sequence $\{a_n\}$), and a sequence of permutations $\sigma = (\sigma_1, \sigma_2, \ldots)$, where $|\sigma_n| = q_n$. The sequence σ is called the **order type** of f. We remark that this combinatorial information completely describes the infinitely-renormalizable homeomorphism as an ordered dynamical system, the ordering arising from the embedding of K in the interval. When $a_n = 2$ for all n, we say that f is a **period-doubling homeomorphism**.

Example 1.1. The unimodal period-doubling attractor (see, e.g, [CT] and [TC].

Suppose that $\{f_a\}$ is a suitably smooth 1-parameter family of quadratic-like maps on the interval whose critical value increases as the parameter increases. An example is the family $\{x \mapsto ax(1-x)\}$. Then the family will undergo a process of bifurcation of its periodic orbits (called period-doubling) as the parameter value increases. At a distinguished parameter value $\bar{a}$, the map will have an attracting Cantor set with a $\{2^n\}$-presentation on which $f_{\bar{a}}$ is a period-doubling homeomorphism (see [S]).

Example 1.1 illustrates how an infinitely-renormalizable homeomorphism can arise as the restriction of a continuous map on the interval to an invariant Cantor set. We take the synthetic perspective, starting with a map on an embedded Cantor set. With this information alone, it is possible to reconstruct the behavior of such a map outside the Cantor set. Given a map $f : K \to K$, we say that $g : I \to \mathbb{R}$ is an **extension** of f to I if $g|_K = f$. Any extension of an infinitely-renormalizable map to a continuous endomorphism of the interval will have distinguished periodic points whose existence is forced by the dynamics on the Cantor Set. Their period and ordering is determined by the order type σ.

Lemma 1.2: *Let $f : K \to K$, $K \subset I$, be an infinitely-renormalizable homeomorphism with order type σ. If g is an extension of f to a continuous endomorphism of I, then:*

(a) for every $n \geq 0$, there exists an $x_0 \in K_{n,0}$ such that $g^{q_n}(x_0) = x_0$ and for $k < q_n$, $g^k(x_0) \in K_{n,\sigma_n^k(0)}$. In particular, x_0 is periodic with period q_n.

(b) g acts minimally on K. In particular, g has no periodic points in K.

The proof relies on elementary applications of the intermediate value theorem, and is found in [BORT].

With infinitely-renormalizable maps, there is no obstruction to extending continuously from the Cantor set to the entire interval. Given any infinitely-renormalizable homeomorphism f with order type σ, we can construct a presented Cantor set K and a map $g_{\sigma,K} : I \to \mathbb{R}$ such that $g_{\sigma,K}|_K$ is infinitely-renormalizable with order type σ.

In this construction, the Cantor set K is uniformly-branched with the ratio of the size of an interval at level $n+1$ to that of an interval at level n determined by the sequence $\{a_n\}$. We start with

$$K_{0,0} = [0,1].$$

At level n, we divide each of the q_{n-1} intervals at level $n-1$ into $2a_{n-1} - 1$ intervals of equal length. From each interval at level $n-1$ we remove every second open interval in the division, leaving a total of $a_{n-1} \cdot q_{n-1} = q_n$ closed subintervals. Each of these subintervals is now an interval at level n. We label these intervals $K_{0,0}, \ldots, K_{0,q_n-1}$, from left to right in I.

The intersection of these intervals is a Cantor set K. The presentation of K described above and the order type σ of f induces the action of our map $g_{\sigma,K}$ on K. To define $g_{\sigma,K}$ at a point x in a gap of K, interpolate linearly between the values of $g_{\sigma,K}$ at the two nearest points in K to the left and to the right of x. This procedure yields a map from the interval to itself.

Proposition 1.3:

(a) $g_{\sigma,K}$ is continuous for all σ, K.

(b) If $\{a_n\}$ is bounded above, then $g_{\sigma,K}$ is Lipschitz.

(c) $a_n = 2$ for all n if and only if $g_{\sigma,K}$ has topological entropy zero.

Proof: The proofs of (a) and (b) are not difficult, so we prove only (c).To show that $h_{g_{\sigma,K}} = 0$, we use the fact that an endomorphism of the interval has positive topological entropy if and only if it has a periodic point of period $k \cdot 2^n$, where k is an odd integer greater than one and n is a nonnegative integer. (See [BF], [M]). It is straightforward to check that if f is period-doubling, then $g_{\sigma,K}$ has no periodic points of this form. Lemma 1.2 implies that period-doubling is a necessary condition.$\diamondsuit$

Remark: It is possible to modify the construction of $g_{\sigma,K}$ to get a map $g_{\sigma,K'}$ with bounded variation on the interval; it is enough to enlarge the gaps $G_{n,i}$ and shrink the $K_{n,i}$ according to the magnitude of a_{n+1}. Details are left to the reader.

