Hölder Regularity of Horocycle Foliations

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1 Introduction

Let M be a C^∞, nonpositively curved manifold. A *horosphere* in M is the projection to M of a limit of metric spheres in the universal cover $\tilde{M}$ (see §2). A *horospherical foliation* $\mathcal{H}$ is a foliation of the unit tangent bundle T^1M whose leaves consist of unit normal vector fields to horospheres.[1]

While regularity of horospherical foliations has been studied extensively for negatively curved manifolds M, considerably less is known in the non-positively curved case. The most general result is due to P. Eberlein: if M is complete and nonpositively curved, then horospheres are C^2, which implies that the individual leaves of $\mathcal{H}$ are C^1. Further, the tangent distribution $T\mathcal{H}$ depends continuously the basepoint $v \in T^1M$ (see [9]).

Beyond Eberlein's theorem, smoothness results have consisted mainly of counterexamples ([2],[5]); in particular, the best one could hope for in the case of a general compact, nonpositively curved M is for $T\mathcal{H}$ to be Hölder-continuous. In this paper we prove

Theorem I′: *Let S be a compact, real-analytic, nonpositively curved surface. Then $T\mathcal{H}$ is Hölder.*

Theorem I′ is actually a corollary of a more general result, Theorem I below.

The problem of finding the regularity of horospherical foliations has a long history, which we briefly summarize here.

[1]As we explain in §2, there are two such foliations, $\mathcal{H}^-$ and $\mathcal{H}^+$, called *stable* and *unstable* horospherical foliations, respectively. In this discussion, we use $\mathcal{H}$ to denote either of these.

E. Hopf showed in [7] that if M is a compact, negatively curved surface, then $T\mathcal{H}$ is C^1. Under the assumption that the sectional curvatures of M are 1/4-pinched, Hopf's result was generalized by M. Hirsch and C. Pugh [10] to any dimension. It follows from the work of D. V. Anosov [1] that $T\mathcal{H}$ is always Hölder, when M is compact and negatively curved.

In Anosov's theorem, the conclusion "Hölder" cannot be improved to "C^1". In fact, B. Hasselblatt [8] has found open sets of metrics, with negative curvature arbitrarily close to 1/4-pinched, for which the horospherical foliations fail to be C^1. Related bounds on the smoothness of $T\mathcal{H}$ beyond C^1, in the context of 3-dimensional Anosov flows, were found by S. Hurder and A. Katok [11]. An example of W. Ballman, M. Brin and K. Burns shows that compactness is necessary in Anosov's result; they construct in [2] a complete, finite volume surface whose curvature is arbitrarily close to -1 but for which $T\mathcal{H}$ is not Hölder.

Returning to the compact, nonpositive curvature case, Gerber and V. Niţică [5] have examples of real-analytic surfaces showing that $T\mathcal{H}$ in Theorem I$'$ can fail to have a Hölder exponent greater than 1/2. In particular, $T\mathcal{H}$ can be non-Lipschitz.

A related issue is that of the regularity of $T\mathcal{H}$ along the leaves of $\mathcal{H}$; that is, how smooth are the leaves of $\mathcal{H}$? For M compact and negatively curved, Anosov [1] showed that the leaves of $\mathcal{H}$ are C^∞. In the case of nonpositive curvature, Eberlein's "C^1" conclusion cannot be improved to "C^2"; Ballman, Brin and Burns construct in [2] a compact, real-analytic surface of nonpositive curvature for which the leaves of $\mathcal{H}$ fail to be C^2. However, the non-C^2 leaves in their example are $C^{1+\text{Lipschitz}}$, (i.e., $T\mathcal{H}$ is Lipschitz along leaves) and this suggested to us the question of whether this is always the case for compact, real-analytic surfaces of nonpositive curvature. As a corollary of our Theorem II below, we have

Theorem II$'$: *Let S be a compact, real-analytic, nonpositively curved surface. Then the leaves of $\mathcal{H}$ are uniformly $C^{1+\mathit{Lipschitz}}$.*

Our interest in these questions arose while studying the ergodic properties of the geodesic flow for analytic, nonpositively curved surfaces. We asked whether the time-one map of such a flow remains ergodic under suitable perturbations. Related results for negatively curved manifolds use Hölder continuity of the horospherical foliations in a central way ([6], [13], [12]). We hope that Theorems I and II can be used to establish similar results for certain nonpositively curved surfaces.

1.1 Statement of Results

Throughout this paper we always assume that manifolds are boundaryless. We follow the usual convention of referring to horospheres as "horocycles" when M has dimension 2.

Theorem I: *Let S be a compact surface with a C^∞ metric of nonpositive curvature K satisfying the following conditions:*

1) *If γ is a geodesic that is not closed, then there is no infinite time interval I for which $K(\gamma(t)) = 0$, for all $t \in I$.*

2) *If γ is a closed geodesic, then there exists a t such that K does not vanish to infinite order at $\gamma(t)$.*

Then the leaves of the horocycle foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly $C^{1+Lipschitz}$.

Theorem II: *Let S be a compact surface with a C^∞ metric of nonpositive curvature satisfying the conditions of Theorem I. Then the tangent distributions $T\mathcal{H}^+$ and $T\mathcal{H}^-$ are Hölder-continuous.*

Proof of Theorems I′ and II′ from I and II. If S is real-analytic, then the set of points in S where K vanishes is a real-analytic subvariety in S. In particular, K cannot vanish on an infinite time interval on a non-closed geodesic nor can K vanish to all orders at a point, unless it vanishes identically on the surface. In this case, S is a flat torus or Klein bottle and the horocycle foliations are analytic. □

Remark: It is an open question whether Theorems I and II hold without hypotheses 1) and 2). It is also not known whether there exist C^∞ surfaces that fail to satisfy hypothesis 1), except if the curvature vanishes identically. There are Lipschitz metrics with this property [3]. At the end of §3 we give an example to show that the estimates on the curvatures of the horocycles that are used in our proofs do not hold without hypothesis 2).

We also have an easier version of Theorem I, with a weaker conclusion, but which holds without the assumptions 1) and 2).

Proposition III: *Let S be a compact surface with a C^4 metric of nonpositive curvature. Then the leaves of the horocylic foliations $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly $C^{1+1/2}$; that is, $T\mathcal{H}^\pm$ is uniformly $1/2$-Hölder along leaves.*

1.2 Outline of the proofs

To prove these results, we study the dependence on $v \in T^1S$ of solutions to the scalar Riccati equation

$$u'(t) + u(t)^2 + K(\sigma_v(t)) = 0,$$

where $\sigma_v(t)$ is the unit-speed geodesic determined by v, and $K : S \to \mathbf{R}$ is the curvature. In §2 we explain how Hölder regularity of $T\mathcal{H}$ amounts to Hölder dependence on v of the "unstable" solutions to the Riccati equations.

