On the ergodicity of partially hyperbolic systems

Keith Burns and Amie Wilkinson

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Abstract

Pugh and Shub [PS3] have conjectured that essential accessibility implies ergodicity, for a C^2, partially hyperbolic, volume-preserving diffeomorphism. We prove this conjecture under a mild center bunching assumption, which is satisfied in particular by all partially hyperbolic systems with 1-dimensional center bundle. We also obtain ergodicity results for $C^{1+\delta}$ partially hyperbolic systems.

Introduction

In [Ho] Eberhard Hopf introduced a simple argument that proved the ergodicity (with respect to Liouville measure) of the geodesic flow of a compact, negatively curved surface. The argument has since been applied to increasingly general classes of dynamical systems. The key feature that these systems possess is hyperbolicity. The strongest form of hyperbolicity is uniform hyperbolicity. A diffeomorphism $f : M \to M$ of a compact manifold M is *uniformly hyperbolic* or *Anosov* if there exists a splitting of the tangent bundle into Tf-invariant subbundles:

$$TM = E^s \oplus E^u,$$

and a continuous Riemannian metric, such that for every unit vector $v \in TM$:

$$\|Tfv\| < 1 \qquad \text{if } v \in E^s, \tag{1}$$

$$\|Tfv\| > 1 \qquad \text{if } v \in E^u. \tag{2}$$

Anosov flows are defined similarly, with $E^s \oplus E^u$ complementary to the bundle E^0 that is tangent to the flow direction. The bundles E^s and E^u of an Anosov system are tangent to the *stable* and *unstable* foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. The properties of these foliations play a crucial role in the Hopf argument.

Hopf's original argument established ergodicity for volume-preserving uniformly hyperbolic systems under the assumption that the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are C^1. While the leaves of these foliations are always as smooth as the diffeomorphism, the foliations are usually only only Hölder continuous in the direction transverse to the leaves. In particular, for geodesic flows on arbitrary compact manifolds of negative sectional curvature, these foliations are not always C^1.

Anosov and Sinai [AS, A] observed that the C^1 condition on the stable and unstable foliations in the Hopf argument could be replaced by the weaker condition of absolute continuity, which we discuss in Section 2.2. They showed that $\mathcal{W}^s$ and $\mathcal{W}^u$ are absolutely continuous if the system is C^2, thereby establishing ergodicity of all C^2 volume-preserving uniformly hyperbolic systems, including geodesic flows for compact manifolds of negative sectional curvature.

At this point, it became clear that the Hopf argument should extend to even more general settings. Two natural generalizations of uniform hyperbolicity are:

- nonuniform hyperbolicity, which requires hyperbolicity along almost every orbit, but allows the expansion of E^u and the contraction of E^s to weaken near the exceptional set where there is no hyperbolicity; and
- partial hyperbolicity, which requires uniform expansion of E^u and uniform contraction of E^s, but allows central directions at each point, in which the expansion and contraction is dominated by the behavior in the hyperbolic directions.

The first direction is Pesin theory; the second is the subject of this paper.

Brin and Pesin [BP] and independently Pugh and Shub [PS1] first examined the ergodic properties of partially hyperbolic systems soon after the work of Anosov and Sinai. The current definition of partial hyperbolicity is more general than theirs, but has the same basic features.[1] We say that a

[1]The difference is that in [BP] the functions $\nu, \hat{\nu}, \gamma$ and $\hat{\gamma}$ in the definition of partial hyperbolicity are assumed to be constant.

diffeomorphism $f : M \to M$ of a compact manifold M is partially hyperbolic if the following conditions hold. There is a nontrivial splitting of the tangent bundle, $TM = E^s \oplus E^c \oplus E^u$, that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{3}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{4}$$
$$\gamma(p) < \|Tfv\| < \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{5}$$
$$\hat{\nu}(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{6}$$

Partial hyperbolicity is a C^1-open condition: any diffeomorphism sufficiently C^1-close to a partially hyperbolic diffeomorphism is itself partially hyperbolic. Partially hyperbolic flows are defined similarly. For an extensive discussion of examples of partially hyperbolic dynamical systems, see the survey article [BPSW] and the book [P]. Among these examples are: the time-1 map of an Anosov flow, the frame flow for a compact manifold of negative sectional curvature, and many affine transformations of compact homogeneous spaces.

All of these examples preserve the volume induced by a Riemannian metric on M. The methods based on Hopf's argument actually apply to a slightly larger class of invariant measures, namely those that lie in the measure class of a volume, meaning that they have the same sets of measure 0 as a volume (note that the volumes of any two Riemannian metrics lie in the same measure class). By a slight abuse of notation, we say that f is *volume-preserving* if it preserves a probability measure in the measure class of a volume. In this paper, m will denote a Riemannian volume on M (not necessarily invariant), and μ will denote an invariant probability measure that lies in the measure class of m. Measurability is with respect to the Borel σ-algebra on M.

As in the Anosov case, the stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are tangent to foliations, which we again denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively [BP]. Brin-Pesin and Pugh-Shub proved that these foliations are absolutely continuous.

By its very nature, the Hopf argument shows that for almost every $p \in M$, almost every point of $\mathcal{W}^s(p)$ and almost every point of $\mathcal{W}^u(p)$ lies in the ergodic component of p. Thus we can only hope to prove ergodicity using a Hopf argument if something close to the following condition holds.

Definition: A partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if any point in M can be reached from any other along an *su-path*, which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$.

The *accessibility class of* $p \in M$ is the set of all $q \in M$ that can be reached from p along an *su*-path. Accessibility means that there is one accessibility class, which contains all points. The following notion is a natural weakening of accessibility.

Definition: A partially hyperbolic diffeomorphism $f : M \to M$ is *essentially accessible* if every measurable set that is a union of entire accessibility classes has either full or zero volume.

Pugh and Shub have conjectured that essential accessibility implies ergodicity, for a C^2, partially hyperbolic, volume-preserving diffeomorphism [PS2]. We prove this conjecture under one, rather mild additional assumption.

Definition: A partially hyperbolic diffeomorphism is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\nu < \gamma\hat{\gamma} \qquad \text{and} \qquad \hat{\nu} < \gamma\hat{\gamma}. \tag{7}$$

Our main result is:

Theorem 0.1 *Let f be C^2, volume-preserving, partially hyperbolic and center bunched. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

This result extends earlier results about ergodicity of partially hyperbolic systems. Brin and Pesin [BP] proved in the early 1970's that a C^2 volume-preserving partially hyperbolic diffeomorphism that is essentially accessible is ergodic if it satisfies the following additional conditions:

- *Center bunching:* Inequalities (7) hold.
- *Dynamical coherence:* There are foliations $\mathcal{W}^c$, $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to E^c, $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively.

- *Lipschitzness of $\mathcal{W}^c$:* There are Lipschitz foliation charts for $\mathcal{W}^c$.

While the Brin-Pesin argument applies to many examples of partially hyperbolic diffeomorphisms, the third condition is in some ways very restrictive: Lipschitzness of $\mathcal{W}^c$ can be destroyed by arbitrarily small perturbations [SW2]. Brin and Pesin's theorem applies in particular to the time-1 map φ_1 of the geodesic flow for a compact surface of constant negative curvature. If we make a C^1 small perturbation to φ_1, all of Brin and Pesin's hypotheses continue to hold, except Lipschitzness of $\mathcal{W}^c$ (this fact follows from combining results in [D] with an argument of Mañé — see [BPSW], p. 352).

It was not until the 1990's that Grayson, Pugh and Shub [GPS] were able to show that any small perturbation of φ_1 is ergodic; in other words, φ_1 is *stably ergodic:* any C^2 volume-preserving diffeomorphism sufficiently C^1-close to φ_1 is ergodic. The ideas in this groundbreaking paper have been generalized in several stages [W, PS2, PS3], culminating in [PS3]. The main result of [PS3] assumes dynamical coherence and uses a significantly stronger version of center bunching than inequalities (7). The center bunching hypothesis in [PS3] requires that the action of Tf on E^c be close to *isometric* — that is, both γ and $\hat{\gamma}$ (and not just their product) must be close to 1.

By contrast, our center bunching hypothesis requires only that the action of Tf on E^c be close enough to *conformal* that the hyperbolicity of f dominates the nonconformality of Tf on E^c. Center bunching always holds when $Tf|_{E^c}$ is conformal. For then we have $\|T_pfv\| = \|T_pf|_{E^c(p)}\|$ for any unit vector $v \in E^c(p)$, and hence we can choose $\gamma(p)$ slightly smaller and $\hat{\gamma}(p)^{-1}$ slightly bigger than

$$\|T_pf|_{E^c(p)}\|.$$

By doing this we may make the ratio $\gamma(p)/\hat{\gamma}(p)^{-1} = \gamma(p)\hat{\gamma}(p)$ arbitrarily close to 1, and hence larger than both $\nu(p)$ and $\hat{\nu}(p)$.

In particular, center bunching holds whenever E^c is one-dimensional. As a corollary, we obtain:

Corollary 0.2 *Let f be C^2, volume-preserving and partially hyperbolic with* $\dim(E^c) = 1$. *If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

This establishes the Pugh-Shub Conjecture mentioned above in the case where E^c is 1-dimensional.

Corollary 0.2 has also been recently proved by F. Rodríguez Hertz, J. Rodríguez Hertz, and R. Ures [HHU]. Their argument is mainly based on techniques in an earlier version[2] of the present paper [BW1]. They prove in addition that stable accessibility is C^r dense among the C^r partially hyperbolic diffeomorphisms with 1-dimensional center, which implies, for $r \geq 2$, that stable ergodicity is C^r dense among the volume-preserving C^r partially hyperbolic diffeomorphisms with one-dimensional center. Their work establishes the main stable ergodicity conjectures of Pugh and Shub ([PS3], Conjectures 1-3) in the case where E^c is one-dimensional.

There is only one place in the proof of Theorem 0.1 where we need the diffeomorphism to be C^2 as opposed to $C^{1+\delta}$, for some $\delta > 0$. This is when we use the fact that center bunching implies that the stable and unstable holonomies between center leaves are Lipschitz. This fact is proved using a graph transform argument in [BP] and also (in a slightly more general setting) in [PSW, PSWc]. In [BW2], we show that the same result about holonomies holds when C^2 is replaced by $C^{1+\delta}$, at the expense of a more stringent bunching hypothesis. Plugging this result into the proof of Theorem 0.1, we obtain:

Theorem 0.3 *Let f be $C^{1+\delta}$, volume-preserving, and partially hyperbolic. Let $\mu, \hat{\mu}$ be continuous functions satisfying:*

$$\mu(p) < \|Tfv\|, \qquad \textit{if } v \in E^s(p), \tag{8}$$

$$\|Tfv\| < \hat{\mu}(p)^{-1}, \qquad \textit{if } v \in E^u(p). \tag{9}$$

Suppose that f satisfies the strong center bunching condition*:*

$$\nu^\theta < \gamma\hat{\gamma} \qquad \textit{and} \qquad \hat{\nu}^\theta < \gamma\hat{\gamma}, \tag{10}$$

where $\theta \in (0, \delta)$ satisfies the inequalities:

$$\nu\gamma^{-1} < \mu^\theta, \qquad \hat{\nu}\hat{\gamma}^{-1} < \hat{\mu}^\theta. \tag{11}$$

If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.

[2] This earlier version proved the same result as the present paper but under the additional hypothesis of dynamical coherence, i.e. the existence of foliations tangent to the bundles $E^c \oplus E^u$ and $E^c \oplus E^s$.

We remark that any θ satisfying the conditions in (11) is a Hölder exponent for the central distribution E^c (see, e.g., Theorem A in [PSW]). It would be interesting to know whether (10) could be replaced by (7) in Theorem 0.3. The strong center bunching condition in (10) is automatically satisfied when E^c is one-dimensional, since, as above, we may then choose γ and $\hat{\gamma}^{-1}$ so that $\gamma\hat{\gamma}$ is arbitrarily close to 1. As a corollary of Theorem 0.3, we obtain:

Corollary 0.4 *Let f be $C^{1+\delta}$, volume-preserving and partially hyperbolic with $\dim(E^c) = 1$. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

Our arguments rely upon the same basic strategy as those of Pugh and Shub in [PS3]. We use a similar notion of center-unstable juliennes, and show that they are quasi-preserved by stable holonomies. But our analysis of the relationship between juliennes and density points is fundamentally different from theirs. One essential novelty is the use of a version of Cavalieri's principle, introduced in Subsection 2.3. Another is the use of the fake foliations constructed in in Section 3. The use of Cavalieri's principle greatly simplifies the analysis of the relationship between juliennes and Lebesgue density points of invariant sets, which is the heart of the proof. It allows us to weaken the center bunching hypothesis to the minimum condition needed to obtain smoothness of holonomy along stable leaves inside a leaf of a center stable foliation. The fake foliations and the simplifications using Cavalieri's principle make it possible to eliminate the hypothesis of dynamical coherence. It is interesting to note that both advances, fake foliations and Cavalieri's principle, are used in an essential way to remove the dynamical coherence assumption; the technique of fake foliations cannot be used directly to modify or improve the Pugh Shub approach to remove dynamical coherence.

Our ergodicity result, Theorem 0.1, is proved in Section 5 as a consequence of Theorem 5.1, which is really the central result of the paper. Theorem 5.1 is proved in Sections 6, 7 and 8.

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1 Preliminaries

1.1 Notational conventions

We use the convention that if q is a point in M and j is an integer, then q_j denotes the point $f^j(q)$, with $q_0 = q$. If $\alpha : M \to \mathbf{R}$ is a positive function, and $j \geq 1$ is an integer, let

$$\alpha_j(p) = \alpha(p)\alpha(p_1)\cdots\alpha(p_{j-1}),$$

and

$$\alpha_{-j}(p) = \alpha(p_{-j})^{-1}\alpha(p_{-j+1})^{-1}\cdots\alpha(p_{-1})^{-1}.$$

We set $\alpha_0(p) = 1$. Observe that α_j is a multiplicative cocycle; in particular, we have $\alpha_{-j}(p)^{-1} = \alpha_j(p_{-j})$. Note also that $(\alpha\beta)_j = \alpha_j\beta_j$, and if α is a constant function, then $\alpha_n = \alpha^n$.

The notation $\alpha < \beta$, where α and β are continuous functions, means that the inequality holds pointwise, and the function $\min\{\alpha, \beta\}$ takes the value $\min\{\alpha(p), \beta(p)\}$ at the point p.

As usual $P = O(Q)$ means that there is a constant $C > 0$ such that $|P| \leq CQ$. Usually P and Q will depend on an integer n and one or more points in M. The constant C must be independent of n and the choice of the points.

1.2 Foliation boxes and local leaves

Let $\mathcal{F}$ be a foliation of an n-manifold M with d-dimensional smooth leaves. For $r > 0$, we denote by $\mathcal{F}(x, r)$ the connected component of x in the intersection of $\mathcal{F}(x)$ with the ball $B(x, r)$.

A *foliation box for* $\mathcal{F}$ is the image U of $\mathbf{R}^{n-d} \times \mathbf{R}^d$ under a homeomorphism that sends each vertical $\mathbf{R}^d$-slice into a leaf of $\mathcal{F}$. The images of the vertical $\mathbf{R}^d$-slices will be called *local leaves of* $\mathcal{F}$ *in* U.