III. An Obstruction to Smoothability

If there exists a homeomorphism $h : I \to I$ and a C^k map $g : I \to I\!R$ such that $g \circ h = h \circ f$, then we say that f is C^k **- smoothable**. Alternately, we say that g is a C^k**-smoothing** of f. Observe that we can transport the presentation of K to a presentation of $h(K)$ so that $g|_{h(K)}$ is infinitely-renormalizable.

Proposition 1.3 implies that any infinitely renormalizable map f of bounded type (that is, one in which $\{a_n\}$ is bounded) is Lipschitz-smoothable. By contrast, if we allow f to have unbounded type, we have:

Proposition 2.1:

(a) There exists an infinitely-renormalizable map $f : K \to K$ (of unbounded type) such that any C^0-smoothing of f has infinite topological entropy. Hence f cannot be Lipschitz-smoothed.

(b) There exists an infinitely-renormalizable map $f : K \to K$ (of unbounded type) that cannot be C^α-smoothed for any $0 < \alpha \leq 1$.

The proof of Propostion 2.1 is left to the reader: a simple way is to use examples from [MN] and [GT].

The order type of an infinitely-renormalizable map $f : K \to K$ forces any continuous extension of f to have turning points, and so forces a smoothing of f to have critical points. Hence the order type might present a combinatorial obstruction to smoothability. In this section, we formulate this notion more precisely.

Let σ be the order type of f and let $\{q_n = |\sigma_n|\}_0^\infty$ be the associated renormalization sequence. Let $\mathcal{E}(f)$ denote the set of all continuous extensions of f to the interval. We define the **critical sequence** $\{k_n\}$ of f by

$$k_n = \inf_{g \in \mathcal{E}(f)} \#\{0 \leq i \leq q_n - 1 \mid g \text{ is not monotone on } K_{n,i}\}.$$

Theorem 2.2: *Let $f : K \to K$ have critical sequence $\{k_n\}$. If f is C^1-smoothable, then*

$$\lim_{n \to \infty} \frac{k_n}{q_n} = 0.$$

Remark: Recently, de Melo and van Strien [MS] have shown that when the sequence $\{k_n\}$ is bounded above, the map f is smoothable by a polynomial map. Our result also applies to those maps which are, by necessity, infinitely modal (i.e., have infinitely many turning points) in any continuous extension. Some of these maps can be quite smooth; using methods from [T], it is easy to construct examples of C^∞ maps with infinitely many turning points, in particular a one-dimensional analogue to the embeding of the disk constructed in [BF].

A strictly weaker obstruction to smoothness, but one with a "finite depth" description, is the measure of the growth of turning points in f determined by its order type σ. For each n, consider those continuous maps from the interval $[0, q_n - 1]$ to itself that agree with σ_n on the integer points $\{0, \ldots, q_n - 1\}$. Let c_n be the minimal number of turning points of such an extension of σ_n. We have the following Corollary:

Corollary 2.3: *If f is C^1-smoothable, then*

$$\lim_{n \to \infty} \frac{c_n}{q_n} = 0.$$

We need the following proposition, whose proof we defer until after the proof of the Theorem.

Recall that the orbit of a periodic point x of a smooth map g of period k is **attracting** if $|Dg^k(x)| < 1$, **repelling** if $|Dg^k(x)| > 1$ and **indifferent** if $|Dg^k(x)| = 1$.

Proposition 2.4: *Suppose that $g : I \to \mathbb{R}$ is C^1 and there exists a $\{q_n\}$-presented Cantor Set K such that $g|_K$ is infinitely renormalizable. Then for every $n \geq 0$, g has a periodic point z with period $p_n \cdot q_n$, where $1 \leq p_n \leq a_n$, so that the orbit of z is either repelling or indifferent. Further, every gap $G_{n,i}$ of the presented Cantor set K contains precisely p_n points of the orbit of z, for $i = 0, \ldots, q_n - 1$.*

Proof of Theorem 2.2: Suppose that g is a C^1-smoothing of f. Since it will not alter the dynamics, we may assume without loss of generality that g is an extension of $f : K \to K$ to the interval. Let k'_n be the number of intervals $K_{n,i}$ at level n that contain a critical point of g. Note that $k_n \leq k'_n$ for all n. We show that $\frac{k'_n}{q_n}$ converges to 0 as n tends to infinity.