In §3 we turn to a study of these Riccati equations. The analysis begins by taking the difference of two Riccati solutions u_0 and u_1 along geodesics determined by v_0 and v_1, to obtain

$$(u_1 - u_0)' = -(u_1 + u_0)(u_1 - u_0) + (K_0 - K_1), \tag{1.1}$$

where K_0 and K_1 are the curvatures of S along these geodesics. To obtain our regularity results, we need $|(u_1 - u_0)(0)|$ to be small relative to the distance between v_0 and v_1. It is apparent from (1.1) that $|u_1 - u_0|$ decreases rapidly if $u_1 + u_0$ is large relative to $|K_0 - K_1|$. The remainder of the proofs is devoted to estimating the sizes of these terms.

The proof of Proposition III depends only on Lemmas 3.1 and 3.2. For the proofs of Theorems I and II, we need the additional Lemmas 3.3 - 3.5. The proof of the lower bound in Lemma 3.3 is presented in §4.

1.3 Acknowledgements

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2 Preliminaries

Let M be a complete n-dimensional manifold of nonpositive sectional curvatures and let $\tilde{M}$ be its universal cover. We now define the horospherical foliations discussed in the introduction. For a unit vector v let σ_v denote the geodesic in M (or $\tilde{M}$) with $\sigma_v'(0) = v$. Vectors $v, w \in T^1\tilde{M}$ are *asymptotic* if there exists a constant $C > 0$ such that for all $t > 0$, $\mathrm{dist}(\sigma_v(t), \sigma_w(t)) \leq C$. Nonpositive curvature and simple connectivity imply that for every $v \in T^1\tilde{M}$ and $p \in \tilde{M}$, there is a unique vector $Z_v(p) \in T_p^1\tilde{M}$ such that $Z_v(p)$ is asymptotic to v. Fixing v, this defines a radial vector field Z_v on $\tilde{M}$. The vector v also determines a *Busemann function* $F_v : \tilde{M} \to \mathbf{R}$ by:

$$F_v(p) = \lim_{t\to\infty}(\mathrm{dist}(p, \sigma_v(t)) - t).$$

It is well-known (see, e.g. [9]) that F_v is C^1, Z_v is the gradient of $-F_v$, and each level set $F_v^{-1}(c)$ is the limit of geodesic spheres of radius $t + c$ centered at $\sigma_v(t)$. Moreover, as was shown by Eberlein, Busemann functions are C^2, and consequently their level sets are C^2 [9].

For $v \in T^1\tilde{M}$, define the *stable* and *unstable horospheres* $h^-(v)$ and $h^+(v)$ determined by v to be the level sets $F_v^{-1}(0)$ and $F_{-v}^{-1}(0)$, respectively. The leaves of the stable and unstable horospherical foliations $\mathcal{H}^-$ and $\mathcal{H}^+$ of $T^1\tilde{M}$ are defined by:

$$\mathcal{H}^-(v) = \{Z_v(p) \,:\, p \in h^-(v)\}$$

and

$$\mathcal{H}^+(v) = \{-Z_{-v}(p) \,:\, p \in h^+(v)\}.$$

Since Busemann functions are C^2, the leaves of $\mathcal{H}^\pm$ are C^1, and the tangent distributions $T\mathcal{H}^\pm$ are defined.

We project the horospheres from $\tilde{M}$ into M to obtain horospheres for vectors in T^1M. Similarly, we obtain the horospherical foliation of T^1M.

We are interested in the regularity of $T\mathcal{H}^\pm$, which reduces to the regularity of the sectional curvature of the horospheres. These sectional curvatures are determined by solutions to certain Riccati and Jacobi equations. We now restrict to the case where M is a surface, S, and these equations can be reduced to scalar ones.

Let $v \in T_p^1\tilde{S}$ and $w \in T_p\tilde{S}$, and let J_- [J_+] be the stable [unstable] Jacobi field along σ_v with $J_-(0) = w$ [$J_+(0) = w$]. (The stable Jacobi field is defined $J_- = \lim_{n\to\infty} J_n$, where J_n is the Jacobi field along σ_v with $J_n(0) = w$ and

$J_n(n) = 0$. The unstable Jacobi field J_+ is defined by the same formula, except replacing $\lim_{n\to\infty}$ by $\lim_{n\to-\infty}$.) If Z_v is the radial vector field defined above, then $\nabla_w Z_v = J'_-(0)$, by Proposition 3.1 in [9]. Now assume w is a unit vector perpendicular to v and let E be the continuous, unit-length vector field along σ_v that is perpendicular to σ_v and satisfies $E(0) = w$. Then $J_-(t) = j_-(t)E(t)$, where j_- is a real-valued function that satisfies the scalar Jacobi equation:

$$j''_-(t) = -K(\sigma_v(t))j_-(t).$$

Let $u_- = j'_-/j_-$. Then u_- satisfies the scalar Riccati equation

$$u'_-(t) + u_-(t)^2 + K(\sigma_v(t)) = 0.$$

Since $j_-(0) = 1$,

$$u_-(0) = j'_-(0) = \langle \nabla_w Z_v, w\rangle = k_-(v), \tag{2.1}$$

where $-k_-(v)$ is the geodesic curvature of $h^-(v)$ at v. The function u_- is called the *stable* solution to the Riccati equation along σ_v; since J_- was constructed as $\lim_{n\to\infty} J_n$, it follows that $u_-(t) = \lim_{n\to\infty} u_n(t)$, where, for $n > 0$, u_n is the solution to the Riccati equation along σ_v with $u_n(n) = -\infty$. The unstable Riccati solution u_+ along σ_v is similarly defined in terms of J_+ and satisfies $u_+(t) = \lim_{n\to-\infty} u_n$, where, for $n < 0$, u_n is the solution to the Riccati equation along σ_v with $u_n(n) = +\infty$. A similar argument to the one summarized in equation (2.1) shows that $u_+(0) = k_+(v)$, where $k_+(v)$ is the geodesic curvature of $h^+(v)$ at v. Since $K \leq 0$, it follows that $u_-(t) \leq 0$ for all t, and $u_+(t) \geq 0$ for all t. Moreover, if $K(\sigma_v(t_0)) < 0$ for some t_0, then $u_-(t) < 0$, for all $t < t_0$, and $u_+(t) > 0$, for all $t > t_0$. (These inequalities are easy consequences of Lemma 3.1 below.)