A *smooth transversal* to $\mathcal{F}$ in U is a smooth codimension-d disk in U that intersects each local leaf in U exactly once and whose tangent bundle is uniformly transverse to $T\mathcal{F}$. If τ_1 and τ_2 are two smooth transversals to $\mathcal{F}$ in U, we have the *holonomy map* $h_{\mathcal{F}} : \tau_1 \to \tau_2$, which takes a point in τ_1 to the intersection of its local leaf in U with τ_2.

1.3 Adapted metrics

We assume that the Riemannian metric on M is chosen so that the inequalities (3)–(6) involving $\nu, \gamma, \hat{\nu}, \hat{\gamma}$ in the Introduction hold. Such a metric will be called adapted. Note that a rescaling of an adapted metric is still adapted. It will be convenient to assume that the metric is scaled so that the geodesic balls of radius 1 are very small neighborhoods of their centers. Distance with respect to the metric will be denoted by d.

There is no harm in increasing ν and $\hat{\nu}$ and decreasing $\gamma, \hat{\gamma}$ slightly, provided that the inequalities (3)–(6) still hold. If f is center bunched, the change must also be small enough so that inequalities (7) still hold. Similarly, if f is strongly center bunched, the change must also be small enough so that inequalities (10) still hold.

By rescaling the metric on M, we may assume that for some $R > 1$, and any $x \in M$, the Riemannian ball $B(x, R)$ is contained in foliation boxes for both $\mathcal{W}^s$ and $\mathcal{W}^u$. We assume that R is large enough so that all the objects considered in the sequel are small compared with R. Having fixed such an R, we define, for $a = s$ or u, the *local leaf of* $\mathcal{W}^a$ *through* x by:

$$\mathcal{W}^a_{loc}(x) = \mathcal{W}^a(x, R).$$

Any foliation box U for either $\mathcal{W}^s$ or $\mathcal{W}^u$ that we consider in the rest of the paper will be small enough so that $\mathcal{W}^a_{loc}(x) \cap U$ is a local leaf of $\mathcal{W}^a$ in U for each $x \in U$. By (if necessary) further rescaling the metric to make the local leaves smaller, we may assume that our metric is still adapted, and that for all $p \in M$, and $q, q' \in B(p, R)$, we have the following:

$$q \in \mathcal{W}^s_{loc}(q') \quad \Longrightarrow \quad d(f(q), f(q')) \leq \nu(p) d(q, q'), \tag{12}$$

and similarly,

$$q \in \mathcal{W}^u_{loc}(q') \quad \Longrightarrow \quad d(f^{-1}(q), f^{-1}(q')) \leq \hat{\nu}(f^{-1}(p)) d(q, q').$$

In particular, $f(\mathcal{W}^s_{loc}(p)) \subset \mathcal{W}^s_{loc}(f(p))$ and $f^{-1}(\mathcal{W}^u_{loc}(p)) \subset \mathcal{W}^u_{loc}(f^{-1}(p))$, for all $p \in M$. This is possible because ν and $\hat{\nu}$ are continuous, and the inequalities (4) and (6) that they satisfy are strict.

An inductive argument then gives:

Lemma 1.1 *If* $q_j, q'_j \in B(p_j, R)$ *for* $j = 0, \dots, n-1$, *and* $q \in \mathcal{W}^s_{loc}(q')$, *then*

$$d(q_n, q'_n) \leq \nu_n(p) d(q, q').$$

If $q_{-j}, q'_{-j} \in B(p_{-j}, R)$ *for* $j = 0, \ldots, n-1$, *and* $q \in \mathcal{W}^u_{loc}(q')$, *then*

$$d(q_{-n}, q'_{-n}) \leq \hat{\nu}_{-n}(p)^{-1} d(q, q').$$

Proof. We prove the first claim; the second follows from the first, with f replaced by f^{-1}. The proof is by induction on n. The claim is vacuously true for $n = 0$. Suppose the claim holds for $n = k$. The inductive assumption gives that

$$d(q_k, q'_k) \leq \nu_k(p) d(q, q')$$

and $q'_k \in \mathcal{W}^s_{loc}(q_k)$. Then (12), applied at p_k, implies that $q'_{k+1} \in \mathcal{W}^s_{loc}(q_{k+1})$, and

$$\begin{aligned} d(q_{k+1}, q'_{k+1}) &\leq \nu(p_k) d(q_k, q'_k) \\ &\leq \nu(p_k)\nu_k(p) d(q, q') \\ &= \nu(p_{k+1}) d(q, q'). \end{aligned}$$

$\diamond$

2 Density points and absolute continuity

2.1 Volume and density

Recall that when we say that the diffeomorphism f is volume-preserving, we mean that f preserves a probability measure μ that lies in the measure class of a Riemannian volume m on M.

If $S \subseteq M$ is a smooth submanifold, we denote by m_S the volume of the induced Riemannian metric on S. If $\mathcal{F}$ is a foliation with smooth leaves, and A is contained in a single leaf of $\mathcal{F}$ and is measurable in that leaf, then we denote by $m_{\mathcal{F}}(A)$ the induced Riemannian volume of A in that leaf. A set is said to be *saturated by a foliation* $\mathcal{F}$ or $\mathcal{F}$*-saturated* if it is a union of entire leaves of $\mathcal{F}$. A set A is *essentially* $\mathcal{F}$*-saturated* if there exists a measurable $\mathcal{F}$-saturated set A', which we call an *essential* $\mathcal{F}$*-saturate* of A, with $m(A \,\Delta\, A') = 0$.

If ν is a measure and A and B are ν-measurable sets with $\nu(B) > 0$, we define the *density of* A *in* B by:

$$\nu(A : B) = \frac{\nu(A \cap B)}{\nu(B)}.$$

A point $x \in M$ is a *Lebesgue density point* of a measurable set $X \subseteq M$ if

$$\lim_{r \to 0} m(X : B_r(x)) = 1.$$

Notice that the notion of Lebesgue density point depends only on the smooth structure of M, because any two Riemannian metrics have the same Lebesgue density points.

The Lebesgue Density Theorem implies that if A is a measurable set and $\widehat{A}$ is the set of Lebesgue density points of A, then $m(A \,\Delta\, \widehat{A}) = 0$.

Lebesgue density points can be characterized using nested sequences of measurable sets. We say that a sequence of measurable sets Y_n *nests* at a point x if $Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset \{x\}$, and

$$\bigcap_n Y_n = \{x\}.$$

A sequence Y_n, with $m(Y_n) > 0$, that nests at x is a *Lebesgue density sequence at* x if for every measurable set X, x is a Lebesgue density point of X if and only if:

$$\lim_{n \to \infty} m(X : Y_n) = 1.$$

It is easily shown that a Lebesgue density sequence Y_n must be regular, a term we now define.

Definition: A nested sequence of measurable sets Y_n is *regular* if there exists $\delta > 0$ such that, for all $n \geq 0$, we have $m(Y_n) > 0$, and

$$m(Y_{n+1}) \geq \delta m(Y_n).$$

The simplest example of a Lebesgue density sequence at x is the sequence of balls $B(x, \rho^n)$, where $\rho \in (0, 1)$.

In our proof of Theorem 0.1, we characterize the Lebesgue density points of a special class of measurable sets, those that are both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated. Such sets will be called *bi essentially saturated*. These sets arise when we consider Birkhoff averages of continuous test functions; the sublevel sets of such averages are bi essentially saturated.

We say that Y_n is a *Lebesgue density sequence at* x *for bi essentially saturated sets* if Y_n nests at x, Y_n is regular, and, for every bi essentially saturated set X, x is a Lebesgue density point of X if and only if:

$$\lim_{n \to \infty} m(X : Y_n) = 1.$$

Note that Lebesgue density sequence for bi essentially saturated sets is not necessarily a Lebesgue density sequence, because only certain measurable sets are considered. In fact, many of the Lebesgue density sequences for bi essentially saturated sets constructed in this paper are not Lebesgue density sequences.

In our proof we will frequently have to pass the property of being a Lebesgue density sequence for bi essentially saturated sets from one sequence Y_n that nests at x to another sequence Z_n that nests at x. In order to do so, we have to show that Z_n is also regular and that, for every measurable set X that is bi essentially saturated,

$$\lim_{n\to\infty} m(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} m(X : Z_n) = 1. \tag{13}$$

We have two techniques for doing this. Which technique we use depends on the construction of Y_n and Z_n.

The first technique is very simple. Two nested sequences of sets Y_n and Z_n are *internested* if there exists a $k \geq 1$ such that, for all $n \geq 0$, we have

$$Y_{n+k} \subseteq Z_n, \qquad \text{and} \qquad Z_{n+k} \subseteq Y_n.$$

Internestedness is an equivalence relation. The following lemma is a straightforward consequence of the definitions.

Lemma 2.1 *Let Y_n and Z_n be internested sequences of measurable sets, with Y_n regular. Then Z_n is also regular. If the sets Y_n have positive measure, then so do the Z_n, and, for any measurable set X,*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} m(X : Z_n) = 1.$$

Corollary 2.2 *Suppose that Y_n and Z_n both nest at x and are internested. Then Y_n is a Lebesgue density sequence for bi essentially saturated sets if and only if Z_n is a Lebesgue density sequence for bi essentially saturated sets.*

The second technique uses absolute continuity of the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, plus the saturation properties of X, and is developed in the next two subsections.

2.2 Absolute continuity

Our arguments in this paper use two versions of the property of absolute continuity of a foliation.

The first version of absolute continuity involves holonomy maps between transversals. A foliation $\mathcal{F}$ with smooth leaves is *transversely absolutely continuous with bounded Jacobians* if for every angle $\alpha \in (0, \pi/2]$, there exists $C \geq 1$ and $R_0 > 0$ such that, for every foliation box U of diameter less than R_0, any two smooth transversals τ_1, τ_2 to $\mathcal{F}$ in U of angle at least α with $\mathcal{F}$, and any m_{τ_1}–measurable set A contained in τ_1:

$$C^{-1} m_{\tau_1}(A) \leq m_{\tau_2}(h_{\mathcal{F}}(A)) \leq C m_{\tau_1}(A). \tag{14}$$

The second version involves a Fubini-like property. A foliation $\mathcal{F}$ with smooth leaves is *absolutely continuous with bounded Jacobians* if, for every $\alpha \in (0, \pi/2]$, there exist $C \geq 1$ and $R_0 > 0$ such that, for every foliation box U of diameter less than R_0, any smooth transversal τ to $\mathcal{F}$ in U of angle at least α with $\mathcal{F}$, and any measurable set A contained in U, we have the inequality:

$$C^{-1} m(A) \leq \int_{\tau} m_{\mathcal{F}}(A \cap \mathcal{F}_{loc}(x))\, dm_{\tau}(x) \leq C m(A). \tag{15}$$

It is not difficult to verify that these two properties are equivalent. Note that the minimal C for which (14) holds is not necessarily the same as the minimal C for which (15) holds.

The foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ for a partially hyperbolic diffeomorphism are transversely absolutely continuous with bounded Jacobians. This was shown in the Anosov case by Anosov [A], and in the case of partial hyperbolicity by Brin-Pesin and Pugh-Shub [BP, PS1]. Their proofs were written under the assumption that the function ν, $\hat{\nu}$, γ and $\hat{\gamma}$ are constant. In the general case of partial hyperbolicity, where these functions are not constant, absolute continuity of $\mathcal{W}^s$ and $\mathcal{W}^u$ follows from Pesin theory. A direct proof in this context has been given by Abdenur and Viana [AV]. All of these results show that the Jacobians are continuous functions, and so are bounded, since M is compact. In general, $\mathcal{W}^c$ does not have either absolute continuity property, even when f is dynamically coherent (examples were first constructed by Katok [Mi]; open sets of examples by Shub-Wilkinson [SW2]).

2.3 Saturated sets and Cavalieri's principle

The second technique mentioned in subsection 2.1 involves decomposing the sets in a sequence nesting at x along the leaves of an absolutely continuous foliation. The method is reminiscent of Cavalieri's Principle, which states that the volume of a 3-dimensional solid may computed by taking 2-dimensional slices of the solid. If the areas of these slices are approximately equal, then the volume of the solid is approximately equal to the product of the slice area and the total height of the the slices.

In our setting, we use the leaves of an absolutely continuous foliation to decompose a subset of a foliation box into slices, which we call fibers. The idea of approximately equal area of the slices translates into c-uniformity of the fibers, which we define below. When the foliation is absolutely continuous, the volume of a subset of the foliation box with c-uniform fibers will be approximately the volume of one of the fibers times the volume of the projection of the set onto a transversal.

Let $\mathcal{F}$ be an absolutely continuous foliation and let U be a foliation box for $\mathcal{F}$. Let τ be a smooth transversal to $\mathcal{F}$ in U. Let $Y \subseteq U$ be a measurable set. For a point $q \in \tau$, we define the *fiber* $Y(q)$ *of* Y *over* q to be the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base* τ_Y *of* Y is the set of all $q \in \tau$ such that the fiber $Y(q)$ is $m_{\mathcal{F}}$-measurable and $m_{\mathcal{F}}(Y(q)) > 0$. The absolute continuity of $\mathcal{F}$ implies that τ_Y is m_τ-measurable. We say "Y fibers over Z" to indicate that $Z = \tau_Y$.

If, for some $c \geq 1$, the inequalities

$$c^{-1} \leq \frac{m_{\mathcal{F}}(Y(q))}{m_{\mathcal{F}}(Y(q'))} \leq c$$

hold for all $q, q' \in \tau_Y$, then we say that Y *has c-uniform fibers.* A sequence of measurable sets Y_n contained in U has *c-uniform fibers* if each set in the sequence has c-uniform fibers, with c independent of n.

Proposition 2.3 *Suppose that the foliation $\mathcal{F}$ is absolutely continuous with bounded Jacobians. Let U be a foliation box for $\mathcal{F}$, and let τ be a smooth transversal to $\mathcal{F}$ in U. Then there is a constant $C \geq 1$ such that for any $c \geq 1$, any measurable set $Y \subset U$ with c-uniform fibers, and any point $q_0 \in \tau_Y$, we have*

$$(Cc)^{-1} m(Y) \leq m_{\mathcal{F}}(Y(q_0)) m_\tau(\tau_Y) \leq Ccm(Y).$$

Proof. Absolute continuity of $\mathcal{F}$ with bounded Jacobians implies that there exists a $C \geq 1$, that depends only on $\mathcal{F}$, U, and τ, and does not depend on Y, such that

$$C^{-1}m(Y) \leq \int_\tau m_\mathcal{F}(Y(q))\, dm_\tau(q) \leq Cm(Y).$$

Since the fibers of Y are c-uniform, we have

$$c^{-1}m_\mathcal{F}(Y(q_0)) \leq m_\mathcal{F}(Y(q)) \leq cm_\mathcal{F}(Y(q_0)),$$

for any $q \in \tau_Y$. Combing these two sets of inequalities gives the desired result. $\diamond$

Regularity of a sequence with c-uniform fibers can be obtained from regularity of its fibers and bases.