Because I is compact and g is C^1, we can pick $\delta > 0$ such that, for all $x, y \in I$, if $|x - y| < \delta$ then $|g'(x) - g'(y)| < \epsilon$. For the same reason, there exists $M > 0$ such that $| g'(x) | \leq M$ for all $x \in I$. Pick N so that when $n \geq N$, we have: $| G_{n,j} | < | K_{n,j} | < \delta$.

The sequence $\{\frac{k'_n}{q_n}\}$ is nonincreasing and bounded below, so the limit

$$\lim_{n\to\infty} \frac{k'_n}{q_n}$$

exists. We need only show that it converges to 0. Assume the contrary to the conclusion; that is, assume that there exists a $k > 0$ such that the number of critical intervals at stage n is bounded below by $k \cdot q_n$, for every $n \geq 0$.

By Proposition 2.4, we can pick $n > N$ such that g has a periodic point $x_n \in G_{n,0}$ of period $p_n \cdot q_n$, where $p_n \leq a_n$, whose orbit is either repelling or indifferent. Further, p_n points of the orbit of x_n lie in $G_{n,i}$, for $i = 0, \ldots, q_n - 1$. By our assumption, at least $k \cdot p_n \cdot q_n$ points on the orbit of x_n are within δ of a critical point of g. That is, there exist distinct $j_1, \ldots, j_{k \cdot p_n \cdot q_n} \in \{1, \ldots, q_n\}$ such that

$$|g'(g^{j_i}(x_n))| < \epsilon$$

for $i = 1, \ldots, k \cdot p_n \cdot q_n$. Since x_n is repelling or indifferent,

$$|Dg^{p_n \cdot q_n}(x_n)| = |\prod_{i=1}^{p_n \cdot q_n} g'(g^{i-1}(x_n))| \geq 1.$$

Hence,

$$M^{p_n \cdot q_n - k \cdot p_n \cdot q_n} \cdot \epsilon^{k \cdot p_n \cdot q_n} > |Dg^{p_n \cdot q_n}(x_n)| \geq 1.$$

This implies that

$$M \geq (\frac{1}{\epsilon})^{\frac{k}{1-k}}.$$

Since ϵ was arbitrary, this gives a contradiction.$\diamondsuit$

We use the following lemma to prove Proposition 2.4. Assume that $g|_K = f$ is infinitely renormalizable, and K is presented with intervals $K_{n,i}$ and gaps $G_{n,j}$ at level n.

Lemma 2.5: *Given g as above. For every n there is a map $\tilde{g} : I \to R$ satisfying the following two properties:*

(a) $\tilde{g}(K_{k,i}) = K_{k,\sigma_k(i)}$, for $k = 0, \ldots, n+2$ and $i = 0, \ldots, q_k - 1$

(b) There exists a $\gamma > 0$, such that for every $k = 0, \ldots, n+2$,

$$|\ \tilde{g}^{q_k}(x) - x\ | < \gamma \Longrightarrow g^{q_k}(x) = \tilde{g}^{q_k}(x).$$

Proof of Lemma 2.5: For all i, j, let $l_{i,j}$ and $r_{i,j}$ be the smallest and largest elements, respectively, of $K_{i,j}$. Note that, for all i, j, we have $l_{i,j}, r_{i,j} \in K$. Define $\tilde{g}$ as follows. Let

$$g_0(x) = \begin{cases} 1 & \text{if } g(x) \geq 1 \\ 0 & \text{if } g(x) \leq 0 \\ g(x) & \text{otherwise.} \end{cases}$$

Now define, inductively, for $k = 1, \ldots, n+2$:

$$g_k(x) = \begin{cases} l_{k,\sigma_k(j)} & \text{if } x \in K_{k,j} \text{ and } g_{k-1}(x) \leq l_{k,\sigma_k(j)} \\ r_{k,\sigma_k(j)} & \text{if } x \in K_{k,j} \text{ and } g_{k-1}(x) \geq r_{k,\sigma_k(j)} \\ g_{k-1}(x) & \text{otherwise.} \end{cases}$$

Let $\tilde{g} = g_{n+2}$. By construction, $\tilde{g}$ satisfies property (a).