A function f from a metric space (X_1, d_1) to a metric space (X_2, d_2) is *Hölder-continuous of exponent* $\alpha \in (0, 1]$ if there exists a constant $C > 0$ such that for all $p, q \in X_1$,

$$d_2(f(p), f(q)) \;\; \leq \;\; C(d_1(p, q))^\alpha. \tag{2.2}$$

The function f is *Lipschitz* if it is Hölder with exponent 1. We say that f is *Hölder (or Lipschitz) at a point* $p \in X_1$ if there is a constant $C = C(p) > 0$ such that inequality (2.2) holds for all $q \in X_1$. A family of functions $\mathcal{F}$ from

X_1 to X_2 is *uniformly Hölder (or Lipschitz)* if there is a single constant C such that (2.2) holds for all $p, q \in X_1$ and for all $f \in \mathcal{F}$.

Throughout this paper all geodesics have unit speed. We will use Fermi coordinates (s, x) along a geodesic γ in $\tilde{S}$, where s is the time parameter along γ, and x is the signed distance to γ. Then the curves $s =$ constant are unit-speed geodesics perpendicular to γ. We will frequently use ϕ to denote the angle between a vector v and the curve $x =$ constant; unless stated otherwise, such angles will be signed angles in $[-\pi/2, \pi/2]$ chosen so that $\sphericalangle(\partial/\partial x, x = a) = \pi/2$.

3 Proofs of Theorems I and II

This section contains the proofs of Theorems I and II, with the exception of the proof of the lower bound on the curvatures of horocycles in Lemma 3.3. This lower bound is proved in §4.

The following lemma contains facts which are routinely used in the study of Riccati and Jacobi equations. For example, part (iii) is the Comparison Lemma in [2] and it is also a special case of a well–known differential inequality ([7], Chapter III, Corollary 4.2). Part (v) is a special case of the Sturm Comparison Theorem.

Lemma 3.1 *Let A and B be real numbers, let $K, K_0, K_1 : \mathbf{R} \to \mathbf{R}$ be continuous functions, and let u, u_0, u_1 satisfy the Riccati equations $u' = -u^2 - K$, $u_i' = -u_i^2 - K_i$, $i = 0, 1$ respectively. Let $y = u_1 - u_0$. Let J_0, J_1 satisfy the Jacobi equations $J_i'' = -K_i J_i$, $i = 0, 1$. Then*

(i) $y' = -(u_0 + u_1)y + K_0 - K_1$

(ii) $y(B) = \int_A^B (K_0(t)) - K_1(t))\ exp\left[-\int_t^B (u_0(\tau) + u_1(\tau))\,d\tau\right] dt$
$\qquad + y(A) exp\left[-\int_A^B (u_0(\tau) + u_1(\tau)) d\tau\right]$

(iii) If $A \leq B, u_0(A) \leq u_1(A)$ and $K_1(t) \leq K_0(t)$ for $t \in [A, B]$, then $u_0(B) \leq u_1(B)$.

(iv) If $A \leq B, K(t) \leq 0$ for $t \in [A, B]$, and $u(A) \geq 0$, then

$$u(B) \geq \frac{u(A)}{(B - A)u(A) + 1}$$

(v) If $A \le B$, $J_0(A) = J_1(A) = 0$, $0 \le J_0'(A) \le J_1'(A)$ and $K_1(t) \le K_0(t) \le 0$ for $t \in [A, B]$, then $J_0(B) \le J_1(B)$.

Proof. Property (i) is obtained by subtracting the Riccati equation for u_0 from the Riccati equation for u_1, and (ii) follows from the formula for the solution of first order linear differential equations. It is clear from (ii) that if $A \le B, y(A) \ge 0$ and $K_0(t) - K_1(t) \ge 0$ for $t \in [A, B]$, then $y(B) \ge 0$. This proves (iii). Now (iv) is a special case of (iii), because if $K_0 \equiv 0$ and $u_0(A) = u(A)$, then $u_0(t) = u(A)/((t - A)u(A) + 1)$. For the proof of (v), see §10.2 of [4]. $\square$

The following lemma will be applied to the curvature function $f = K$. In this lemma the complete surface S could easily be replaced by a complete Riemannian manifold.

Lemma 3.2 *If f is a nonpositive function on a complete surface S such that $|(d^2/dt^2)(f(\sigma(t)))|$ exists and is uniformly bounded from above along all geodesics σ, then there exist constants $L_1, L_2 > 0$ such that for all $p, q \in S$,*

$$|f(p) - f(q)| \le L_1 \epsilon \sqrt{-f(p)} + L_2 \epsilon^2, \tag{3.1}$$

where $\epsilon = dist(p, q)$.

Proof. Let $L = \sup\{|(d^2/dt^2)|_{t=0}(f(\sigma(t)))| : \sigma \text{ is a geodesic on } S\}$. Let $p \in S$, and let σ be a geodesic on S such that $\sigma(0) = p$ and $\sigma'(0)$ is in a direction of the greatest increase of f at p; i.e., $(d/dt)|_{t=0}(f(\sigma(t))) = \|Df_p\|$. Let $g : \mathbf{R} \to \mathbf{R}$ satisfy $g(0) = f(\sigma(0)), g'(0) = (d/dt)|_{t=0}(f(\sigma(t)))$ and $g''(t) = -L$ for all t.

Then for $t \ge 0$,

$$0 \ge f(t) \ge g(t) = f(p) + \|Df_p\|t - \frac{1}{2}Lt^2.$$

Setting $t = \|Df_p\|/L$, we obtain $\|Df_p\| \le \sqrt{-2Lf(p)}$. Let $L_1 = \sqrt{2L}, L_2 = L/2$. Then the lemma follows from Taylor's Theorem.$\square$

In the following lemmas, we begin invoking our hypotheses 1) and 2) on the surface S.

Lemma 3.3 *Suppose S is a complete surface of nonpositive curvature K and $\gamma(s)$ is a closed geodesic on S such that K vanishes to order $m-1$ on γ, where $m \in \{2, 4, 6, \dots\}$. (I.e., if (s, x) are the Fermi coordinates along the lift $\tilde{\gamma}$ of γ to $\tilde{S}$, then*

$$\frac{\partial^k}{\partial x^k}\bigg|_{x=0} K(s, x) = 0$$

for $k = 0, 1, \dots, m-1$ and all s.) Also assume that there is at least one point, say $\gamma(0)$, such that K does not vanish to order m at $\gamma(0)$. (I.e., $(\partial^m/\partial x^m)K(0, x) \neq 0$.) Then there exists a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ and a positive constant C such that for any $v \in \mathcal{U}$ with footpoint having second Fermi coordinate $x = a$ the curvatures $k_-(v)$ and $k_+(v)$ of the stable and unstable horocycles satisfy

$$C \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right) \leq -k_-(v) \leq C^{-1} \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right)$$
$$C \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right) \leq k_+(v) \leq C^{-1} \max\left(|a|^{m/2}, |\phi|^{m/(m+2)}\right),$$

where $\phi = \sphericalangle(v, x = a)$.