Proposition 2.4 *Suppose that the foliation $\mathcal{F}$ is absolutely continuous with bounded Jacobians. Let U be a foliation box for $\mathcal{F}$, and let τ be a smooth transversal to $\mathcal{F}$ in U. Let Y_n be a sequence of subsets of U with c-uniform fibers. Suppose that there exists $\delta > 0$ such that:*

1. *for all $n \geq 0$,*
$$m_\tau(\tau_{Y_{n+1}}) \geq \delta m_\tau(\tau_{Y_n});$$
2. *for all $n \geq 0$, there are points $z \in \tau_{Y_{n+1}}, z' \in \tau_{Y_n}$ with*
$$m_\mathcal{F}(Y_{n+1}(z)) \geq \delta m_\mathcal{F}(Y_n(z')).$$

Then Y_n is regular.

Proof. It follows from Proposition 2.3 that

$$m(Y_{n+1}) \geq (cC)^{-1}m_\mathcal{F}(Y_{n+1}(z))m_\tau(\tau_{Y_{n+1}}),$$

and

$$m(Y_n) \leq cCm_\mathcal{F}(Y_n(z'))m_\tau(\tau_{Y_n}).$$

Using the two properties of δ, we then obtain

$$m(Y_{n+1}) \geq (cC)^{-2}\delta^2 m(Y_n),$$

which says that Y_n is regular. $\diamond$

We now turn to the second technique mentioned above for proving an equivalence of the form (13). The main result we prove in this section is:

Proposition 2.5 *Let $\mathcal{F}$ be absolutely continuous with bounded Jacobians, and let U be a foliation box for $\mathcal{F}$ with smooth transversal τ. Let Y_n and Z_n be sequences of measurable subsets of U with c-uniform fibers, for some $c \geq 1$. Suppose that $\tau_{Y_n} = \tau_{Z_n}$, for all n. Then, for any essentially $\mathcal{F}$-saturated set $X \subseteq U$, we have the equivalence:*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m(X : Z_n) = 1.$$

Corollary 2.6 *Fix $\mathcal{F} = \mathcal{W}^s$ or $\mathcal{W}^u$. Suppose that Y_n and Z_n satisfy the hypotheses of Proposition 2.5, and that Y_n and Z_n both nest at x. If Y_n is a Lebesgue density sequence at x for bi essentially saturated sets, and Z_n is regular, then Z_n is a Lebesgue density sequence at x for bi essentially saturated sets.*

Before proving Proposition 2.5, we establish a related result, which will also be used in the proof of Theorem 0.1.

Proposition 2.7 *Let $\mathcal{F}$ be absolutely continuous with bounded Jacobians, and let U be a foliation box for $\mathcal{F}$ with smooth transversal τ. Suppose that $\{Y_n\}_{n\geq 0}$ is a sequence of measurable sets in U with c-uniform fibers, for some $c \geq 1$. Then, for every $\mathcal{F}$-saturated measurable set X, we have the equivalence:*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m_\tau(\tau_X : \tau_{Y_n}) = 1.$$

Remark: The hypothesis that X is $\mathcal{F}$-saturated can be weakened: it suffices for $X \cap U$ to be a union of local leaves of $\mathcal{F}$ in U.

Proof of Proposition 2.7. Let X^* be the complement of X in M. Then X^* is also $\mathcal{F}$-saturated. The proposition can be reformulated in terms of X^*. We have to prove the equivalence:

$$\lim_{n\to\infty} m(X^* : Y_n) = 0 \iff \lim_{n\to\infty} m_\tau(\tau_{X^*} : \tau_{Y_n}) = 0.$$

For each n, let

$$m_n = \inf_{q\in\tau_{Y_n}} m_{\mathcal{F}}(Y_n(q)).$$

Since the fibers of Y_n are c-uniform, it follows that $m_n > 0$ for all n, and:

$$m_n \leq m_{\mathcal{F}}(Y_n(q)) \leq c m_n,$$

for all $q \in \tau_{Y_n}$. Absolute continuity implies that there exists a $C \geq 1$ such that

$$C^{-1}m(Y_n) \leq \int_\tau m_{\mathcal{F}}(Y_n(q))\, dm_\tau(q) \leq Cm(Y_n).$$

Together, these inequalities imply that

$$C^{-1}m_n m_\tau(\tau_{Y_n}) \leq m(Y_n) \leq Ccm_n m_\tau(\tau_{Y_n}). \tag{16}$$

Since X^* is $\mathcal{F}$-saturated, the $\mathcal{F}$-fiber of $X^* \cap Y_n$ over a point $q \in \tau_{Y_n}$ is either empty or equal to $Y_n(q)$. Thus $X^* \cap Y_n$ also has c-uniform fibers, and, as above, we obtain:

$$C^{-1}m_n m_\tau(\tau_{X^*\cap Y_n}) \leq m(X^* \cap Y_n) \leq Ccm_n m_\tau(\tau_{X^*\cap Y_n}). \tag{17}$$

Noting that $\tau_{X^*\cap Y_n} = \tau_{X^*} \cap \tau_{Y_n}$ and dividing the inequalities in (17) by those in (16), we obtain:

$$(C^2c)^{-1}m_\tau(\tau_{X^*} : \tau_{Y_n}) \leq m(X^* : Y_n) \leq C^2cm_\tau(\tau_{X^*} : \tau_{Y_n}).$$

The result follows easily from this. $\diamond$

Proof of Proposition 2.5. Let X' be an essential $\mathcal{F}$-saturate of X. Using Proposition 2.7, we have the equivalences:

$$\begin{aligned}
\lim_{n\to\infty} m(X : Y_n) = 1 \quad &\iff \quad \lim_{n\to\infty} m(X' : Y_n) = 1 \\
&\iff \quad \lim_{n\to\infty} m_\tau(\tau_{X'} : \tau_{Y_n}) = 1 \\
&\iff \quad \lim_{n\to\infty} m_\tau(\tau_{X'} : \tau_{Z_n}) = 1 \\
&\iff \quad \lim_{n\to\infty} m(X' : Z_n) = 1 \\
&\iff \quad \lim_{n\to\infty} m(X : Z_n) = 1. \qquad \diamond
\end{aligned}$$

3 Fake invariant foliations

The Lebesgue density sequences for bi essentially saturated sets that we use in this proof will be constructed using dynamical foliations with uniform continuity properties. If f happens to be dynamically coherent, then we are free to use the foliations $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^c, \mathcal{W}^{cs}$, and $\mathcal{W}^{cu}$ for these constructions.

Since we are not assuming dynamical coherence, we must find substitutes for $\mathcal{W}^c, \mathcal{W}^{cs}$, and $\mathcal{W}^{cu}$ to make our proof work in general. It turns out to be simplest to find substitutes for *all* invariant foliations $\mathcal{W}^s, \mathcal{W}^u, \mathcal{W}^c, \mathcal{W}^{cs}$, and $\mathcal{W}^{cu}$. We call these substitutes "fake invariant foliations." There are a few key places in the argument where we will have to use the real invariant foliations $\mathcal{W}^u$ and $\mathcal{W}^s$, rather than their fake counterparts. We will indicate where this is the case. The reader should recall the choice of the constant R from Section 1.3.

Proposition 3.1 *Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism. For any $\varepsilon > 0$, there exist constants r and r_1 with $R > r > r_1 > 0$ such that, for every $p \in M$, the neighborhood $B(p, r)$ is foliated by foliations $\widehat{\mathcal{W}}^u_p$, $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^c_p$, $\widehat{\mathcal{W}}^{cu}_p$ and $\widehat{\mathcal{W}}^{cs}_p$ with the following properties, for each $\beta \in \{u, s, c, cu, cs\}$:*

1. **Almost tangency to invariant distributions:** *For each $q \in B(p, r)$, the leaf $\widehat{\mathcal{W}}^\beta_p(q)$ is C^1 and the tangent space $T_q\widehat{\mathcal{W}}^\beta_p(q)$ lies in a cone of radius ε about $E^\beta(q)$.*

2. **Local invariance:** *for each $q \in B(p, r_1)$,*
$$f(\widehat{\mathcal{W}}^\beta_p(q, r_1)) \subset \widehat{\mathcal{W}}^\beta_{f(p)}(f(q)), \text{ and } f^{-1}(\widehat{\mathcal{W}}^\beta_p(q, r_1)) \subset \widehat{\mathcal{W}}^\beta_{f^{-1}(p)}(f^{-1}(q)).$$

3. **Exponential growth bounds at local scales:** *The following hold for all $n \geq 0$.*

   (a) *Suppose that $q_j \in B(p_j, r_1)$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^s_p(q, r_1)$, then $q'_n \in \widehat{\mathcal{W}}^s_p(q_n, r_1)$, and
   $$d(q_n, q'_n) \leq \nu_n(p) d(q, q').$$
   If $q'_j \in \widehat{\mathcal{W}}^{cs}_p(q_j, r_1)$ for $0 \leq j \leq n-1$, then $q'_n \in \widehat{\mathcal{W}}^{cs}_p(q_n)$, and
   $$d(q_n, q'_n) \leq \hat{\gamma}_n(p)^{-1} d(q, q').$$

   (b) *Suppose that $q_{-j} \in B(p_{-j}, r_1)$ for $0 \leq j \leq n-1$.*
   If $q' \in \widehat{\mathcal{W}}^u_p(q, r_1)$, then $q'_{-n} \in \widehat{\mathcal{W}}^u_p(q_{-n}, r_1)$, and
   $$d(q_{-n}, q'_{-n}) \leq \hat{\nu}_{-n}(p)^{-1} d(q, q').$$
   If $q'_{-j} \in \widehat{\mathcal{W}}^{cu}_p(q_{-j}, r_1)$ for $0 \leq j \leq n-1$, then $q'_{-n} \in \widehat{\mathcal{W}}^{cu}_p(q_{-n})$, and
   $$d(q_{-n}, q'_{-n}) \leq \gamma_{-n}(p) d(q, q').$$

4. **Coherence:** *$\widehat{\mathcal{W}}^s_p$ and $\widehat{\mathcal{W}}^c_p$ subfoliate $\widehat{\mathcal{W}}^{cs}_p$; $\widehat{\mathcal{W}}^u_p$ and $\widehat{\mathcal{W}}^c_p$ subfoliate $\widehat{\mathcal{W}}^{cu}_p$.*

5. **Uniqueness:** *$\widehat{\mathcal{W}}^s_p(p) = \mathcal{W}^s(p,r)$, and $\widehat{\mathcal{W}}^u_p(p) = \mathcal{W}^u(p,r)$.*

6. **Regularity:** *If f is $C^{1+\delta}$, then the foliations $\widehat{\mathcal{W}}^u_p$, $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^c_p$, $\widehat{\mathcal{W}}^{cu}_p$ and $\widehat{\mathcal{W}}^{cs}_p$ and their tangent distributions are uniformly Hölder continuous.*

7. **Regularity of the strong foliation inside weak leaves:** *If f is C^2 and center bunched, then each leaf of $\widehat{\mathcal{W}}^{cs}_p$ is C^1 foliated by leaves of the foliation $\widehat{\mathcal{W}}^s_p$, and each leaf of $\widehat{\mathcal{W}}^{cu}_p$ is C^1 foliated by leaves of the foliation $\widehat{\mathcal{W}}^u_p$. If f is $C^{1+\delta}$ and strongly center bunched, then the same conclusion holds.*

The regularity statements in 6. and 7. hold uniformly in $p \in M$.

Proof. Suppose that f is C^k, for some $k \geq 1$. After possibly reducing ε, we can assume that inequalities (3)–(6) hold for unit vectors in the ε-cones around the spaces in the partially hyperbolic splitting.

The construction of the leaves of $\widehat{\mathcal{W}}^{cu}_p$ and $\widehat{\mathcal{W}}^{cs}_p$ through p is essentially the same as the proof of the existence of pseudo-hyperbolic plaque families in [HPS]. They are obtained as fixed points of graph transforms of a map that coincides with f in a neighborhood of the orbit of p. We take the argument one step further and consider all fixed points of these graph transforms in the entire neighborhood of p.

Our construction will be performed in two steps. In the first, we construct foliations of each tangent space T_pM. In the second step, we use the exponential map $\exp_p$ to project these foliations from a neighborhood of the origin in T_pM to a neighborhood of p.

Step 1. We choose an $r_0 > 0$ such that $\exp_p^{-1}$ is defined on $B(p, 2r_0)$. For $r \in (0, r_0]$, we define, in the standard way, a map:

$$\begin{array}{ccc} TM & \xrightarrow{F_r} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

which is uniformly C^k on fibers, satisfying:

1. $F_r(p,v) = \exp^{-1}_{f(p)} \circ f \circ \exp_p(v)$, for $\|v\| \leq r$;

2. $F_r(p,v) = T_pf(v)$, for $\|v\| \geq 2r$;

3. $\|F_r(p,\cdot) - T_pf(\cdot)\|_{C^1} \to 0$ as $r \to 0$, uniformly in p.

Endowing M with the discrete topology, we regard TM as the disjoint union of its fibers. Property 3. implies that, if r is small enough, then F_r is partially hyperbolic, and each bundle in the partially hyperbolic splitting for F_r at $v \in T_pM$ lies within the $\varepsilon/2$-cone about the corresponding subspace of T_pM in the partially hyperbolic splitting for f at p (we are making the usual identification of T_vT_pM with T_pM). If r is small enough, the equivalents of inequalities (3)–(6) will hold with Tf replaced by TF_r.

If r is sufficiently small, standard graph transform arguments give stable, unstable, center-stable, and center-unstable foliations for F_r inside each T_pM. These foliations are uniquely determined by the extension F_r and the requirement that their leaves be graphs of bounded functions. We obtain a center foliation by intersecting the leaves of the center-unstable and center-stable foliations. While TM is not compact, all of the relevant estimates for F_r are uniform, and it is this, not compactness, that counts. The proof of the Hadamard-Perron Theorem in [KH] contains many of the details of this argument.

The uniqueness of the stable and unstable foliations implies, via a standard argument (see, e.g. [HPS], Theorem 6.1 (e)), that the stable foliation subfoliates the center-stable, and the unstable subfoliates the center-unstable.

We now discuss the regularity properties of these foliations of TM. Recall the standard method for determining the regularity of invariant bundles and foliations. Suppose that $TX = E_1 \oplus E_2$ is a Tg-invariant splitting of the tangent bundle for a C^k diffeomorphism $g : X \to X$ satisfying, for every $p \in X$, and every unit vector $v \in T_pX$:

$$\alpha_1(p) < \|Tgv\| < \beta_1(p), \qquad \text{if } v \in E_1(p), \tag{18}$$

$$\alpha_2(p) < \|Tgv\| \qquad\qquad \text{if } v \in E_2(p), \tag{19}$$

where $0 < \alpha_1(p) < \beta_1(p) < \alpha_2(p)$. Then the bundle E_2 is C^a, for any $a \leq k-1$ that satisfies:

$$\sup_{p\in M} \frac{\beta_1(p)}{\alpha_1(p)^a \alpha_2(p)} < 1.$$

When the functions $\alpha_1, \alpha_2, \beta_2$ are constant, this fact is classical — see, e.g. the C^r Section Theorem in [HPS]. The general case of this result appears more recently in the literature [SS, Ha, W, PSW]. This result extends, at least in part, to give regularity of invariant foliations. In particular, when there is a foliation $\mathcal{F}_2$ tangent to E_2 that arises at the unique fixed point of a nonlinear graph transform, then $\mathcal{F}_2$ is a C^a foliation [PSW, PSWc]. These results are proved in the compact case, but compactness is used only to obtain uniform estimates on the functions $\alpha_1, \beta_1, \alpha_2$ and the derivative of g; the results carry over as long as such uniform estimates hold.