To show that $\tilde{g}$ satisfies (b), we first observe that $\tilde{g}|_K = g|_K$; this means that, $\tilde{g}$ is an infinitely-renormalizable map with the same order type and presented Cantor set as g. Suppose that $g(x) \neq \tilde{g}(x)$, for some x. By definition of $\tilde{g}$, this implies that there exists k, $0 \leq k \leq n+2$, such that $x \in K_{k,j}$ but $g(x) \notin K_{k,\sigma_k(j)}$. We also know that $g(\tilde{K}_{k,j}) = \tilde{K}_{k,\sigma_k(j)}$, so if $g(x) \notin K_{k,\sigma_k(j)}$, then $x \notin K$. Hence, $\tilde{g}$ and g coincide on K, and, in particular, $\tilde{g}$ acts minimally on K. Note further that by our construction, if $\tilde{g}(x) \neq g(x)$, then $\tilde{g}(x) \in K$. This fact, combined with the minimality of $\tilde{g}$ on K, implies that if x is a periodic point for $\tilde{g}$, then x is also a periodic point for g (of the same period).

Now consider the following subset of I:

$$\mathcal{C} = \bigcup_{i=1,k=0,j=0}^{q_{n+2},n+2,q_k-1} \tilde{g}^{-i}(l_{k,j}) \cup \bigcup_{i=1,k=0,j=0}^{q_{n+2},n+2,q_k-1} \tilde{g}^{-i}(r_{k,j}).$$

The closed set $\mathcal{C}$ contains the set of points in the interval where g^{q_k} and $\tilde{g}^{q_k}$ fail to coincide, for $k = 0, \ldots, n+2$. By the above considerations, $\mathcal{C}$ contains no periodic points for $\tilde{g}$. So, for every $0 \leq k \leq n+2$, the set $\mathcal{C}$ is disjoint from the set

$$\mathcal{D}_k = \{x \in I \mid \tilde{g}^{q_k}(x) = x\}.$$

Because $\mathcal{C}$ is a closed subset of I, the $\mathcal{D}_k$ are uniformly bounded away from $\mathcal{C}$; thus, there exists a $\gamma > 0$ so that $\mathcal{C}$ is disjoint from a γ-neighborhood of $\mathcal{D}_k$, for every $0 \leq k \leq n+2$. This proves (b).$\diamondsuit$

Proof of Proposition 2.4: By Lemma 1.2, for all $k \geq 0$ there exists $x_k \in G_{k,0}$ of period q_k. Assume that for all but finitely many n, the orbit of $x_n \in G_{n,0}$ is attracting of period q_n and the orbit of $x_{n+1} \in G_{n+1,0}$ is attracting of period q_{n+1}. Fix such an n, and fix x_{n+1} and x_n (note that $x_{n+1} < x_n$). For this n, we may find a map $\tilde{g}$ satisfying the conditions of Lemma 2.5.

For $k = 0, \ldots, n+2$, we have that $\tilde{g}(K_{k,i}) = K_{k,\sigma_k(i)}$, for $i = 0, \ldots, q_k - 1$. Consider the map $h = \tilde{g}^{q_{n+1}} |_{K_{n,0}}$. Assume without loss of generality that x_n and x_{n+1} are also periodic points for $\tilde{g}$ of period q_n and q_{n+1}, respectively. We have that $h(x_n) = x_n$ and $h(x_{n+1}) = x_{n+1}$. Since x_n and x_{n+1} both have attracting orbits, we also have that $| h'(x_n) | < 1$ and $| h'(x_{n+1}) | < 1$. This implies that there exists δ such that for any $y \in K_{n,0}$, if $0 < x_n - y < \delta$, then $h(y) > y$, and if $0 < y - x_{n+1} < \delta$, then $h(y) < y$. Hence there exists $z \in K_{n,0}$ such that $h(z) = z$ and, for some ϵ with $\gamma > \epsilon > 0$, (where γ is as in Lemma 2.5, part (b)) we have that $h(w) \leq w$ for $0 < z - w < \epsilon$, and $h(w) \geq w$ for $0 < w - z < \epsilon$.

Thus:

$$\frac{h(z_1) - h(z_2)}{z_1 - z_2} \geq \frac{z_1 - z_2}{z_1 - z_2} = 1,$$

for all z_1, z_2 such that $z - \epsilon < z_1 < z < z_2 < z + \epsilon$. In this neighborhood, h is differentiable (by Lemma 2.5, part (b)), and $h'(z) \geq 1$.