The upper bounds on $-k_-(v)$ and $k_+(v)$ are proved in [5], Theorem 3.1. The assumption that there is a point on γ where the curvature does not vanish to order m is not needed to obtain these upper bounds. The lower bounds are proved in §4.

Note that the hypothesis of Lemma 3.3 could not hold for odd m, because K does not change sign.

Lemma 3.4 *If S is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,*

$$-k_-(v) \geq C\sqrt{-K(p)} \tag{3.2}$$

and

$$k_+(v) \geq C\sqrt{-K(p)}, \tag{3.3}$$

where p is the footpoint of v.

Proof. If γ is a closed geodesic on S along which K vanishes to order $m-1$, for $m \in \{2,4,6,\dots\}$, then there is a constant $C_1 > 0$ such that $-K(p) \leq C_1|a|^m$ for p in a neighborhood of γ with Fermi coordinates (s,a). If, in addition, there is a point on γ at which K does not vanish to order m, then the lower bounds on $-k_-(v)$ and $k_+(v)$ in Lemma 3.3 imply that (3.2) and (3.3) are satisfied for v with footpoint in some neighborhood of $T^1\gamma$, for some constant $C > 0$.

By hypotheses 1) and 2) of Theorems I and II, there are at most finitely many closed geodesics along which K vanishes. Therefore (3.2) and (3.3) hold for v in a neighborhood $\mathcal{U}$ of the union of the unit tangent bundles of such geodesics.

Now for $v \in T^1S, k_-(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \geq 0$, and $k_+(v)$ vanishes only if $K(\sigma_v(t)) = 0$ for all $t \leq 0$. Thus by hypothesis 1), $k_-(v)$ and $k_+(v)$ each vanish only for v in the unit tangent bundle of a closed geodesic along which K vanishes. Then (3.2) and (3.3) extend to the complement of $\mathcal{U}$ in T^1S for some $C > 0$ by the continuity of k_-, k_+, and K.$\square$

Lemma 3.5 *Let S be a surface satisfying the hypotheses of Theorems I and II. Let γ_0 and γ_1 be geodesics on S or $\tilde{S}$, let $K_i(t) = K(\gamma_i(t))$, for $i = 0,1$ and $t \in \mathbf{R}$, and let A and B be real numbers with $A \leq B$. Let u_i be a solution to the Riccati equation $u_i' = -u_i^2 - K_i, i = 0,1$, where u_0 is the unstable solution and $u_1(A) \geq 0$. Let $y = u_1 - u_0$. Then there exist positive constants C_1 and C_2, which depend only on S, such that*

$$|y(B)| \leq C_1\epsilon + C_2(B-A)\epsilon^2 + |y(A)|exp\Big[-\int_A^B (u_0(\tau) + u_1(\tau))\, d\tau\Big],$$

where $\epsilon = \max\{dist(\gamma_0(t), \gamma_1(t)) : A \leq t \leq B\}$.

Proof. By part (ii) of Lemma 3.1, we have

$$\begin{aligned}|y(B)| \leq \int_A^B |K_0(t) - K_1(t)| \exp\Big[-\int_t^B (u_0(\tau) + u_1(\tau))\, d\tau\Big]\, dt \\ + |y(A)| \exp\Big[-\int_A^B (u_0(\tau) + u_1(\tau))\, d\tau\Big]\end{aligned}$$

Moreover, if we apply Lemma 3.2 to $f = K$, we obtain constants $L_1, L_2 > 0$ such that

$$|K_0(t) - K_1(t)| \leq L_1 \epsilon \sqrt{-K_0(t)} + L_2 \epsilon^2,$$

where ϵ is as in the statement of the present lemma. Also, $u_1(\tau) \geq 0$ for $\tau \geq A$. Thus

$$\begin{aligned} |y(B)| \leq L_1 \epsilon \int_A^B \sqrt{-K_0(t)} \exp\Big[- \int_t^B u_0(\tau)\, d\tau \Big]\, dt \\ + L_2(B-A)\epsilon^2 + |y(A)| \exp\Big[- \int_A^B (u_0(\tau) + u_1(\tau))\, d\tau \Big]. \end{aligned} \tag{3.4}$$

Now, by Lemma 3.4, we obtain

$$u_0(\tau) \geq C_3 \sqrt{-K_0(\tau)}$$

for some constant $C_3 > 0$, for all $\tau \in [A, B]$. Therefore the first term on the right hand side of inequality (3.4) is bounded above by

$$\frac{L_1 \epsilon}{C_3} \int_A^B g(t) \exp\Big[- \int_t^B g(\tau)\, d\tau \Big]\, dt, \tag{3.5}$$

where $g(\tau) = C_3 \sqrt{-K_0(\tau)}$. The expression in (3.5) is equal to

$$\frac{L_1 \epsilon}{C_3} \Big(1 - \exp\Big[- \int_A^B C_3 \sqrt{-K_0(\tau)}\, d\tau \Big] \Big),$$

which is less than or equal to $L_1 \epsilon / C_3$.
This proves the lemma with $C_1 = L_1/C_3$ and $C_2 = L_2$.□

Proof of Theorem I. Let γ_0 and γ_1 be two geodesics on $\tilde{S}$ such that $\gamma_0'(0)$ and $\gamma_1'(0)$ are on the same unstable horocycle; i.e.,

$$\lim_{t \to -\infty} \mathrm{dist}_{\tilde{S}}(\gamma_0(t), \gamma_1(t)) = 0.$$

To prove that the leaves of the unstable horocycle foliation of $T^1\tilde{S}$ (or $T^1 S$) are uniformly $C^{1+\mathrm{Lipschitz}}$, it suffices to show that there exists a constant $C > 0$ (depending only on S) such that

$$|k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))| \leq C\epsilon,$$

where $\epsilon = \mathrm{dist}_{\tilde{S}}(\gamma_0(0), \gamma_1(0))$. Since $K \leq 0$, $\mathrm{dist}_{\tilde{S}}(\gamma_0(t), \gamma_1(t)) \leq \epsilon$, for $t \leq 0$. Let u_i be unstable Riccati solutions along γ_i, $i = 0, 1$. Let $y = u_1 - u_0$. Then $y(0) = k_+(\gamma_1'(0)) - k_+(\gamma_0'(0))$. Now apply Lemma 3.5 with $A = -1/\epsilon$ and $B = 0$. We obtain

$$|y(0)| \leq C_1\epsilon + C_2\epsilon + |y(A)|\mathrm{exp}\Big[-\int_A^0 (u_0(\tau) + u_1(\tau))\, d\tau\Big].$$