Our foliations of TM have been constructed as the unique fixed points of graph transform maps. We can apply the above results to the F_r-invariant splittings of TTM as the sum of the stable and center-unstable bundles for F_r and as the sum of the center-stable and unstable bundles for F_r. It follows immediately that both the center-unstable and unstable bundles and the corresponding foliations are Hölder continuous as long as F_r is $C^{1+\delta}$ for some $\delta > 0$. We obtain the Hölder continuity of the center-stable and stable bundles for F_r and the corresponding foliations by thinking of the same splittings as F_r^{-1}-invariant. Hölder regularity of the center bundle and foliation is obtained by noticing the center is the intersection of the center-stable and center-unstable.

When $k \geq 2$, a similar estimate gives the C^1 regularity of the unstable bundle along the leaves of the center-unstable foliation. The manifold X is the disjoint union of the leaves of the center-unstable foliation for F_r, E_2 is the unstable bundle, and E_1 is the center bundle. We have:

$$\alpha_1 = \gamma, \quad \beta_1 = \hat{\gamma}^{-1}, \quad \text{and} \quad \alpha_2 = \hat{\nu}^{-1}.$$

The center bunching hypothesis $\hat{\nu} < \gamma\hat{\gamma}$ implies that

$$\sup_{p \in M} \frac{\hat{\nu}(p)}{\gamma(p)^a\, \hat{\gamma}(p)} < 1,$$

for some $a > 1$. It follows that $E_2 = E^u$ is a C^1 bundle over X, the leaves of the center-unstable foliation. Similarly, this argument shows that the center bunching hypothesis $\nu < \gamma\hat{\gamma}$ implies that E^s is a C^1 bundle over the leaves of the center-stable foliation. There is an additional difficulty in the proof of this estimate, which is that X is not a C^k manifold in general, even when F_r is C^k. This difficulty is dealt with in [PSW, PSWc].

When $1 < k < 2$, this type of estimate does not work at all. A different argument, working with holonomy maps instead of bundles, is presented in [BW2]. The main result there implies that under the strong center bunching hypothesis on f, which carries over to F_r, the unstable foliation C^1-subfoliates the center-unstable, and the stable foliation C^1-subfoliates the center-stable.

Step 2. We now have foliations of T_pM, for each $p \in M$. We obtain the foliations $\widehat{\mathcal{W}}_p^u, \widehat{\mathcal{W}}_p^c, \widehat{\mathcal{W}}_p^s, \widehat{\mathcal{W}}_p^{cu}$, and $\widehat{\mathcal{W}}_p^{cs}$ by applying the exponential map $\exp_p$ to the corresponding foliations of T_pM inside the ball around the origin of radius r.

If r is sufficiently small, then the distribution $E^\beta(q)$ lies within the angular $\varepsilon/2$-cone about the parallel translate of $E^\beta(p)$, for every $\beta \in \{u, s, c, cu, cs\}$ and all p, q with $d(p, q) \leq r$. Combining this fact with the preceding discussion, we obtain that property 1. holds if r is sufficiently small.

Property 2. — local invariance — follows from invariance under F_r of the foliations of TM and the fact that $\exp_{f(p)}(F_r(p, v)) = f(\exp_p(p, v))$ provided $\|v\| \leq r$.

Having chosen r, we now choose r_1 small enough so that $f(B(p, 2r_1)) \subset B(f(p), r)$ and $f^{-1}(B(p, 2r_1)) \subset B(f^{-1}(p), r)$, and so that, for all $q \in B(p, r_1)$,

$$
\begin{aligned}
q' \in \widehat{\mathcal{W}}_p^s(q, r_1) &\implies d(f(q), f(q')) \leq \nu(p)\, d(q, q'),\\
q' \in \widehat{\mathcal{W}}_p^u(q, r_1) &\implies d(f^{-1}(q), f^{-1}(q')) \leq \hat{\nu}(f^{-1}(p))\, d(q, q'),\\
q' \in \widehat{\mathcal{W}}_p^{cs}(q, r_1) &\implies d(f(q), f(q')) \leq \hat{\gamma}(p)^{-1}\, d(q, q'), \qquad \textit{and}\\
q' \in \widehat{\mathcal{W}}_p^{cu}(q, r_1) &\implies d(f^{-1}(q), f^{-1}(q')) \leq \gamma(f^{-1}(p))^{-1}\, d(q, q').
\end{aligned}
$$

Property 3. — exponential growth bounds at local scales — is now proved by an inductive argument similar to the proof of Lemma 1.1.

Properties 4.– 7. — coherence, uniqueness, regularity and regularity of the strong foliation inside weak leaves — follow immediately from the corresponding properties of the foliations of TM discussed above. $\diamond$

Remark: Note that the system of local foliations constructed in Proposition 3.1 is not unique; it depends on the extension of F_r outside of a neighborhood of the zero-section of TM. Also note that, even when f is dynamically coherent, in general there is no reason to expect the fake invariant foliations $\widehat{\mathcal{W}}_p^{cs}, \widehat{\mathcal{W}}_p^{cu}$, and $\widehat{\mathcal{W}}_p^c$ to coincide with the local leaves of the real invariant foliations $\mathcal{W}^{cs}, \mathcal{W}_p^{cu}$, and $\mathcal{W}_p^c$, even at p. For the leaf $\widehat{\mathcal{W}}_p^{cs}(p)$ to coincide with the local leaf of $\mathcal{W}^{cs}$ through p, it is necessary that every iterate $f^{-n}(\mathcal{W}^{cs}(p_n, r)$ overflow the neighborhood $B(p, r)$.

For the rest of the paper, $\widehat{\mathcal{W}}^s_p, \widehat{\mathcal{W}}^c_p$, $\widehat{\mathcal{W}}^s_p$, $\widehat{\mathcal{W}}^{cs}_p$ and $\widehat{\mathcal{W}}^{cu}_p$ will denote fake invariant foliations given by Proposition 3.1, with $\varepsilon > 0$ much less than the angle between any two of the subspaces in the partially hyperbolic splitting. We may rescale the metric so that the radius r_1 in conclusion 2. of Proposition 3.1 is much bigger than 1. This will ensure that all of the objects used in the rest of the paper are well-defined. We may assume that if $\max\{d(x,p), d(y,p)\} \leq 3$, then $\widehat{\mathcal{W}}^{cs}_p(x) \cap \widehat{\mathcal{W}}^u_p(y)$, $\widehat{\mathcal{W}}^{cs}_p(x) \cap \mathcal{W}^u_{loc}(y)$, $\widehat{\mathcal{W}}^{cu}_p(x) \cap \widehat{\mathcal{W}}^s_p(y)$ and $\widehat{\mathcal{W}}^{cu}_p(x) \cap \mathcal{W}^s_{loc}(y)$ are single points. We denote by $\hat{m}_a$ the measure $m_{\widehat{\mathcal{W}}^a}$.

4 Distortion estimates inside thin neighborhoods

4.1 A simple distortion lemma

The next lemma will be used to compare values of Hölder cocycles at nearby points.

Lemma 4.1 *Let $\alpha : M \to \mathbf{R}$ be a positive Hölder continuous function, with exponent $\theta > 0$. Then there exists a constant $H > 0$ such that the following holds, for all $p, q \in M$, $B > 0$ and $n \geq 1$:*

$$\sum_{i=0}^{n-1} d(p_i, q_i)^\theta \leq B \quad \Longrightarrow \quad e^{-HB} \leq \frac{\alpha_n(p)}{\alpha_n(q)} \leq e^{HB},$$

and

$$\sum_{i=1}^{n} d(p_{-i}, q_{-i})^\theta \leq B \quad \Longrightarrow \quad e^{-HB} \leq \frac{\alpha_{-n}(p)}{\alpha_{-n}(q)} \leq e^{HB}.$$

Proof. We prove the first part of the lemma. The second part is proved similarly. The function $\log \alpha$ is also Hölder continuous with exponent θ. Let $H > 0$ be the Hölder constant of $\log \alpha$, so that for all $x, y \in M$:

$$|\log \alpha(x) - \log \alpha(y)| \leq H d(x,y)^\theta.$$

The desired inequalities are equivalent to:

$$|\log \alpha_n(p) - \log \alpha_n(q)| \leq HB.$$

Expanding $\log \alpha_n$ as a series, we obtain:

$$
\begin{aligned}
|\log \alpha_n(p) - \log \alpha_n(q)| &\leq \sum_{i=0}^{n-1} |\log \alpha(p_i) - \log \alpha(q_i)| \\
&\leq H \sum_{i=0}^{n-1} d(p_i, q_i)^\theta. \\
&\leq HB,
\end{aligned}
$$

since

$$
\sum_{i=0}^{n-1} d(p_i, q_i)^\theta \leq B,
$$

by the hypothesis of the lemma. $\diamond$

4.2 Thin neighborhoods of $\mathcal{W}^s(p, 1)$

We next identify, for each $n \geq 0$ and $p \in M$, a neighborhood of p whose first n iterates remain in a uniform neighborhood of the corresponding iterates of p. We give an exponential estimate of the size of the first n iterates of the nth such neighborhood. In our proof of Theorem 0.1 we construct sequences of geometric objects; the nth term in the sequence of objects for any $x \in \mathcal{W}^s(p, 1)$ will lie in the nth neighborhood of p.

Let $\sigma < 1$ be a continuous function. For $n \geq 0$ and $p \in M$, define the set $S_{n,\sigma}(p)$ by:

$$
S_{n,\sigma}(p) = \bigcup_{x \in \mathcal{W}^s(p,1)} \widehat{\mathcal{W}}^c_p(x, \sigma_n(p)).
$$

Lemma 4.2 *Suppose that σ satisfies $\sigma < \min\{\hat{\gamma}, 1\}$. Then*

$$
f^j(S_{n,\sigma}(p)) \subset B(p_j, 2),
$$

for $j = 0, \ldots, n$.

Further, there exist positive constants $\kappa < 1$ and $C > 0$ such that, for every $n \geq 0$,

$$
f^j(S_{n,\sigma}(p)) \subset B(p_j, C\kappa^j),
$$

for $j = 0, \ldots, n$.

Proof. Suppose that $x \in \mathcal{W}^s(p, 1)$ and $y \in \widehat{\mathcal{W}}^c_p(x, \sigma_n(p))$. By part 3(a) of Proposition 3.1, we then have

$$y_j \in \widehat{\mathcal{W}}^c_{p_j}(x_j, \hat{\gamma}_j^{-1}(p)\sigma_n(p)) \subset \widehat{\mathcal{W}}^c_{p_j}(x_j, 1) \subset B(p_j, 2),$$

for $0 \leq j \leq n$. In fact, since $\sigma < \min\{\hat{\gamma}, 1\}$, the quantity $\hat{\gamma}_j^{-1}(p)\sigma_n(p) < \hat{\gamma}_j^{-1}(p)\sigma_j(p)$ is exponentially small in j, as is the diameter of $f^j(\mathcal{W}^s(p, 1))$. This implies the second conclusion. $\diamond$

Now let $\tau \leq 1$ be another continuous function. For every $x \in S_{n,\sigma}(p)$, we have that $B(x_n, \tau_n(p)) \subset B(p_n, r)$, and so the set

$$T_{n,\sigma,\tau}(p) = f^{-n}\left(\bigcup_{z \in f^n(S_{n,\sigma}(p))} \widehat{\mathcal{W}}^u_{p_n}(z, \tau_n(p)) \cup \mathcal{W}^u(z, \tau_n(p))\right)$$

is well-defined. Proposition 3.1 and Lemma 1.1 imply that the leaves of $\widehat{\mathcal{W}}^u_{p_j}$ and $\mathcal{W}^u_{loc}$ are uniformly contracted by f^{-1} as long as they stay near the orbit of p; combining these facts with Lemma 4.2, we get:

Lemma 4.3 *For every continuous function σ satisfying $\sigma < \min\{\hat{\gamma}, 1\}$, the set $T_{n,\sigma,1}(p)$ satisfies*

$$f^j(T_{n,\sigma,1}(p)) \subset B(p_j, 3),$$

for $j = 0, \ldots, n$.

Further, for every such σ and every continuous function $\tau < 1$, there exist positive constants $\kappa < 1$ and $C > 0$ such that, for every $n \geq 0$,

$$f^j(T_{n,\sigma,\tau}(p)) \subset B(p_j, C\kappa^j),$$

for $j = 0, \ldots, n$.

The dimensions of the neighborhoods $T_{n,\sigma,\tau}(p)$ and their iterates are illustrated in Figure 4.2, in the case where $\sigma, \tau < 1$.

As simple corollary of this lemma and Lemma 4.1, we then obtain:

Lemma 4.4 *Let $\alpha : M \to \mathbf{R}$ be a positive, uniformly Hölder continuous function, and let σ, τ be continuous functions satisfying*

$$\sigma < \min\{\hat{\gamma}, 1\}, \quad \tau < 1.$$

Then there is a constant $C \geq 1$ such that, for all $n \geq 0$ and all $x, y \in T_{n,\sigma,\tau}(p)$,

$$C^{-1} \leq \frac{\alpha_n(y)}{\alpha_n(x)} \leq C.$$

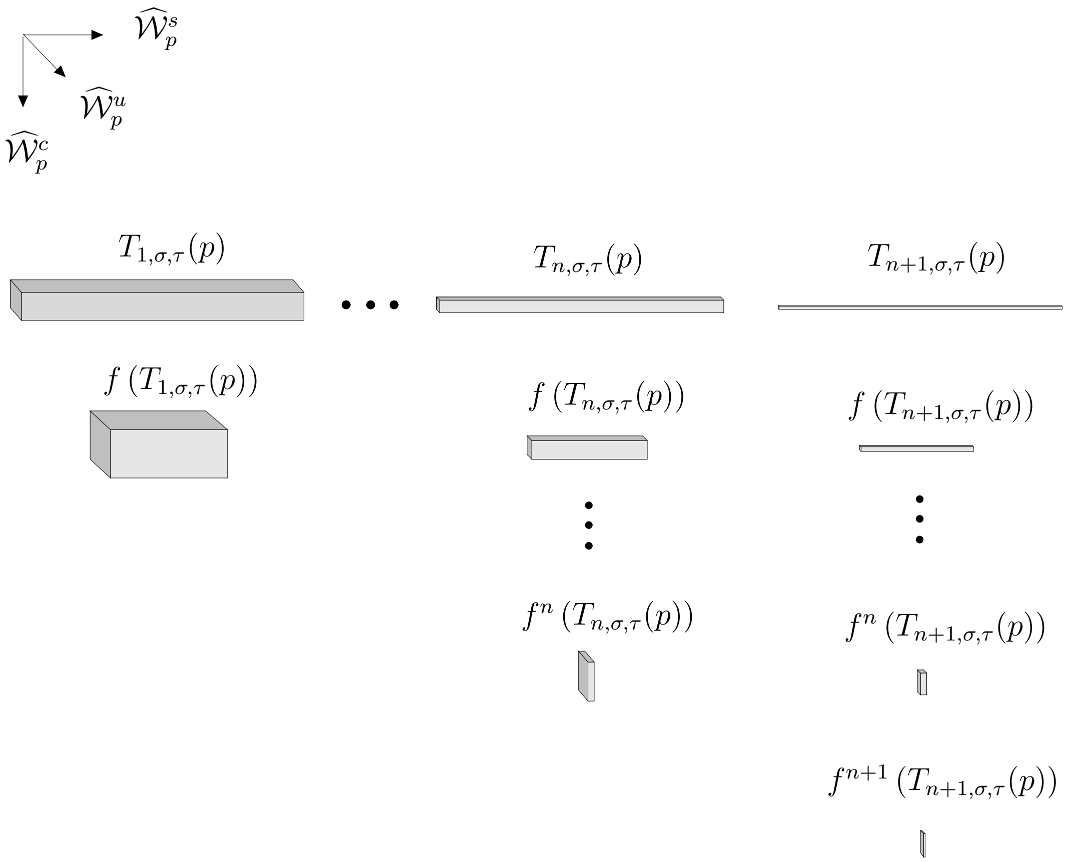


Figure 1: Dimensions of the neighborhoods $T_{n,\sigma,\tau}(p)$ and their iterates

5 The main theorem

The properties of accessibility and essential accessibility can be reformulated using the notion of saturation. We say that a set is *bisaturated* if it is both $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated. Accessibility means that a set which is bisaturated must be either empty or all of M. Essential accessibility means that a measurable set which is bisaturated must have either 0 or full volume.