Since z is a fixed point of h, z is a periodic point of $\tilde{g}$ (and hence, of g) of period at most q_{n+1}. Because $\tilde{g}$ is aperiodic on K, the point z must lie in some gap $G_{k,j}$ of K, for some k. Since z lies in $K_{n,0}$, we know that k is at least n. If k is greater than $n+1$, then z must also lie in an interval $K_{n+2,j'}$. By the construction of $\tilde{g}$, however, the q_{n+2} intervals at level $n+2$ are cyclically permuted by $\tilde{g}$, so if z is in $K_{n+2,j'}$, then the orbit of z contains at least q_{n+2} distinct points. This is impossible, so k is either n or $n+1$. Therefore, z is a periodic point for g of period $p_n \cdot q_n$, where $1 \leq p_n \leq a_n$. In either case, the orbit of z is either repelling or indifferent.$\diamondsuit$

The following two facts are useful:

(1) If an interval of length r contains a zero of an α-Hölder function, then the values of the function on that interval are bounded by $c \cdot r^{\alpha}$, where c is a uniform constant.

(2) The product of the lengths of k'_n disjoint intervals contained in the unit interval is at most $\left(\frac{1}{k'_n}\right)^{k'_n}$.

Using (1) and (2), we can modify the proof of Theorem 2.2 to show Theorem 2.6.

Theorem 2.6: *Let f be infinitely renormalizable with critical sequence $\{k_n\}$. If f is $C^{1+\alpha}$-smoothable for some $0 < \alpha \leq 1$,then there exists a $c > 0$ such that $n \cdot \frac{k_n}{q_n} \leq c$, for all n. That is, the sequence $\{\frac{k_n}{q_n}\}$ is $O(\frac{1}{n})$.*

This result allows to construct a wealth of examples of period-doubling maps that are C^1-smoothable but not $C^{1+\alpha}$-smoothable for any $\alpha > 0$.

Question: *Are there further obstructions to higher smoothability in terms of the sequence $\{\frac{k_n}{q_n}\}$? For $k > 1$ are there infinitely-renormalizable maps that are C^k-smoothable, but not C^{k+1}-smoothable?*

IV. Cascades of Periodic Orbits and Homeomorphisms of the 2-disk

Let f be an orientation-preserving embedding of the 2-disk $\mathbb{D}^2$. A **cascade of periodic orbits** for f is an infinite sequence of periodic orbits $\{O_n\}$ of f with periods $\{q_n\}$ such that, for each $n \geq 1$, we have:
- $q_n = a_n \cdot q_{n-1}$ with $q_0 = 1$ and $a_n > 1$,
- there exists a collection of disjoint, simple closed curves $\mathcal{C}_n^0, \ldots, \mathcal{C}_n^{q_{n-1}-1}$ bounding the disjoint disks $\mathcal{D}_n^0, \ldots, \mathcal{D}_n^{q_{n-1}-1}$, with the following properties:
  - each $\mathcal{D}_n^i$ contains exactly one point of O_{n-1}, and a_n points of O_n,
  - $f(\mathcal{C}_n^i)$ is isotopic to $\mathcal{C}_n^{i+1 \, mod \, q_{n-1}}$ in the punctured disk $\mathbb{D}^2 \setminus \bigcup_{i \leq n} O_i$,
  - the union of the $\mathcal{D}_n^i$'s is contained in the union of the $\mathcal{D}_{n-1}^i$'s,
  - the diameters of the $\mathcal{D}_n^i$'s converge uniformly to 0 with n.

Let $\{f_t\}_{t\in[0,\,1]}$ be an arc of embeddings joining the identity map to $f = f_1$, and $\{f_t\}_{t\in\mathbb{R}}$ be the extended arc of embeddings joining the identity map to all iterates of f, with $f_t = f^{[t]} \circ f_{\{t\}}$. To each cascade of periodic orbits $\{O_n\}$, we associate a **signature** $\{(\tilde{l}_n, q_n)\}$, where $\tilde{l}_n = \frac{l_n}{q_n}$, and l_n is defined as follows:
In one of the $\mathcal{D}_n^i$'s, pick the point x_{n-1} of O_{n-1}, and a point x_n of O_n; then l_n is the algebraic number of loops that the vector $\frac{f_t(x_n)-f_t(x_{n-1})}{\|f_t(x_n)-f_t(x_{n-1})\|}$ performs on the unit circle when t goes from 0 to q_n. The number l_n is independent of the choice of $\mathcal{D}_n^i$, and of the choice of the point x_n in $\mathcal{D}_n^i$.