If $u_0(A) = u_1(A)$, then $y(A) = 0$ and we have the desired estimate. Thus we may assume $u_0(A) \neq u_1(A)$, say $u_1(A) > u_0(A)$. Then

$$\begin{aligned}
|y(A)|\mathrm{exp}\Big[-\int_A^0 (u_0(\tau) + u_1(\tau))\, d\tau\Big] &\leq u_1(A)\mathrm{exp}\Big[-\int_A^0 u_1(\tau)\, d\tau\Big] \\
&= u_1(A)\mathrm{exp}\left[-\int_0^{-A} u_1(\tau + A)\, d\tau\right] \\
&\leq u_1(A)\mathrm{exp}\left[-\int_0^{-A} \frac{u_1(A)\, d\tau}{\tau u_1(A) + 1}\right] \quad \text{by Lemma 3.1, part (iv)} \\
&= u_1(A)\left[\frac{1}{-Au_1(A) + 1}\right] \leq \frac{-1}{A} = \epsilon.
\end{aligned}$$

This completes the proof.□

The following proof is similar to the proof of Theorem I, but is independent of Lemmas 3.3 - 3.5.

Proof of Proposition III. Let S be a compact surface with a C^4 metric of nonpositive curvature. Then K is C^2 and Lemma 3.2 applies to $f = K$. We use the notation in the proof of Theorem I, but let $A = -1/\sqrt{\epsilon}$ and $B = 0$. By Lemmas 3.1(ii) and 3.2, we have

$$\begin{aligned}
|y(0)| &\leq \int_A^0 |K_0(t) - K_1(t)|\, dt + |y(A)|\mathrm{exp}\left[-\int_A^0 (u_0(t) + u_1(t))\, dt\right] \\
&\leq \epsilon L_1 \int_A^0 \sqrt{-K_0(t)}\, dt + L_2\epsilon^{3/2} + |y(A)|\mathrm{exp}\left[-\int_A^0 (u_0(t) + u_1(t))\, dt\right].
\end{aligned}$$

By the argument at the end of the proof of Theorem I, the third term in the preceding expression is bounded above by $-1/A = \sqrt{\epsilon}$. Therefore, if $\epsilon \leq 1$, we obtain

$$\begin{aligned}
|y(0)| &\leq \sqrt{\epsilon}L_1\left(\max_{p\in M} \sqrt{-K(p)}\right) + L_2\epsilon^{3/2} + \sqrt{\epsilon} \\
&\leq C\sqrt{\epsilon}
\end{aligned}$$

for some $C > 0$. This proves that the leaves of $\mathcal{H}^+$ are uniformly $C^{1+1/2}$.□
We now turn to the proof of Theorem II. We need an additional lemma.

Lemma 3.6 *If S is a surface satisfying the hypotheses of Theorems I and II, then there is a constant $C > 0$ such that for all $v \in T^1S$,*

$$Ck_+(v) \leq -k_-(v) \leq C^{-1}k_+(v).$$

Proof. This lemma follows from Lemma 3.3 in the same way that Lemma 3.4 follows from Lemma 3.3.□

Proof of Theorem II. We must show that there are constants $\alpha, C > 0$ such that

$$|k_+(v_1) - k_+(v_0)| \leq C(\mathrm{dist}(v_0, v_1))^\alpha, \text{ for all } v_0, v_1 \in T^1S. \tag{3.6}$$

Let $\Gamma \subset T^1S$ be the set of unit vectors which are tangent to closed geodesics along which K vanishes.

Step 1. We first show that it suffices to prove (3.6) in the case v_0 and v_1 have the same footpoint. I.e., we will show that (3.6) will follow for some $C > 0$ if there is a constant $\tilde{C} > 0$ such that for all $p \in S$ and v_0, $v_1 \in T^1_pS$,

$$|k_+(v_1) - k_+(v_0)| \leq \tilde{C}\theta^\alpha, \text{ where } \theta = \sphericalangle(v_0, v_1). \tag{3.7}$$

We now define a "horocyclic coordinate system," $(\mathcal{S}, X, \Phi) = (\mathcal{S}, X, \Phi)_{v_0}$ about a vector $v_0 \in T^1S$ with footpoint p_0. Let $W = W_{v_0} = -Z_{-v_0}$ where Z_{-v_0} is the radial vector field consisting of vectors asymptotic to $-v_0$. (See §2.) A vector v in a neighborhood of v_0 has coordinates $(\mathcal{S}, X, \Phi)$ if v makes signed angle Φ with W_{v_0} and the footpoint p of v can be obtained as follows. Let $h^+(v_0)$ be the unstable horocycle determined by v_0, as defined in §2. Start at p_0 and go along $h^+(v_0)$ for a signed distance X, to a point $\tilde{p}$. Then apply the geodesic flow to $W(\tilde{p})$ for time $\mathcal{S}$ to arrive at the vector $W(p)$. (See Figure 3.1.) Note that the curves $X =$ constant, $\Phi = 0$ are geodesics and the curves $\mathcal{S} =$ constant, $\Phi = 0$ are unstable horocycles. Also, along any geodesic $X =$ constant, $\partial/\partial\mathcal{S}$ is a Jacobi field corresponding to the variation of geodesics defined by $\mathcal{S} \to (X = \mathcal{S})$. Therefore the $(\mathcal{S}, X)$ coordinate system is C^2 in a neighborhood of p_0 in S and the $(\mathcal{S}, X, \Phi)$ coordinate system is C^1 in a neighborhood of v_0 in T^1S. Moreover, there is an $\eta > 0$ such that the $(\mathcal{S}, X, \Phi)_{v_0}$ coordinate system is defined on an open ball $\mathcal{B}(v_0, \eta)$ about v_0, for each v_0 in T^1S. We may also assume that η is chosen sufficiently small so that the images of the maps $(\mathcal{S}, X, \Phi)_{v_0} : \mathcal{B}(v_0, \eta) \to \mathbf{R}^3$ lie in $(-1, 1) \times (-1, 1) \times (-1, 1)$. The C^1 norms of these maps are uniformly bounded by some positive constant C_3.