The central result of this paper is:

Theorem 5.1 *Let f be C^2, partially hyperbolic and center bunched (or $C^{1+\delta}$ and strongly center bunched). Let A be a measurable set that is both essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated. Then the set of Lebesgue density points of A is $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated.*

Theorem 5.1 does not assume that f is volume-preserving.

The central result of Pugh and Shub in [PS3] is a version of Theorem 5.1, which involves a different notion of density point (defined in [PS3]) and a slightly different hypothesis:

For $a = u$ or s, if A is essentially $\mathcal{W}^a$-saturated, then the set of julienne density points of A is $\mathcal{W}^a$-saturated.

In contrast, Theorem 5.1 requires that A be *both* essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated in order to conclude anything.

With the obvious definitions, Theorem 5.1 has the corollary:

Corollary 5.2 *If f is as in Theorem 5.1, then every bi essentially saturated set is essentially bisaturated.*

Proof. Let A be a bi essentially saturated set. Theorem 5.1 implies that the set $\hat{A}$ of Lebesgue density points of A is bisaturated. The Lebesgue Density Theorem implies that $m(A\Delta\hat{A}) = 0$. Thus A is essentially bisaturated: it differs by a zero set from a bisaturated set.$\diamond$

Essential accessibility tells us that essentially bi saturated sets have 0 or full measure. *A priori*, the notion of bi essential saturation is weaker than the notion of essential bisaturation. Corollary 5.2 says that the two concepts coincide when f is center bunched, so that if f is essentially accessible, then bi essentially saturated sets must have 0 or full measure.

Theorem 0.1 follows easily from Corollary 5.2 by a version of the Hopf argument and a result of Brin and Pesin (see Section 2 of [BPSW] for more details).

Proof of Theorem 0.1. Let μ be the f-invariant probability measure in the measure class of m. The concepts of μ-almost everywhere and m-almost everywhere are the same. To prove that f is ergodic with respect to μ, it suffices to show that the Birkhoff averages of continuous functions are almost everywhere constant. Let φ be a continuous function, and let

$$\hat{\varphi}_s(p) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \varphi(f^i(p)) \quad \text{and } \hat{\varphi}_u(p) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \varphi(f^{-i}(p))$$

be the forward and backward Birkhoff averages of φ under f. The function $\hat{\varphi}_s$ is constant along $\mathcal{W}^s$-leaves, and $\hat{\varphi}_u$ is constant along $\mathcal{W}^u$-leaves. It follows that for any $a \in \mathbf{R}$, the sets

$$A^s(a) = \hat{\varphi}_s^{-1}(-\infty, a] \quad \text{and} \quad A^u(a) = \hat{\varphi}_u^{-1}(-\infty, a].$$

are $\mathcal{W}^s$-saturated and $\mathcal{W}^u$-saturated, respectively.

Since f preserves μ, the Birkhoff Ergodic Theorem implies that $\hat{\varphi}_s = \hat{\varphi}_u$ almost everywhere. Consequently $m(A^s(a)\,\Delta\,A^u(a)) = 0$, so that the set $A(a) = A^u(a) \cap A^s(a)$ has $A^u(a)$ as an essential $\mathcal{W}^u$-saturate and $A^s(a)$ as an essential $\mathcal{W}^s$-saturate. Thus $A(a)$ is bi essentially saturated.

It follows from Corollary 5.2 that $A(a)$ is essentially bisaturated. Essential accessibility implies that $A(a)$ has 0 or full measure. Since a was arbitrary, it follows that $\hat{\varphi}_s$ and $\hat{\varphi}_u$ are almost everywhere constant, and so f is ergodic.

To prove that f has the Kolmogorov property, it suffices to show that all sets in the Pinsker subalgebra $\mathcal{P}$ have 0 or full measure. According to Proposition 5.1 of [BP], if f is partially hyperbolic, then any set in $P \in \mathcal{P}$ is bi essentially saturated. It again follows from Corollary 5.2 and essential accessibility that P has 0 or full measure. $\diamond$

In order to prove Theorem 5.1 it suffices to show that the set of Lebesgue density points of A is $\mathcal{W}^s$-saturated; applying this result with f replaced by f^{-1} then shows that the set of Lebesgue density points of A is also $\mathcal{W}^u$-saturated. More precisely, it suffices to show that, for any $p \in M$, if $x, x' \in \mathcal{W}^s(p, 1)$, and x is a Lebesgue density point of A, then so is x'.

Let

$$N = \bigsqcup_{j\geq 0} B(p_j, r)$$

be the disjoint union, over $j \geq 0$, of the balls $B(p_j, r)$, where $r \gg 1$ is given by Proposition 3.1. Everything we do in the rest of this paper takes place inside N, and we drop the dependence on p where it is not confusing.

We let $\widehat{\mathcal{W}}^u$ be the locally invariant foliation of N whose restriction to $B(p_j, r)$ is $\widehat{\mathcal{W}}^u_{p_j}$. Similarly we define foliations $\widehat{\mathcal{W}}^c, \widehat{\mathcal{W}}^s, \widehat{\mathcal{W}}^{cu}$ and $\widehat{\mathcal{W}}^{cs}$. By Proposition 3.1, all of these foliations are *uniformly* Hölder continuous, $\widehat{\mathcal{W}}^u$ uniformly C^1 subfoliates $\widehat{\mathcal{W}}^{cu}$, and $\widehat{\mathcal{W}}^s$ uniformly C^1 subfoliates $\widehat{\mathcal{W}}^{cs}$. The foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ induce foliations of N, which will again be denoted by $\mathcal{W}^u$ and $\mathcal{W}^s$. Hence $\mathcal{W}^s(p)$ is used to denote the local leaf $\mathcal{W}^s(p, r)$. Note that $\widehat{\mathcal{W}}^s(p_j) = \mathcal{W}^s(p_j)$ and $\widehat{\mathcal{W}}^u(p_j) = \mathcal{W}^u(p_j)$ for all $j \geq 0$. Note also that, since we are not assuming that f is dynamically coherent, there are no foliations $\mathcal{W}^c, \mathcal{W}^{cu}$, or $\mathcal{W}^{cs}$.

Most leaves of these foliations (with the notable exception of the leaves of $\mathcal{W}^s$ and $\widehat{\mathcal{W}}^s$ passing through the orbit of p) are invariant under only finitely many iterates of f, until their orbits leave N. In our proof, the geometric objects in N that we need to iterate n times always start out in a neighborhood $T_{n,\sigma,\tau}$ of the type defined in Section 4.2 with $\tau < 1$ and $\sigma < \min\{\hat{\gamma}, 1\}$. By Lemma 4.3, these objects remain inside of N for n iterates. As long as their orbits remain inside of N, the locally invariant foliations $\widehat{\mathcal{W}}^u, \widehat{\mathcal{W}}^c, \widehat{\mathcal{W}}^s, \widehat{\mathcal{W}}^{cu}$ and $\widehat{\mathcal{W}}^{cs}$ used to construct these geometric objects are nearly indistinguishable from their invariant counterparts $\mathcal{W}^u, \mathcal{W}^c, \mathcal{W}^s, \mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ in the dynamically coherent setting.

These locally invariant foliations differ from true invariant foliations in one key respect. The measurable set A in Theorem 5.1 is essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated. The property of (essential) $\mathcal{W}^u$-saturation neither implies nor is implied by (essential) $\widehat{\mathcal{W}}^u$-saturation. Similarly, (essential) $\mathcal{W}^s$-saturation neither implies nor is implied by (essential) $\widehat{\mathcal{W}}^s$-saturation. While it is possible to prove a version of Theorem 5.1 in which $\mathcal{W}^u$ and $\mathcal{W}^s$ are replaced by $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$, such a theorem is not enough to prove ergodicity using a Hopf argument. This is because the Hopf argument uses infinitely many iterates of f and is thus global in nature. Therefore, wherever we use explicitly the fact that our set A is $\mathcal{W}^u$-saturated, in particular, in our arguments that use Proposition 2.5, we must switch from using $\widehat{\mathcal{W}}^u$ to using $\mathcal{W}^u$, and similarly for stable foliations.

Following Pugh and Shub, we consider for each $x \in \mathcal{W}^s(p, 1)$ a sequence of sets, called center-unstable juliennes, that lie in the fake center-unstable manifold $\widehat{\mathcal{W}}^{cu}(x)$ and shrink exponentially as $n \to \infty$ while becoming increasingly thin in the $\widehat{\mathcal{W}}^u$-direction. In the Pugh-Shub construction, dynamical coherence is assumed, and the true invariant foliations $\mathcal{W}^{cu}$, $\mathcal{W}^c$ and $\mathcal{W}^u$ are used; here we use their fake counterparts. While objects we work with, such as center-unstable juliennes, will depend on the fake invariant foliations given by Proposition 3.1, the final conclusion of Theorem 5.1 does not, since we always have $\widehat{\mathcal{W}}^s(p) = \mathcal{W}^s(p)$.

Recall the center bunching assumptions (7):

$$\nu < \gamma\hat{\gamma} \qquad \text{and} \qquad \hat{\nu} < \gamma\hat{\gamma}.$$

In what follows we will use only the first of these inequalities. The second inequality is used to prove $\mathcal{W}^u$ saturation of density points.

We choose continuous functions τ and σ such that

$$\nu < \tau < \sigma\gamma \quad \text{and} \quad \sigma < \min\{\hat{\gamma}, 1\}.$$

Note that these inequalities also imply that

$$\tau\hat{\nu} < \sigma\gamma\hat{\nu} < \sigma\gamma\hat{\gamma} \leq \sigma.$$

The choice of σ and τ with the desired properties is possible because of the center bunching assumption. The reader should think of τ as being just a little bigger than ν and σ as just a little bit less than $\min\{\hat{\gamma}, 1\}$. The reader might also choose to keep in mind the case where the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ are constants, and where τ and σ can be chosen to be constant. In this case the cocycles τ_n and σ_n are just the constants τ^n and σ^n.

Using the notation defined in Section 4.2, let $S_n = S_{n,\sigma}(p)$, and let $T_n = T_{n,\sigma,\tau}(p)$. For the rest of the paper, except where we indicate otherwise, cocycles will be evaluated at the point p. We will also drop the dependence on p from the notation; thus, if α is a cocycle, then $\alpha_n(p)$ will be abbreviated to α_n.

The center-unstable juliennes $\widehat{J}^{cu}_n(x)$ that we construct will be contained in T_n. We now describe the construction of center-unstable juliennes.

Define, for all $x \in \mathcal{W}^s(p, 1)$,

$$\widehat{B}^c_n(x) = \widehat{\mathcal{W}}^c(x, \sigma_n).$$

Note that

$$S_n = \bigcup_{x \in \mathcal{W}^s(p,1)} \widehat{B}^c_n(x).$$

For $y \in S_n$, we may then define two types of *unstable juliennes*:

$$\widehat{J}^u_n(y) = f^{-n}(\widehat{\mathcal{W}}^u(y_n, \tau_n))$$

and

$$J^u_n(y) = f^{-n}(\mathcal{W}^u(y_n, \tau_n)).$$

Observe that for all $y \in S_n$, the sets $\widehat{J}^u_n(y)$ and $J^u_n(y)$ are contained in T_n.

For each $x \in \mathcal{W}^s(p, 1)$ and $n \geq 0$, we then define the *center-unstable julienne* centered at x of order n:

$$\widehat{J}^{cu}_n(x) = \bigcup_{y \in \widehat{B}^c_n(x)} \widehat{J}^u_n(y).$$

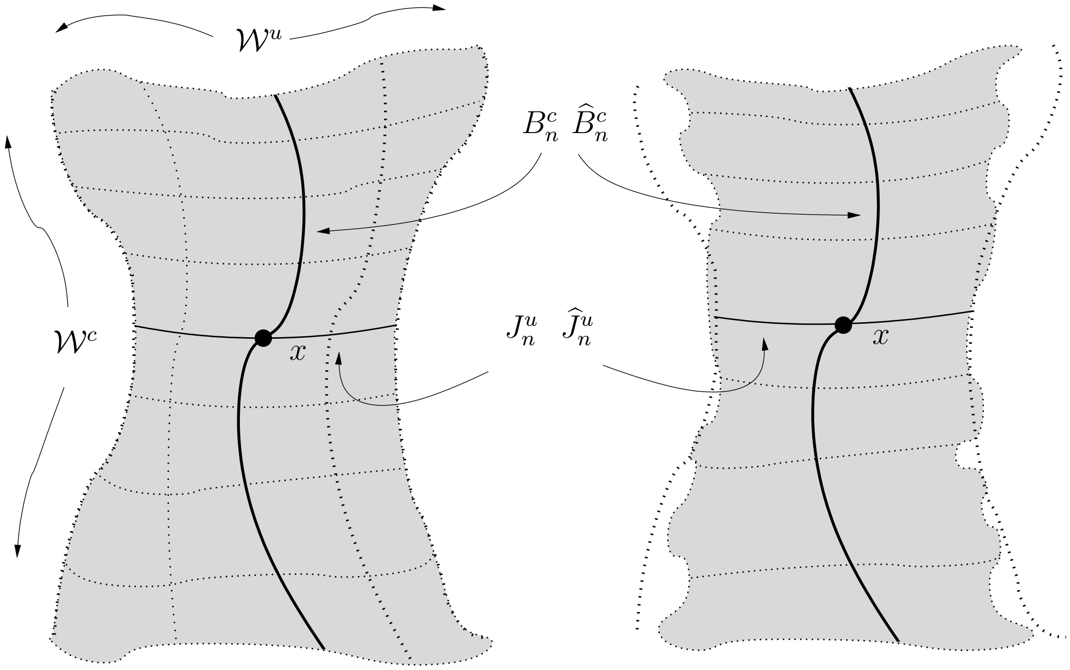
Note that, by their construction, the sets $\widehat{J}^{cu}_n(x)$ are contained in T_n, for all $n \geq 0$ and $x \in \mathcal{W}^s(p, 1)$. Note also that $\widehat{J}^{cu}_n(x)$ is contained in the smooth submanifold $\widehat{\mathcal{W}}^{cu}(x)$, which carries the restricted Riemannian volume $\hat{m}_{cu} = m_{\widehat{\mathcal{W}}^{cu}}$, and $\widehat{J}^{cu}_n(x)$ has positive $\hat{m}_{cu}$–measure. The reason for using $\widehat{J}^u$ here instead of J^u is that the distribution $T\widehat{\mathcal{W}}^c \oplus E^u$ is not necessarily integrable. The set defined by replacing $\widehat{J}^u_n$ by J^u_n in the definition of $\widehat{J}^{cu}_n$ might not be a C^1 submanifold of M.