For each signature $\{(\tilde{l}_n, q_n)\}$ with $\tilde{l}_n = \frac{l_n}{q_n}$, the sequence $\{q_n\}$ is strictly increasing and the integers $a_n = \frac{q_n}{q_{n-1}}$ and l_n are coprime for each n.

Remark. *When changing the isotopy $\{f_t\}_{t\in[0,\,1]}$, all the l_n are changed by adding the same integer.*

The following result is proven in [GST].

Theorem 4.1: *If $\{O_n\}$ is a cascade of periodic orbits for a C^1 diffeomorphism f of the 2-disk, then $\lim_{n\to\infty}(\tilde{l}_n)$ exists.*

Let σ be the order type of a period-doubling homeomorphism. By Lemma 1.2, the map $g_{\sigma,K}$ has a cascade of periodic orbits $\{O_n\}$, where $O_n = \{x_0, x_1, \ldots, x_{2^n-1}\}$, with $x_0 < x_1, \ldots < x_{2^n-1}$. For each n we associate weights to the points of O_n as follows: we consider the minimal (piecewise linear) interpolation f_n of the finite cascade $\{O_i\}_{0\leq i\leq n}$, and set $w(x_{2j}) = w(x_{2j+1}) = 0$ if f_n is increasing on $[x_{2j}, x_{2j+1}]$, and $w(x_{2j}) = w(x_{2j+1}) = 1$ otherwise (then f_n is decreasing on $[x_{2j}, x_{2j+1}]$). Then we have

$$l_n = \sum_{x_j \in O_n} w(x_j)$$

and

$$\tilde{l}_n = \frac{\sum_{x_j \in O_n} w(x_j)}{2^n}.$$

In the two-dimensional case when the embedding is smooth, there is also the notion of an infinitessimal winding number: it is the winding number of a vector in the tangent space around an orbit (see [R] and [GST]). Given a cascade $\{O_n\}$, this yields a sequence of infinitessimal twisting numbers ω_n, and average infinitessimal twisting numbers $\tilde{\omega}_n = \frac{\omega_n}{q_n}$. The average infinitessimal twisting number can also be shown to be a well-defined number ω on the Cantor set in the closure of $\{O_n\}$, with the following property

Theorem 4.2([GST]). *If $\{O_n\}$ is a cascade of periodic orbits for a C^1 diffeomorphism f of the 2-disk, then $\lim_{n\to\infty}(\tilde{l}_n) = \omega$.*

These infinetessimal quantities can be computed in one dimension by introducing a new system of weights.

Let $\{O_n\}$ be the cascade associated to the period-doubling map $g_{\sigma,K}$. The map $g_{\sigma,K}$ is the minimal interpolation (in terms of turning points) of $\{O_n\}$. For $x_i \in O_n$, set $s(x_i) = 0$ if $g_{\sigma,K}$ is increasing at x_i and $s(x_i) = 1$ if $g_{\sigma,K}$ is decreasing at x_i (again, one of these two possibilities holds true). Then

$$w_n = \sum_{x_j \in O_n} s(x_j)$$

and

$$\tilde{w}_n = \frac{\sum_{x_j \in O_n} s(x_j)}{2^n}.$$

On the Cantor set K, the s-weight $s(x)$ is not defined if $g_{\sigma,K}$ is not monotone at x, but otherwise is defined in the same way as for periodic orbits. In the smooth case, we know by Theorem 1 that the critical points of any smoothing have zero measure for the (unique) invariant probability measure ν on K, and so the set of points in K where s is not defined have zero measure. By the ergodic theorem, the average infinitessimal winding number along orbits

$$\sum_{i=0}^{\infty} s(g^i_{\sigma,K}(x))$$

is thus defined almost everywhere on K, and is equal to the average value of s:

$$\int_K s(x) d\nu(x).$$

Theorem 2.2 and Corollary 2.3 thus imply the following

Corollary 4.3: *If $\{O_n\}$ is a cascade of periodic orbits with $a_n \equiv 2$ for a C^1 endomorphism f of the interval, then $\lim_{n\to\infty}(\tilde{l}_n) = \omega$.*

We leave to the reader to check that the restriction to the case $a_n \equiv 2$ is indeed superfluous in Corollaries 4.2 and 4.3. Since Theorem 2.2 implies Corollary 2.3, but is strictly stronger, Theorem 2.3 is stronger than the one-dimensional version of Theorems 4.1 and 4.2.

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