Suppose $\mathrm{dist}(v_0, v_1) < \eta$ and $v_0, v_1 \in T^1S$. Let $(\mathcal{S}, X, \Phi) = (\mathcal{S}, X, \Phi)_{v_0}$. Assume that (3.7) holds. Note that k_+ is uniformly Lipschitz along unstable horocycles (by Theorem I) and k_+ is uniformly C^1 (in fact, C^∞) along the orbits of the geodesic flow. Thus there exist positive constants C_4, C_5, C_6 such that

$$\begin{aligned} |k_+(v_1) - k_+(v_0)| &\leq C_4|X| + C_5|\mathcal{S}| + \tilde{C}|\Phi|^\alpha \\ &\leq 3 \max(C_4, C_5, \tilde{C})\|(X, \mathcal{S}, \Phi)\|^\alpha \\ &\leq 3 \max(C_4, C_5, \tilde{C})C_3^\alpha\Big(\mathrm{dist}(v_0, v_1)\Big)^\alpha \\ &\leq C_6\Big(\mathrm{dist}(v_0, v_1)\Big)^\alpha. \end{aligned} \tag{3.8}$$

By increasing C_6, if necessary, we may assume that $|k_+(v_1) - k_+(v_0)|$ is less than or equal to the expression in (3.8) even if we drop the assumption that $\mathrm{dist}(v_0, v_1) < \eta$. This completes the reduction of (3.6) to (3.7), and we proceed with the proof of (3.7).

Step 2. Application of Lemma 3.5.

Let $p \in S$, let $v_0, v_1 \in T_p^1S$ and let $\theta = \sphericalangle(v_0, v_1)$. For $i = 0, 1$, let γ_i be a geodesic on S with $\gamma_i'(0) = v_i$. For $0 \leq r \leq 1$, let γ_r be a continuous variation of geodesics with $\gamma_r(0) = \gamma_0(0) = \gamma_1(0) = p$ and $\sphericalangle(\gamma_0'(0), \gamma_r'(0)) = r\theta$. Let

$$\begin{aligned} T = \max\{T_0 : &\text{ length of curve } r \to \gamma_r(t),\ 0 \leq r \leq 1, \\ &\text{is less than or equal to } \sqrt{\theta}\ ,\ \text{for } -T_0 \leq t \leq 0\}. \end{aligned} \tag{3.9}$$

Let $K_r(t) = K(\gamma_r(t))$ and let J_r be the perpendicular Jacobi field along γ_r defined by $J_r(t) = (d/dr)(\gamma_r(t))$. Let $J_r(t) = j_r(t)E_r(t)$, where E_r's are unit normal fields along γ_r's oriented so that $j_r(t) > 0$ for $t < 0$. Then $j_r(0) = 0$ and $j_r'(0) = -\theta$. By comparing with the $K \equiv 0$ case and applying Lemma 3.1 (v) (with t replaced by $-t$), we have $j_r(-T) \geq \theta T$. From the definition of T we have

$$\sqrt{\theta} = \int_0^1 j_r(-T)dr \geq \theta T. \tag{3.10}$$

Therefore $T \leq 1/\sqrt{\theta}$. (See Figure 3.2.) Similarly, by comparing with the $K \equiv K_{\min}$ case, where $K_{\min} < 0$ is the minimum value of the curvature function on S, we obtain $j_r(-T) \leq (\theta/\sqrt{|K_{\min}|})\sinh(\sqrt{|K_{\min}|}T)$. Thus there exists $\theta_0 > 0$ such that if $\theta < \theta_0$, then $T > 1$. Since it is clear that there is a $\tilde{C}$ such that (3.7) holds for $\theta \geq \theta_0$, we will henceforth assume that $T > 1$. (This will be used in Step 3.)

Let $u_i, i = 0, 1$, be the unstable solution of $u_i' = -u_i^2 - K_i$. Let $y = u_1 - u_0$ and apply Lemma 3.5 with $A = -T, B = 0$ and $\epsilon = \sqrt{\theta}$. We obtain constants $C_7, C_8, C_9 > 0$ such that

$$\begin{aligned}|k_+(v_1) - k_+(v_0)| &= |y(0)| \\ &\leq C_7\sqrt{\theta} + C_8T\theta + |y(-T)|\exp\left[-\int_{-T}^{0}(u_0(t) + u_1(t))\,dt\right] \\ &\leq (C_7 + C_8)\sqrt{\theta} + C_9\Big(\exp\left[-\int_{-T}^{0}u_0(t)\,dt\right]\Big)\Big(\exp\left[-\int_{-T}^{0}u_1(t)\,dt\right]\Big).\end{aligned} \tag{3.11}$$

Thus we must estimate $\exp\Big[-\int_{-T}^{0}u_0(t)\,dt\Big]$ and $\exp\Big[-\int_{-T}^{0}u_1(t)\,dt\Big]$ from above. We first estimate a related integral.

Step 3. Let $w_r = j_r'/j_r$. Then $w_r' = -w_r^2 - K_r$, $w_r(0) = -\infty$, and $w_r(t) < 0$ for $t < 0$. In this step, we will show that there is a constant $C_{10} > 0$ such that

$$\exp\left[\int_{-T}^{-1}w_0(t)\,dt\right] \leq C_{10}\sqrt{\theta}. \tag{3.12}$$

We have

$$\frac{j_r(-1)}{j_r(-T)} = \exp\left[\int_{-T}^{-1}\frac{j_r'}{j_r}\,dt\right] = \exp\left[\int_{-T}^{-1}w_r\,dt\right].$$

Thus

$$j_r(-T) = j_r(-1)\exp\left[-\int_{-T}^{-1}w_r\,dt\right]. \tag{3.13}$$

Since $j_r(0) = 0$ and $j_r'(0) = -\theta$, we see that $j_r(-1) \leq C_{11}\theta$, for some $C_{11} > 0$ (by comparing with the case $K \equiv K_{\min}$ and applying Lemma 3.1 (v)). Combining this fact with (3.10) and (3.13), we obtain

$$\theta^{-1/2} \leq C_{11}\int_{0}^{1}\exp\left[-\int_{-T}^{-1}w_r\,dt\right]dr. \tag{3.14}$$

If the average of the quantities $\exp\left[-\int_{-T}^{-1} w_r\,dt\right]$ for $0 \le r \le 1$ in (3.14) could be replaced by $\exp\left[-\int_{-T}^{-1} w_0\,dt\right]$ the desired inequality (3.12) would follow. To make this type of replacement we now show that there is a constant $C_{12} > 0$ such that

$$\exp\left[\int_{-T}^{-1} |w_r - w_0|\,dt\right] \le C_{12}, \quad \text{for all } r,\ 0 \le r \le 1. \tag{3.15}$$