Our *cu*-juliennes are closely related to, but not exactly the same as, those of Pugh and Shub. In the case where σ and τ are constant functions, and f is dynamically coherent, their center-unstable julienne is the foliation product of $\mathcal{W}^c(x, \sigma^n)$ and $f^{-n}(\mathcal{W}^u(x_n, \tau^n))$; see Figure 5. In this case, the image under f^n of our $J^{cu}_n(p)$ appears in [PS3] as a tubelike approximation to the Pugh-Shub center-unstable postjulienne of rank n. The results of [PS3] show that the *cu*-juliennes defined here and in [PS3] are internested. Thus our *cu*-juliennes could be replaced by the Pugh-Shub *cu*-juliennes in Propositions 5.3 and 5.5.

As in [PS3], the *cu*-juliennes have a quasi-conformality property: they are approximately preserved by holonomy along the stable foliation.

Proposition 5.3 *Let $x, x' \in \mathcal{W}^s(p, 1)$, and let $h^s : \widehat{\mathcal{W}}^{cu}(x) \to \widehat{\mathcal{W}}^{cu}(x')$ be the holonomy map induced by the stable foliation $\mathcal{W}^s$. Then the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$ are internested.*

There are estimates on the volumes of unstable and center-unstable juliennes.



Center-unstable julienne $J_n^{cu}(x)$ in [PS3] $\qquad$ $\widehat{J}_n^{cu}(x)$ in this paper.

Figure 2: Two types of center-unstable juliennes, when τ and σ are constant.

Proposition 5.4 *There exist $\delta > 0$ and $c \geq 1$ such that, for all $x \in \mathcal{W}^s(p,1)$, and all $q, q' \in S_n$, the following hold, for all $n \geq 0$:*

$$c^{-1} \leq \frac{\hat{m}_u(\widehat{J}^u_n(q))}{\hat{m}_u(\widehat{J}^u_n(q'))} \leq c,$$

$$c^{-1} \leq \frac{m_u(J^u_n(q))}{m_u(J^u_n(q'))} \leq c,$$

$$\hat{m}_u(\widehat{J}^u_{n+1}(q)) \geq \delta \hat{m}_u(\widehat{J}^u_n(q)),$$

and

$$\hat{m}_{cu}(\widehat{J}^{cu}_{n+1}(x)) \geq \delta \hat{m}_{cu}(\widehat{J}^{cu}_n(x)).$$

The final crucial property of the cu-juliennes is that, *for the sets that appear in the proof of Theorem 5.1*, Lebesgue density points are precisely cu-julienne density points.

Proposition 5.5 *Let X be a measurable set that is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated. Then $x \in \mathcal{W}^s(p)$ is a Lebesgue density point of X if and only if:*

$$\lim_{n\to\infty} \hat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) = 1.$$

Note the asymmetry that X is fully $\mathcal{W}^s$-saturated but only essentially $\mathcal{W}^u$-saturated. This is because X will be an essential stable saturate of a bi essentially saturated set.

Remark: Pugh and Shub show that Lebesgue almost every point of *any* measurable set is a cu-julienne density point. In their argument they prove a Vitali covering lemma for their juliennes. This argument accounts for their definition of cu-juliennes as a foliation product and for the stronger bunching hypothesis in their main result. We do not know whether their result, specifically Theorem 7.1 of [PS3], still holds under our weaker bunching hypothesis, or whether it holds at all in the absence of dynamical coherence (although it can be shown that in the dynamically coherent, symmetrized setting that they consider, their hypothesis can be weakened from $\nu < \gamma^{2+2/\theta_0}$ to $\nu < \gamma^{1/\theta_0}$).

The proof of Proposition 5.3 is essentially contained in [PS3]. For completeness we reproduce the argument in the next section. Proposition 5.4 is proved in Section 7, and the proof of Proposition 5.5 is in the final section. We now use these three propositions to prove the main result.

Proof of Theorem 5.1. As we noted above, it suffices to show that the Lebesgue density points of A are $\mathcal{W}^s$-saturated; to see that the Lebesgue density points of A are $\mathcal{W}^u$-saturated, just consider f^{-1} instead of f. Let A^s be an essential $\mathcal{W}^s$-saturate of A. Since $m(A \,\Delta\, A^s) = 0$, the Lebesgue density points of A are precisely the same as those of A^s. Fix $p \in M$ and suppose that $x \in \mathcal{W}^s(p,1)$ is a Lebesgue density point of A^s. Proposition 5.5 implies that x is a cu-julienne density point of A^s.

To finish the proof, we show that every $x' \in W^s(p,1)$ is a cu-julienne density point of A^s. Then by Proposition 5.5, every $x' \in W^s(p,1)$ is a Lebesgue density point of A^s. The Lebesgue density points of A^s, and hence of A, are therefore $\mathcal{W}^s$-saturated.

Let $h^s : \widehat{\mathcal{W}}^{cu}(x) \to \widehat{\mathcal{W}}^{cu}(x')$ be the holonomy map induced by the stable foliation $\mathcal{W}^s$. The sequence $h^s(\widehat{J}^{cu}_n(x)) \subset \widehat{\mathcal{W}}^{cu}(x')$ nests at x'. Transverse absolute continuity of h^s with bounded Jacobians implies that

$$\lim_{n\to\infty} \hat{m}_{cu}(A^s : \widehat{J}^{cu}_n(x)) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \hat{m}_{cu}(h^s(A^s) : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Since A^s is s-saturated, we then have:

$$\lim_{n\to\infty} \hat{m}_{cu}(A^s : \widehat{J}^{cu}_n(x)) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \hat{m}_{cu}(A^s : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Since we are assuming that x is a cu-julienne density point of A^s, we thus have

$$\lim_{n\to\infty} \hat{m}_{cu}(A^s : h^s(\widehat{J}^{cu}_n(x))) = 1.$$

Working inside of $\widehat{\mathcal{W}}^{cu}(x')$, we will apply Lemma 2.1 to the sequences $h^s(\widehat{J}^{cu}_n(x))$ and $\widehat{J}^{cu}_n(x')$, which both nest at x'. Proposition 5.3 implies that these sequences are internested. Proposition 5.4 implies that $\widehat{J}^{cu}_n(x')$ is regular with respect to the induced Riemannian measure $\hat{m}_{cu}$ on $\widehat{\mathcal{W}}^{cu}(x')$. Lemma 2.1 now tells us that

$$\lim_{n\to\infty} \hat{m}_{cu}(A^s : h^s(\widehat{J}^{cu}_n(x))) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \hat{m}_{cu}(A^s : \widehat{J}^{cu}_n(x')) = 1,$$

and so x' is a cu-julienne density point of A^s. It follows from Proposition 5.5 that x' is a Lebesgue density point of A^s, and thus of A. $\diamond$

6 Julienne quasiconformality

We adapt the proof of Theorem 4.4 in [PS3] to prove Proposition 5.3. It will suffice to show that k can be chosen so that

$$h^s(\widehat{J}^{cu}_n(x)) \subseteq \widehat{J}^{cu}_{n-k}(x'), \tag{20}$$

for all $n \geq k$, whenever x and x' satisfy the hypotheses of the proposition. The hypotheses of the proposition treat x and x' symmetrically, so we can then reverse their roles to obtain:

$$\overline{h}^s(\widehat{J}^{cu}_n(x')) \subseteq \widehat{J}^{cu}_{n-k}(x),$$

for all $n \geq k$, where $\overline{h}^s : \widehat{\mathcal{W}}^{cu}_{loc}(x') \to \widehat{\mathcal{W}}^{cu}(x)$ is the holonomy induced also by the stable foliation $\mathcal{W}^s$. Since $\overline{h}^s$ and h^s are inverses, we then obtain:

$$\widehat{J}^{cu}_n(x') \subseteq h^s(J^{cu}_{n-k}(x)),$$

for all $n \geq k$.

In order to prove that k can be chosen so that (20) holds, we need two lemmas.

Lemma 6.1 *There exists a positive integer k_1 such that, for all $x, x' \in \mathcal{W}^s(p)$,*

$$\hat{h}^s(\widehat{B}^c_n(x)) \subseteq \widehat{B}^c_{n-k_1}(x'),$$

for all $n \geq k_1$, where $\hat{h}^s : \widehat{\mathcal{W}}^c_{loc}(x) \to \widehat{\mathcal{W}}^c(x')$ is the local $\widehat{\mathcal{W}}^s$ holonomy.

Proof. Proposition 3.1 implies that $\hat{h}^s$ is L-Lipschitz, for some $L \geq 1$. Therefore the image of $\widehat{\mathcal{W}}^c(x, \sigma_n)$ under $\hat{h}^s$ is contained in $\widehat{\mathcal{W}}^c(x', L\sigma_n) \subseteq \widehat{\mathcal{W}}^c(x', \sigma_{n-k_1})$, for any k_1 large enough so that $\sigma_{-k_1} > L$. $\diamond$

Lemma 6.2 *There exists a positive integer k_2 such that the following holds for every integer $n \geq k_2$. Suppose $q \in S_n$, $q' \in S_{n-k_1}$ and $q' \in \widehat{\mathcal{W}}^s(q)$. Let $y \in \widehat{J}^u_n(q)$, and let y' be the image of y under $\mathcal{W}^s$ holonomy from $\widehat{\mathcal{W}}^{cu}_{loc}(q)$ to $\widehat{\mathcal{W}}^{cu}(q')$. Then*

$$y' \in \widehat{J}^u_{n-k_2}(z'),$$

for some $z' \in \widehat{\mathcal{W}}^c(q', \sigma_{n-k_2})$.

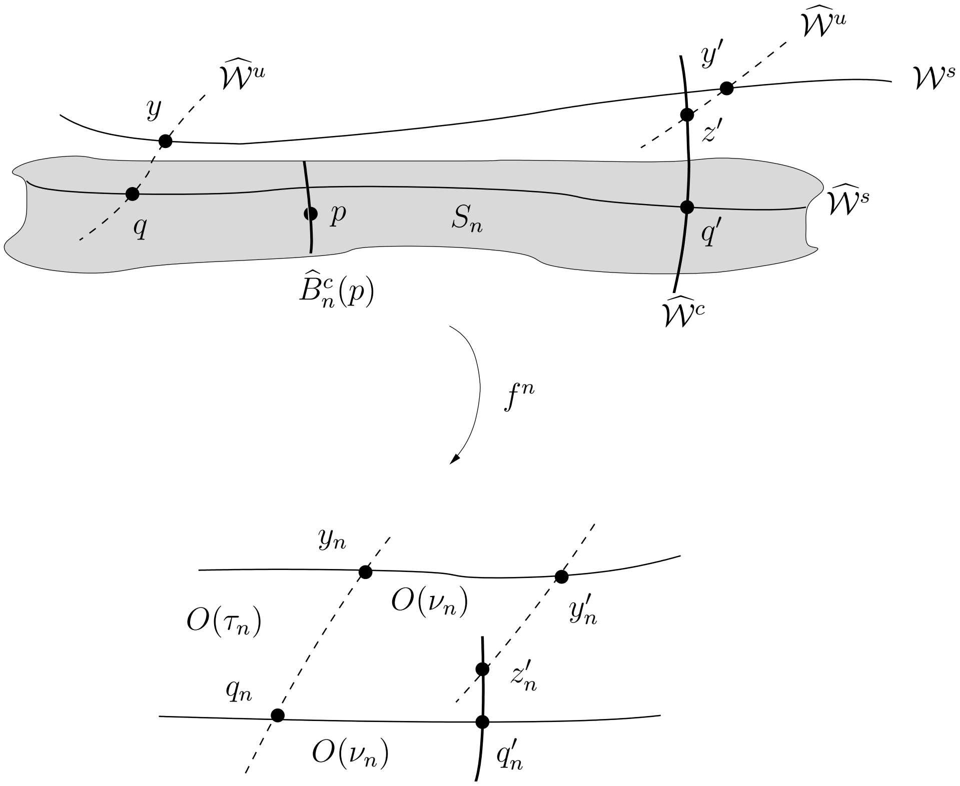


Figure 3: Picture for the proof of Lemma 6.2.

Remark: Note that two types of holonomy maps appear in Lemma 6.2: the point q' is the image of q under $\widehat{\mathcal{W}}^s$ holonomy between $\widehat{\mathcal{W}}^{cu}_{loc}(x)$ and $\widehat{\mathcal{W}}^{cu}(x')$, whereas y' is the image of y under $\mathcal{W}^s$ holonomy.

Proof of Lemma 6.2. Let z' be the unique point in $\widehat{\mathcal{W}}^u(y') \cap \widehat{\mathcal{W}}^c(q')$. It is not hard to see that $z'_j \in N$, for $j = 0, \ldots, n-1$ and that z'_n is the unique point in $\widehat{\mathcal{W}}^u(y'_n) \cap \widehat{\mathcal{W}}^c(q'_n)$. It will suffice to prove that $d(y'_n, z'_n) = O(\tau_n)$ and $d(q', z') = O(\sigma_n)$.

We have $d(q_n, y_n) \le \tau_n$ because $y \in f^{-n}(\widehat{\mathcal{W}}^u(q_n, \tau_n))$. By Proposition 3.1, 3(a), we also have that $d(q_n, q'_n) = O(\nu_n)$ and $d(y_n, y'_n) = O(\nu_n)$, since $d(q, q')$ and $d(y, y')$ are both $O(1)$. Note that q_n and z'_n are, respectively, the images of y_n and y'_n under $\widehat{\mathcal{W}}^u$-holonomy between $\widehat{\mathcal{W}}^{cs}_{loc}(y_n)$ and $\widehat{\mathcal{W}}^{cs}(q_n)$. Uniform transversality of the foliations $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^{cs}$ implies that

$$d(y'_n, z'_n) = O(\max\{d(q_n, y_n), d(y_n, y'_n)\}) = O(\tau_n),$$

since $\nu < \tau$.