Fix r, $0 \le r \le 1$. Let $\tilde{y} = w_r - w_0$, and let t and t_0 satisfy $-T \le t \le -1$ and $-1 < t_0 < 0$. By Lemma 3.1(ii) with $A = t_0$, $B = t$, we have

$$\begin{aligned}\tilde{y}(t) = &-\int_t^{t_0} (K_0(s) - K_r(s)) \exp\left[\int_t^s w_0(\tau) + w_r(\tau)\,d\tau\right] ds \\ &+ \tilde{y}(t_0)\exp\left[\int_t^{t_0} w_0(\tau) + w_r(\tau)\,d\tau\right]\end{aligned}$$

Since $w_0(\tau), w_r(\tau) \le 0$ for $\tau < 0$, this implies

$$|\tilde{y}(t)| \le \int_t^{t_0} |K_0(s) - K_r(s)| \exp\left[\int_t^s w_0(\tau)\,d\tau\right] ds + |\tilde{y}(t_0)|. \tag{3.16}$$

If u_- denotes the stable Riccati solution along γ_0, then for $\tau < 0$,

$$\begin{aligned}|w_0(\tau)| \ge |u_-(\tau)| &= k_-(\gamma_0'(\tau)) \\ &\ge C_{13}\sqrt{-K(\gamma_0(\tau))},\end{aligned}$$

for some $C_{13} > 0$, by Lemma 3.4. Then the same argument as in the proof of Lemma 3.5 shows that the first term on the right side of (3.16) is bounded by $C_{14}\sqrt{\theta}$ for some constant $C_{14} > 0$, where C_{14} does not depend on t_0. Thus

$$|\tilde{y}(t)| \le C_{14}\sqrt{\theta} + |\tilde{y}(t_0)|.$$

By comparing with constant curvature 0 and constant curvature $K_{\min}$ and applying Lemma 3.1(iii),

$$\sqrt{|K_{\min}|}\,\coth \tau\sqrt{|K_{\min}|} \le w_r(\tau) \le \frac{1}{\tau} \text{ for } \tau < 0.$$

A couple of applications of L'Hôpital's rule shows that

$$\lim_{\tau \to 0} \sqrt{|K_{\min}|}\,\coth \tau\sqrt{|K_{\min}|} - \frac{1}{\tau} = 0,$$

and it follows that $\lim_{t_0\to 0}|\tilde{y}(t_0)| = \lim_{t_0\to 0}|w_r(t_0) - w_0(t_0)| = 0$.

Therefore $|\tilde{y}(t)| \le C_{14}\sqrt{\theta}$ for $-T \le t \le -1$. Since $T \le 1/\sqrt{\theta}$, this proves (3.15) and with $C_{12} = C_{14}$. By rewriting (3.14) and applying (3.15) we obtain

$$\begin{aligned}\theta^{-1/2} &\le C_{11}\int_0^1 \exp\left[-\int_{-T}^{-1} w_0\,dt\right]\exp\left[\int_{-T}^{-1} w_0 - w_r\,dt\right]dr\\ &\le C_{11}C_{12}\exp\left[-\int_{-T}^{-1} w_0\,dt\right].\end{aligned}$$

This proves (3.12).

Step 4. We now prove that there are positive constants $\approx\atop{C}$ and β such that

$$\exp\left[-\int_{-T}^{-1} u_0\,dt\right] \le \left(\frac{\overset{\approx}{C}}{k_+(v_0)}\right)^{\beta}\left(\exp\left[\int_{-T}^{-1} w_0\,dt\right]\right)^{\beta}, \tag{3.17}$$

provided $k_+(v_0) \ne 0$.

Suppose $k_+(v_0) \ne 0$ (i.e., $v_0 \notin \Gamma$). Since u_0 and u_- are, respectively, the unstable and stable Riccati solutions along γ_0, it follows from Lemma 3.6 (with $\beta = C$) that

$$-u_0(t) \le \beta u_-(t) \text{ for all } t \in \mathbf{R}.$$

Thus

$$\exp\left[-\int_{-T}^{-1} u_0\,dt\right] \le \left(\exp\left[\int_{-T}^{-1} u_-\,dt\right]\right)^{\beta}. \tag{3.18}$$

We now find an upper bound for the ratio

$$\frac{\exp\left[\int_{-T}^{-1} u_-\,dt\right]}{\exp\left[\int_{-T}^{-1} w_0\,dt\right]} = \exp\left[\int_{-T}^{-1} z\,dt\right], \tag{3.19}$$

where $z(t) = u_-(t) - w_0(t)$, which is positive for $t < 0$. By Lemma 3.1(ii), with equality of the two curvature functions and $A = t, B = -1$, we have

$$\exp\left[\int_{-T}^{-1} z\,dt\right] = \exp\left[z(-1)\int_{-T}^{-1}\exp\left[\int_t^{-1} u_-(\tau) + w_0(\tau)d\tau\right]dt\right] \tag{3.20}$$

Fix $\tau \leq -1$. Given the values $\tilde{A} = u_-(-1)$ and $\tilde{B} = w_0(-1), \tilde{B} < \tilde{A} < 0$, the largest possible value for $u_-(\tau) + w_0(\tau)$ would occur if $K_0(t)$ were identically 0 for $t \leq -1$ (by Lemma 3.1(iii)). It then follows from (3.20) that we can obtain an upper bound for

$$\exp\left[\int_{-T}^{-1} z \ dt\right] = \exp\left[\int_{-T}^{-1} u_- - w_0 \ dt\right] \tag{3.21}$$

by replacing u_- and w_0 in (3.21) by Riccati solutions for the curvature 0 case, with the values $\tilde{A}$ and $\tilde{B}$, respectively at $t = -1$. Thus

$$\begin{aligned}\exp\left[\int_{-T}^{-1} z\, dt\right] &\leq \exp\left[\int_{-T}^{-1} \frac{\tilde{A}}{(t+1)\tilde{A}+1} - \frac{\tilde{B}}{(t+1)\tilde{B}+1}\, dt\right] \\ &= \left|\frac{\tilde{B} + \frac{1}{(-T+1)}}{\tilde{A} + \frac{1}{(-T+1)}}\right| \leq \left|\frac{\tilde{B}}{\tilde{A}}\right| = \left|\frac{w_0(-1)}{u_-(-1)}\right|\end{aligned} \tag{3.22}$$

By the analog of Lemma 3.1(iv) for negative Riccati solutions,

$$u_-(-1) \leq \frac{u_-(0)}{-u_-(0)+1}. \tag{3.23}$$

Also, since K is bounded from below, both $|u_-(0)|$ and $|w_0(-1)|$ are bounded from above by a constant. Therefore, by (3.22) and (3.23), there exists a constant $C_{15} > 0$ such that

$$\exp\left[\int_{-T}^{-1} z\, dt\right] \leq \frac{C_{15}}{|u_-(0)|} = \frac{C_{15}}{|k_-(v_0)|}. \tag{3.24}$$

If we let $\tilde{\tilde{C}}= C_{15} \sup\{(k_+(v))/|k_-(v)|) : v \notin \Gamma\}$, which is finite by Lemma 3.6, then (3.17) follows from (3.18), (3.19) and (3.24).