We next show that $d(q', z') = O(\sigma_n)$. By the triangle inequality,

$$d(q_n', z_n') \leq d(q_n', q_n) + d(q_n, y_n) + d(y_n, y_n') + d(y_n', z_n').$$

All four of the quantities on the right-hand side are easily seen to be $O(\tau_n)$. Since q_n' and z_n' lie in the same $\widehat{\mathcal{W}}^c$-leaf at distance $O(\tau_n)$, Proposition 3.1 now implies that $d(q', z') = O((\gamma_n)^{-1}\tau_n)$. But τ and σ were chosen so that $\tau < \gamma\sigma$. Hence $(\gamma_n)^{-1}\tau_n < \sigma_n$ and $d(q', z') = O(\sigma_n)$, as desired. $\diamond$

Proof of Proposition 5.3. As noted above, it suffices to prove the inclusion (20). For $q \in \widehat{B}_n^c(x)$, let $q' = \hat{h}^s(q)$. Then $q' \in \widehat{B}_{n-k_1}^c(p')$ by Lemma 6.1. Hence $q, q' \in S_{n-k_1}$ and we can apply Lemma 6.2 to obtain

$$h^s(\widehat{J}_n^{cu}(x)) \subseteq \bigcup_{z \in Q} \widehat{J}_{n-k_2}^u(z),$$

where

$$Q = \bigcup_{q' \in \widehat{B}_{n-k_1}^c(x')} \widehat{B}_{n-k_2}^c(q').$$

For $k \geq k_2$, we have:

$$\bigcup_{z \in Q} \widehat{J}_{n-k_2}^u(z) \subseteq \bigcup_{z \in Q} \widehat{J}_{n-k}^u(z).$$

It therefore suffices to find $k \geq k_2$ such that $Q \subseteq \widehat{B}_{n-k}^c(x')$. This latter inclusion holds if:

$$\sigma_{n-k_1} + \sigma_{n-k_2} \leq \sigma_{n-k},$$

which is obviously true for all $n \geq k$, if k is sufficiently large. $\diamond$

7 Julienne measure

We next prove Proposition 5.4. Continuity of $\widehat{\mathcal{W}}^u$ implies that there exists $C_1 \geq 1$ such that

$$C_1^{-1} \leq \frac{\hat{m}_u(\widehat{\mathcal{W}}^u(q_n, \tau_n))}{\hat{m}_u(\widehat{\mathcal{W}}^u(q_n', \tau_n))} \leq C_1, \tag{21}$$

for all $q, q' \in S_n$.

Let $\widehat{E}^s$, $\widehat{E}^c$, and $\widehat{E}^u$ be the tangent distributions to the leaves of $\widehat{\mathcal{W}}^s, \widehat{\mathcal{W}}^c$, and $\widehat{\mathcal{W}}^u$, respectively. They are Hölder continuous by Proposition 3.1, part 6. Furthermore, the restrictions of these distributions to T_n are invariant under Tf^j, for $j = 1, \dots n$. We next observe that the Jacobian $\mathrm{Jac}(Tf^n|_{\widehat{E}^u})$ is nearly constant when restricted to the set T_n. More precisely, we have:

Lemma 7.1 *There exists $C_2 \geq 1$ such that, for all $n \geq 1$, and all $y, y' \in T_n$,*

$$C_2^{-1} \leq \frac{\mathrm{Jac}(Tf^n|_{\widehat{E}^u})(y)}{\mathrm{Jac}(Tf^n|_{\widehat{E}^u})(y')} \leq C_2.$$

Proof. By the Chain Rule, these inequalities follow from Lemma 4.4 with $\alpha = \mathrm{Jac}(Tf|_{\widehat{E}^u})$.$\diamond$

Let $q \in S_n$, and let $X \subseteq \widehat{J}^u_n(q)$ be a measurable set (such as $\widehat{J}^u_n(q)$ itself). Then:

$$\hat{m}_u(f^n(X)) \;\; = \;\; \int_X \mathrm{Jac}(Tf^n|_{\widehat{E}^u})(x)\, d\hat{m}_u(x).$$

From this and Lemma 7.1 we then obtain:

Lemma 7.2 *There exists $C_3 > 0$ such that, for all $n \geq 0$, for any $q, q' \in S_n$, and any measurable sets $X \subset \widehat{J}^u_n(q), X' \subset \widehat{J}^u_n(q')$, we have:*

$$C_3^{-1} \frac{\hat{m}_u(f^n(X))}{\hat{m}_u(f^n(X'))} \leq \frac{\hat{m}_u(X)}{\hat{m}_u(X')} \leq C_3 \frac{\hat{m}_u(f^n(X))}{\hat{m}_u(f^n(X'))}.$$

Recall that $f^n(\widehat{J}^u_n(q)) = \widehat{\mathcal{W}}^u(q_n, \tau_n)$, for $q \in S_n$. The first conclusion of Proposition 5.4 now follows from (21) and Lemma 7.2 with $X = \widehat{J}^u_n(q)$ and $X' = \widehat{J}^u_n(q')$.

The second conclusion is proved similarly.

We next show that there exists $\delta > 0$ such that

$$\frac{\hat{m}_u(\widehat{J}^u_{n+1}(q))}{\hat{m}_u(\widehat{J}^u_n(q))} \geq \delta, \tag{22}$$

for all $n \geq 0$ and all $q \in S_n$. To obtain (22), we will apply Lemma 7.2 with $q = q'$, $X = \widehat{J}^u_{n+1}(q)$, and $X' = \widehat{J}^u_n(q)$. This gives us:

$$\frac{\hat{m}_u(\widehat{J}^u_{n+1}(q))}{\hat{m}_u(\widehat{J}^u_n(q))} \;\; \geq \;\; C_3^{-1} \frac{\hat{m}_u(f^n(\widehat{J}^u_{n+1}(q)))}{\hat{m}_u(f^n(\widehat{J}^u_n(q)))}.$$

But $f^n(\widehat{J}^u_{n+1}(q)) = f^{-1}(\widehat{\mathcal{W}}^u(q_{n+1},\tau_{n+1}))$ and $f^n(\widehat{J}^u_n(q)) = \widehat{\mathcal{W}}^u(q_n,\tau_n)$, and hence:

$$\frac{\hat{m}_u(f^n(\widehat{J}^u_{n+1}(q)))}{\hat{m}_u(f^n(\widehat{J}^u_n(q)))} = \frac{\hat{m}_u(f^{-1}(\widehat{\mathcal{W}}^u(q_{n+1},\tau_{n+1})))}{\hat{m}_u(\widehat{\mathcal{W}}^u(q_n,\tau_n))}.$$

This ratio is uniformly bounded below away from 0, since f^{-1} is a diffeomorphism, the leaves of $\widehat{\mathcal{W}}^u$ are uniformly smooth, and the ratio $\tau_{n+1}/\tau_n = \tau(p_n)$ is uniformly bounded below away from 0.

To prove the final claim, we begin by observing that, considered as a subset of $\widehat{\mathcal{W}}^{cu}(x)$, the set $\widehat{J}^{cu}_n(x)$ fibers over $\widehat{B}^c_n(x)$ with $\widehat{\mathcal{W}}^u$-fibers $\widehat{J}^u_n(q)$. We have just proved that these fibers are c-uniform. Since $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ is uniformly bounded away from 0, the ratio

$$\frac{\hat{m}_c(\widehat{B}^c_{n+1}(x))}{\hat{m}_c(\widehat{B}^c_n(x))} = \frac{\widehat{\mathcal{W}}^c(x,\sigma_{n+1})}{\widehat{\mathcal{W}}^c(x,\sigma_n)}$$

is bounded away from 0, uniformly in x and n. Thus the sequence of bases $\widehat{B}^c_n(x)$ of $\widehat{J}^{cu}_n(x)$ is regular in the induced Riemannian volume $\hat{m}_c$. Proposition 3.1, part 7. implies that $\widehat{\mathcal{W}}^u$ C^1 subfoliates $\widehat{\mathcal{W}}^{cu}$; in particular, considered as a subfoliation of $\widehat{\mathcal{W}}^{cu}(x)$, $\widehat{\mathcal{W}}^u$ is absolutely continuous with bounded Jacobians. Proposition 2.4 implies that the sequence $\widehat{J}^{cu}_n(x)$ is regular, with respect to the induced Riemannian measure $\hat{m}_{cu}$. This proves the final claim. $\diamond$

8 Julienne density

We now come to the proof of Proposition 5.5. We must show that if a measurable set X is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated, then a point $x \in \mathcal{W}^s(p,1)$ is a Lebesgue density point of X if and only if

$$\lim_{n\to\infty} \hat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) = 1.$$

We will establish the following chain of equivalences:

$$\begin{aligned} x \text{ is a Lebesgue density point of } X &\iff \lim_{n\to\infty} m(X : B_n(x)) = 1 \\ &\iff \lim_{n\to\infty} m(X : C_n(x)) = 1 \\ &\iff \lim_{n\to\infty} m(X : D_n(x)) = 1 \\ &\iff \lim_{n\to\infty} m(X : E_n(x)) = 1 \\ &\iff \lim_{n\to\infty} m(X : F_n(x)) = 1 \\ &\iff \lim_{n\to\infty} m(X : G_n(x)) = 1 \\ &\iff \lim_{n\to\infty} \hat{m}_{cu}(X : \widehat{J}^{cu}_n(x)) = 1. \end{aligned}$$

Before defining the sets $B_n(x)$ through $G_n(x)$, we outline the general scheme of the proof. After verifying the first equivalence, we prove that $B_n(x)$ is regular, and that $B_n(x)$ and $C_n(x)$ are internested. The second equivalence then follows from Lemma 2.1. The sets $C_n(x)$ and $D_n(x)$ both fiber over the same base in $\widehat{\mathcal{W}}^{cs}$, with c-uniform fibers in $\mathcal{W}^u$, so the third equivalence follows from Proposition 2.5. We prove that the sets $D_n(x), E_n(x), F_n(x)$, and $G_n(x)$ are all internested, and that $G_n(x)$ is a regular sequence. Equivalences 4-6 then follow from Lemma 2.1. Finally, $G_n(x)$ fibers over $\widehat{J}^{cu}_n(x)$, with c-uniform $\mathcal{W}^s$-fibers, and so the final equivalence follows from Proposition 2.7. This final step uses $\mathcal{W}^s$-saturation of X.

The sets $B_n(x)$ through $G_n(x)$ are defined as follows. The set $B_n(x)$ is a Riemannian ball in M:

$$B_n(x) = B(x, \sigma_n).$$

The sets $C_n(x)$, $D_n(x)$ and $E_n(x)$ will fiber over the same base $D^{cs}_n(x)$, where

$$D^{cs}_n(x) = \bigcup_{x' \in \widehat{\mathcal{W}}^s(x,\sigma_n)} \widehat{B}^c_n(x').$$

Proposition 3.1, part 4. implies that $D^{cs}_n(x)$ is contained in the C^1 submanifold $\widehat{\mathcal{W}}^{cs}(x)$; the sequences $D^{cs}_n(x)$ and $\widehat{\mathcal{W}}^{cs}(x, \sigma_n)$ are internested. Let

$$C_n(x) = \bigcup_{q \in D^{cs}_n(x)} \mathcal{W}^u(q, \sigma_n),$$

and let

$$D_n(x) = \bigcup_{q \in D^{cs}_n(x)} J^u_n(q).$$

The set $E_n(x)$ is nearly identical to $D_n(x)$, with the crucial difference that the J^u_n-fibers are replaced with $\widehat{J}^u_n$-fibers:

$$E_n(x) = \bigcup_{q \in D^{cs}_n(x)} \widehat{J}^u_n(q) = \bigcup_{x' \in \widehat{\mathcal{W}}^s(x,\sigma_n)} \widehat{J}^{cu}_n(x') = \bigcup_{x' \in \mathcal{W}^s(x,\sigma_n)} \widehat{J}^{cu}_n(x').$$

The rightmost equality follows from the fact that $\widehat{\mathcal{W}}^s(x,\sigma_n) = \mathcal{W}^s(x,\sigma_n)$, for all $x \in \mathcal{W}^s(p,1)$ (Proposition 3.1, part 5.)

We define $F_n(x)$ to be the foliation product of $\widehat{J}^{cu}_n(x)$ and $\mathcal{W}^s(x,\sigma_n)$:

$$F_n(x) = \bigcup_{q \in \widehat{J}^{cu}_n(x),\ q' \in \mathcal{W}^s(x,\sigma_n)} \mathcal{W}^s(q) \cap \widehat{\mathcal{W}}^{cu}(q').$$

This definition makes sense since the foliations $\widehat{\mathcal{W}}^{cu}$ and $\mathcal{W}^s$ are transverse. Finally, let

$$G_n(x) = \bigcup_{q \in \widehat{J}^{cu}_n(x)} \mathcal{W}^s(q,\sigma_n).$$

It is in the transition from D_n to E_n that the exchange between the measure-theoretically useful foliation $\mathcal{W}^u$ and the geometrically useful foliation $\widehat{\mathcal{W}}^u$ takes place. The definition of F_n is where we first use the foliation $\mathcal{W}^s$.

Figure 8 is a schematic illustration of the relationship between the sets $E_n(x)$, $F_n(x)$ and $G_n(x)$. All three sets contain $\widehat{J}^{cu}_n(x)$ and $\mathcal{W}^s(x,\sigma_n)$. The set $E_n(x)$ fibers over $\mathcal{W}^s(x,\sigma_n)$ with fibers of the form $\widehat{J}^{cu}_n(\cdot)$. The set $G_n(x)$ fibers over $\widehat{J}^{cu}_n(x)$ with fibers of the form $\mathcal{W}^s(\cdot,\sigma_n)$. The foliation product $F_n(x)$ of $\widehat{J}^{cu}_n(x)$ and $\mathcal{W}^s(x,\sigma_n)$ is, in some sense, intermediate between $E_n(x)$ and $G_n(x)$.

We now prove these equivalences, following the outline described above.

First, recall that $B_n(x)$ is a round ball about x of radius σ_n. The forward implication in the first equivalence is obvious from the definition of $B_n(x)$. The backward implication follows from this definition and the fact that the ratio $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ of successive radii is less than 1, and is bounded away from both 0 and 1 independently of n. From this we also see that $B_n(x)$ is regular.

The set $C_n(x)$ fibers over $D^{cs}_n(x)$, with fiber $\mathcal{W}^u(x',\sigma_n)$ over $x' \in D^{cs}_n(x)$. The sequence $D^{cs}_n(x)$ internests with the sequence of disks $\widehat{\mathcal{W}}^{cs}(x,\sigma_n)$, by continuity and transversality of the foliations $\widehat{\mathcal{W}}^c$ and $\widehat{\mathcal{W}}^s$. Continuity and transversality of the foliations $\mathcal{W}^u$ and $\widehat{\mathcal{W}}^{cs}$ then imply that $C_n(x)$ and $B_n(x)$ are internested.

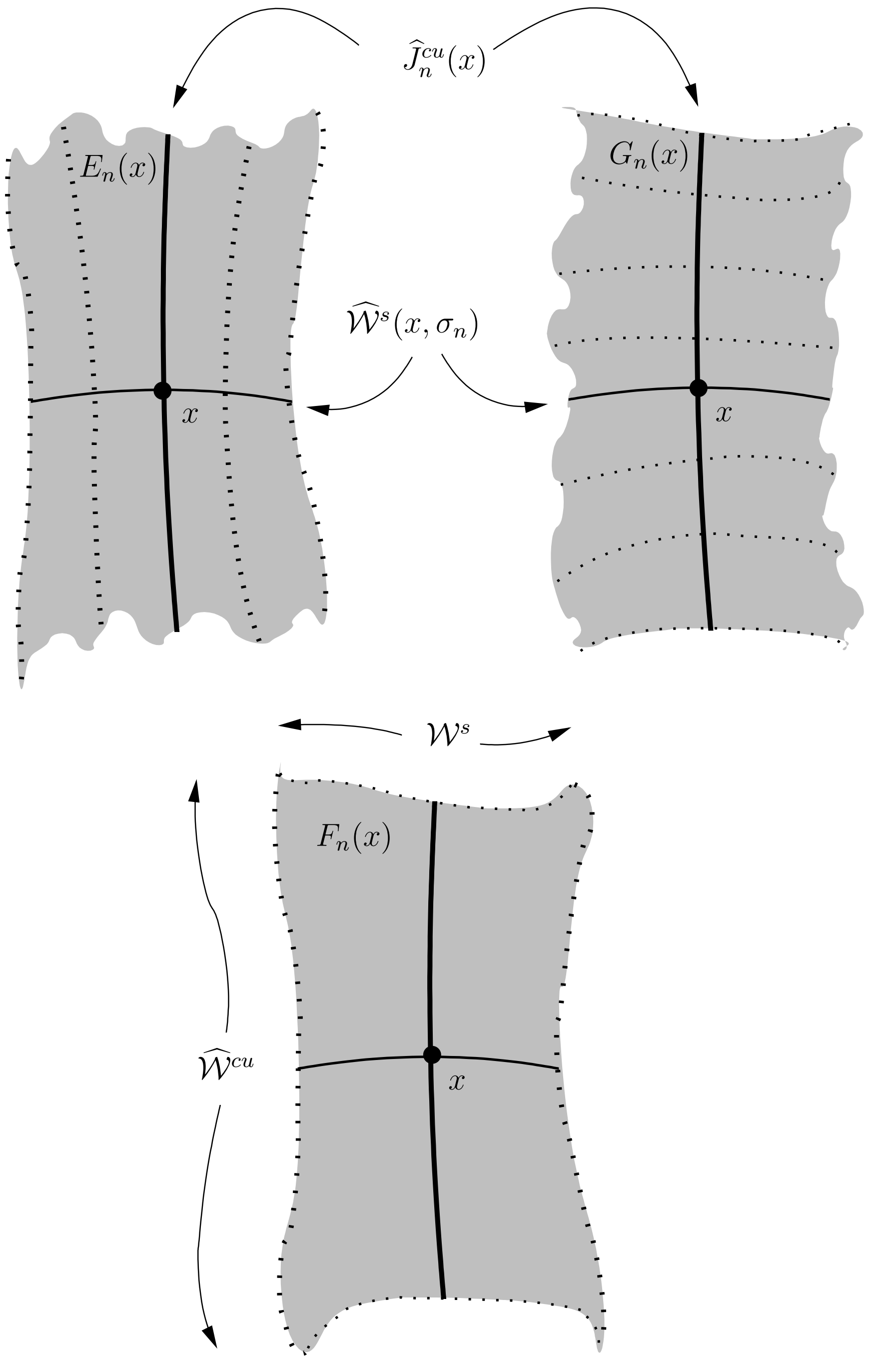


Figure 4: Comparison between $E_n(x)$, $F_n(x)$ and $G_n(x)$.

To prove the equivalence

$$\lim_{n\to\infty} m(X : C_n(x)) = 1 \iff \lim_{n\to\infty} m(X : D_n(x)) = 1,$$

we note that $C_n(x)$ and $D_n(x)$ both fiber over $D_n^{cs}(x)$, with $\mathcal{W}^u$-fibers. Since X is essentially $\mathcal{W}^u$-saturated, Proposition 2.5 implies that it suffices to show that the fibers of $C_n(x)$ and $D_n(x)$ are both c-uniform. The fibers of of $C_n(x)$ are easily seen to be uniform, because they are all comparable to balls in $\mathcal{W}^u$ of fixed radius σ_n. The fibers of $D_n(x)$ are the unstable juliennes $J_n^u(x')$, for $x' \in D_n^{cs}(x)$. Uniformity of these fibers follows from Proposition 5.4.

We next prove:

Lemma 8.1 *The sequences $D_n(x)$ and $E_n(x)$ are internested.*

Proof. Recall that

$$D_n(x) = \bigcup_{q\in D_n^{cs}(x)} J_n^u(x), \qquad \text{and} \qquad E_n(x) = \bigcup_{q\in D_n^{cs}(x)} \widehat{J}_n^u(x).$$

Internesting of the sequences $D_n(x)$ and $E_n(x)$ means that there is a $k \geq 0$ such that, for all $n \geq k$,

$$D_n(x) \subseteq E_{n-k}(x) \quad \text{and} \quad E_n(x) \subseteq D_{n-k}(x).$$

We will show that there is a k for which the first inclusion holds. Reversing the roles of of $\mathcal{W}^u$ and $\widehat{\mathcal{W}}^u$ in the proof gives the second inclusion.

Suppose $y \in D_n(x)$. Then $y \in J_n^u(q) = f^{-n}(\mathcal{W}^u(q_n, \tau_n))$, for some $q \in D_n^{cs}(x)$; in particular,

$$d(y_n, q_n) \quad = \quad O(\tau_n). \tag{23}$$

Let $\hat{q}$ be the unique point of intersection of $\widehat{\mathcal{W}}^u(y)$ with $\widehat{\mathcal{W}}^{cs}(x)$. We will show that $y \in E_{n-k}(x)$, for some k that is independent of n. In order to do this, it suffices to show that $\hat{q} \in D_{n-k}^{cs}(x)$ and $y \in \widehat{J}_{n-k}^u(\hat{q}) = f^{-(n-k)}(\widehat{\mathcal{W}}^u(\hat{q}_{n-k}, \tau_{n-k}))$.

In order to prove that $\hat{q} \in D_{n-k}^{cs}(x)$ it will suffice to show that

$$d(q, \hat{q}) = o(\sigma_n) \tag{24}$$

(in fact, $O(\sigma_n)$ would suffice, but the argument gives $o(\sigma_n)$). In order to prove that $y \in \widehat{J}_{n-k}^u(\hat{q})$ it will suffice to show that

$$d(y_n, \hat{q}_n) = O(\tau_n). \tag{25}$$

Equation (24) follows easily from (25). Since y_n and $\hat{q}_n$ lie in the same $\widehat{\mathcal{W}}^u$ leaf, Proposition 3.1 and (25) imply that

$$d(y, \hat{q}) = O(\hat{\nu}_n \tau_n) = o(\sigma_n), \tag{26}$$

since $\hat{\nu}\tau < \sigma$. Similarly, Proposition 3.1 and (23) imply that

$$d(y, q) = o(\sigma_n). \tag{27}$$

Applying the triangle inequality to (26) and (27) gives (24).

It remains to prove (25). Recall from the construction of the fake foliations in Proposition 3.1 that, at any point z in the neighborhood N of the orbit of p in which the fake foliations are defined, the tangent space $T_z\widehat{\mathcal{W}}^u(z)$ lies in the ε-cone about $T_z\mathcal{W}^u(z) = E^u(z)$. Furthermore, the angle between $T_z\widehat{\mathcal{W}}^{cs}(z)$ and either $T_z\widehat{\mathcal{W}}^u(z)$ or $T_z\mathcal{W}^u(z)$ is uniformly bounded away from 0. Note that $\hat{q}_n$ is the unique point in $\widehat{\mathcal{W}}^u(y_n) \cap \widehat{\mathcal{W}}^{cs}(x_n)$ and q_n is the unique point in $\mathcal{W}^u(y_n) \cap \widehat{\mathcal{W}}^{cs}(x_n)$; combining this with (23) gives:

$$d(y_n, \hat{q}_n) = O(d(y_n, q_n)) = O(\tau_n).$$

This completes the proof.$\diamond$

We next show:

Lemma 8.2 *$E_n(x)$ and $F_n(x)$ are internested, as are $F_n(x)$ and $G_n(x)$.*

Proof. The sets $E_n(x)$ and $F_n(x)$ both fiber over the same base $\widehat{\mathcal{W}}^s(x, \sigma_n)$. The fibers of $E_n(x)$ are the cu-juliennes$\widehat{J}^{cu}_n(x')$, for $x' \in \widehat{\mathcal{W}}^s(x, \sigma_n)$. the fibers of $F_n(x)$ are images of $\widehat{J}^{cu}_n(x)$ under $\mathcal{W}^s$-holonomy from $\widehat{\mathcal{W}}^{cu}(x)$ to $\widehat{\mathcal{W}}^{cu}(x')$, for $x' \in \widehat{\mathcal{W}}^s(x, \sigma_n)$. It follows immediately from Proposition 5.3 that the sequences $E_n(x)$ and $F_n(x)$ are internested.

To see that $F_n(x)$ and $G_n(x)$ are internested, suppose that q' lies in the boundary of the fiber of $F_n(x)$ that lies in $\mathcal{W}^s(q)$ for some $q \in \widehat{J}^{cu}_n(x)$. Then $q' \in \widehat{J}^{cu}_n(x')$ for a point x' that lies in the boundary of $\mathcal{W}^s(x, \sigma_n)$. The diameters of $\widehat{J}^{cu}_n(x)$ and $\widehat{J}^{cu}_n(x')$ are both $O(\sigma_n)$, and $d(x, x') = \sigma_n$. Hence, if k is large enough, we will have

$$\sigma_{n+k} \leq d(q, q') \leq \sigma_{n-k}.$$

Thus all points on the boundary of the fiber of $F_n(x)$ in $\mathcal{W}^s_{loc}(q)$ lie outside $\mathcal{W}^s(q, \sigma_{n+k})$ and inside $\mathcal{W}^s(q, \sigma_{n-k})$. $\diamond$

We now know that any two of $D_n(x), E_n(x), F_n(x)$ and $G_n(x)$ are internested. As discussed above, to prove the fourth through sixth equivalences, it now suffices to show:

Lemma 8.3 *The sequence $G_n(x)$ is regular for each $x \in \mathcal{W}^s(p,1)$.*

Proof. The set

$$G_n(x) = \bigcup_{q \in \widehat{J}_n^{cu}(x)} \mathcal{W}^s(q, \sigma_n)$$

fibers over $\widehat{J}_n^{cu}(x)$, with $\mathcal{W}^s$-fibers $\mathcal{W}^s(q, \sigma_n)$. Since $\mathcal{W}^s$ is absolutely continuous, Proposition 2.4 implies that regularity of $G_n(x)$ follows from regularity of the base sequence and fiber sequence. Proposition 5.4 implies that the sequence $\widehat{J}_n^{cu}(x)$ is regular in the induced measure $\hat{m}_{cu}$. As we remarked above, the ratio $\sigma_{n+1}/\sigma_n = \sigma(p_n)$ is uniformly bounded below away from 0. Consequently, the ratio

$$\frac{m_s(\mathcal{W}^s(q, \sigma_{n+1}))}{m_s(\mathcal{W}^s(q, \sigma_n))}$$

is bounded away 0, uniformly in x, q, and n. The regularity of $G_n(x)$ now follows from Proposition 2.4. $\diamond$

To prove the final equivalence, we use the fact that $G_n(x)$ fibers over $\widehat{J}_n^{cu}(x)$ with c-uniform fibers and apply Proposition 2.7. Here we use the fact that X is $\mathcal{W}^s$-saturated. This completes the proof of Proposition 5.5. $\diamond$

9 Open questions

We do not know whether the bunching assumption can be dropped in Theorem 0.1. It seems clear that a significant new idea would be required to remove this hypothesis. Indeed, even if we impose stringent assumptions on the regularity of the center, center-stable and center-unstable foliations, as Brin and Pesin do in their 1974 work, every known ergodicity result requires the center bunching hypothesis (7). Thus a first step in answering this question might be to answer the following question:

Question: Suppose that f is C^2, volume-preserving and partially hyperbolic. If E^c is absolutely continuous (Lipschitz, smooth, ...) and f is (essentially) accessibile, is f then ergodic?

As a concrete example, consider a diffeomorphism $f_\lambda : \mathbf{T}^2 \times \mathbf{T}^2 \to \mathbf{T}^2 \times \mathbf{T}^2$ of the form:

$$f_\lambda(x, y) = (A(x), g_\lambda(y)),$$

where $A : \mathbf{T}^2 \to \mathbf{T}^2$ is the linear Anosov diffeomorphism given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2,$$

and $g_\lambda : \mathbf{T}^2 \to \mathbf{T}^2$ is a standard map of the form:

$$g_\lambda(z, w) = (z + w, w + \frac{\lambda}{2\pi} \sin(2\pi(z + w))).$$

A straightforward calculation shows that there is an interval $\Lambda \subset \mathbf{R}$ containing $(-4, 4)$ such that, if $\lambda \in \Lambda$, then f_λ is partially hyperbolic with respect to the standard (flat) metric on $\mathbf{T}^2 \times \mathbf{T}^2$. It is also not difficult to show, by examining the spectrum of Tf_λ at the fixed point $(0, 0)$, that f_λ is center bunched if and only if $\lambda \in (-1, 1)$.

This example appears in [SW1], where it is shown that there is a function $\varphi : \mathbf{T}^2 \to \mathbf{T}^2$ with $\varphi(0) = 0$ and an interval $E = (0, \epsilon_0)$ such that, for all $(\lambda, \epsilon) \in \Lambda \times E$, the map

$$f_{\lambda,\epsilon}(x, y) = (A(x), g_\lambda(y) + \epsilon\varphi(x))$$

is both partially hyperbolic and stably accessible. Furthermore, $f_{\lambda,\epsilon}$ is center bunched if and only if $(\lambda, \epsilon) \in (-1, 1) \times E$. For all $(\lambda, \epsilon) \in \Lambda \times E$, the center bundle E^c is tangent to the fibers $\{x\} \times \mathbf{T}^2$ and is C^∞. The foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ are also C^∞.

Theorem 0.1 implies that $f_{\lambda,\epsilon}$ is stably ergodic for all $(\lambda, \epsilon) \in (-1, 1) \times E$. We do not however know whether $f_{\lambda,\epsilon}$ is ergodic for even a single value of $(\lambda, \epsilon) \in (\Lambda \setminus (-1, 1)) \times E$.

We also do not know whether or not the center bunching inequalities (7) imply that a partially hyperbolic diffeomorphism is dynamically coherent. The simplest non dynamically coherent examples are linear Anosov diffeomorphisms of nilmanifolds, viewed as partially hyperbolic by grouping the weaker stable and unstable directions together to form a center distribution. It was first observed in [W] that this construction can provide examples that are not dynamically coherent.

In [BW3], we discuss a family of examples of this type. They are Anosov diffeomorphisms of a compact quotient of the product of the Heisenberg group with itself induced by linear maps whose eigenvalues are

$$\lambda^{a+b} \geq \lambda^{b} \geq \lambda^{a} \geq \lambda^{-a} \geq \lambda^{-b} \geq \lambda^{-a-b},$$

where $b \geq a \geq 0$ and $\lambda > 1$ is the larger of the two eigenvalues of a hyperbolic matrix in $SL(2, \mathbf{Z})$ whose eigenvalues are $\lambda > 1 > \lambda^{-1} > 0$. When $a > 0$, the map can be viewed as a partially hyperbolic diffeomorphism with E^u and E^s being the left invariant one dimensional distributions corresponding to the eigenvalues λ^{a+b} and λ^{-a-b} respectively. The center is the left invariant four dimensional distribution corresponding to the sum of the eigenspaces for the other eigenvalues.

The linear map is chosen so that $[E^c, E^c]$ is spanned by $E^u \oplus E^s$. This ensures that distribution E^c is not integrable. Consequently these examples are not dynamically coherent.

For these examples we have

$$\nu = \hat{\nu} = \lambda^{-a-b} \qquad \text{and} \qquad \gamma = \hat{\gamma} = \lambda^{-b}.$$

Thus center bunching fails, because

$$\nu = \hat{\nu} = \lambda^{-a-b} \geq \lambda^{-2b} = \gamma\hat{\gamma}.$$

Note that center bunching only just fails in the extreme case when $a = b$. Gourmelon has constructed perturbations of the $a = b$ example that are robustly ergodic, but neither Anosov nor dynamically coherent. It does not seem to be known whether or not his perturbations are center bunched.

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