Step 5. Completion of the proof. Combining the results of steps 3 and 4, we obtain

$$\exp\left[-\int_{-T}^{-1} u_0\, dt\right] \leq \left(\frac{C_{10}\ \tilde{\tilde{C}}}{k_+(v_0)}\right)^{\beta} \theta^{\beta/2}.$$

The same argument shows that this inequality also holds with u_0 and v_0 replaced by u_1 and v_1, respectively. These inequalities, together with (3.11), imply that

$$|k_+(v_1) - k_+(v_0)| \leq (C_7 + C_8)\sqrt{\theta} + C_9\Big(C_{10}\ \tilde{\tilde{C}}\ \Big)^{\beta} \min((k_+(v_0))^{-\beta}, (k_+(v_1))^{-\beta})\theta^{\beta/2}$$

If $k_+(v_0)$ and $k_+(v_1)$ are both less than or equal to $\theta^{1/4}$, then $|k_+(v_1) - k_+(v_0)| \leq 2\theta^{1/4}$. If at least one of $k_+(v_0)$ and $k_+(v_1)$ is greater than $\theta^{1/4}$, then

$$|k_+(v_1) - k_+(v_0)| \leq (C_7 + C_8)\sqrt{\theta} + C_9\Big(C_{10}\, \stackrel{\approx}{C}\,\Big)^{\beta}\theta^{\beta/4}$$

In both cases, (3.7) holds for $\alpha = \min(1/4, \beta/4)$ and some positive constant $\tilde{C}$.□

Although we do not have counterexamples to Theorems I and II if hypothesis 2) is omitted, we now give an example to show that the crucial Lemmas 3.4 and 3.6 fail to hold without hypothesis 2). This example satisfies hypothesis 1).

Example. Let S be a compact surface containing a closed right circular cylinder $\mathcal{C}$ with negative curvature on $S \setminus \mathcal{C}$. Let γ be a closed geodesic along the boundary of $\mathcal{C}$ and let S be constructed so that for some $\epsilon > 0$, the ϵ neighborhood of γ in S is a surface of revolution, and in Fermi coordinates (s, x) along γ, we have

$$K(s,x) = \begin{cases} -e^{-1/x}, & \text{for } 0 < x < \epsilon, -\infty < s < \infty \\ 0, & \text{for } -\epsilon < x \leq 0, -\infty < s < \infty. \end{cases}$$

It follows from a minor modification of Theorem 2.3 in [5] that there is a constant $C_1 > 0$ such that if v_ϕ is a vector with footpoint on γ which makes an angle ϕ with γ and which has a positive component in the $\partial/\partial x$ direction, then

$$-k_-(v_\phi) > C_1\phi|\ln\phi|. \tag{3.25}$$

Let $\sigma = \sigma_{v_\phi}$ and let $T_1 = T_1(\phi)$ be chosen so that $\{\sigma(t) : -T_1 \leq t \leq 0\}$ is a component of the intersection of σ and $\mathcal{C}$. Then there is a constant $C_2 > 0$ such that $T_1 > C_2/\phi$. Let u_+ be the unstable Riccati solution along σ. Since $K(\sigma(t)) = 0$ for $-T_1 \leq t \leq 0$,

$$k_+(v_\phi) = u_+(0) = \frac{u_+(-T_1)}{T_1 u_+(-T_1) + 1} \leq \frac{1}{T_1} < C_3\phi, \tag{3.26}$$

where $C_3 = 1/C_2$. By (3.25) and (3.26) we see that the second inequality in the conclusion of Lemma 3.6 does not hold for any constant C.

We now show that the same example also fails to satisfy the conclusion of Lemma 3.4. Let $0 < x_0 < \epsilon$, let $0 < \phi < \pi/2$, and let v_ϕ and $\sigma = \sigma_{v_\phi}$ be as above. Let $T_2 = \sup\{T > 0 : \ \mathrm{dist}(\sigma(t), \gamma)) \le x_0$ for $0 \le t \le T\}$. By comparison with the curvature 0 case (using a particular case of Lemma 4.2 from §4), we have $T_2 \le x_0/\sin\phi \le 2x_0/\phi$. Since $u_+' = -u_+^2 - K$, we obtain

$$\begin{aligned} u_+(T_2) &\le u_+(0) + \int_0^{T_2} -K(\sigma(t))\, dt \\ &\le u_+(0) + T_2 e^{-1/x_0} \\ &\le C_3\phi + \frac{2x_0 e^{-1/x_0}}{\phi}. \end{aligned}$$

Let $\phi = \sqrt{x_0}e^{-1/(2x_0)}$, let $w = \sigma'(T_2)$, and let p be the footpoint of w. Then

$$k_+(w) = u_+(T_2) \le \sqrt{x_0}(C_3 + 2)e^{-\frac{1}{2x_0}}, \tag{3.27}$$

while $\sqrt{K(p)} = e^{-1/(2x_0)}$. Since the coefficient $\sqrt{x_0}(C_3+2)$ in (3.27) can be made arbitrarily small, the second inequality in the conclusion of Lemma 3.4 does not hold for any constant $C > 0$.

4 Lower Bounds on Curvatures of Horocycles

In this section we establish the lower bound given in Lemma 3.3 on the curvatures, $k_+(v)$, of the unstable horocycles. We consider the curvatures of these horocycles at vectors v that are close to $T^1\gamma$, where γ is a closed geodesic along which the curvature K of S vanishes identically. As in hypothesis 2) of Theorems I and II, we will assume that there is a point q on γ such that K does not vanish to infinite order at q. Assume that q is chosen so that the order to which K vanishes is minimized (over points of γ) at q. The geodesic σ_v determined by v wraps around S many times very close to γ. (See Figure 4.1.) In those time intervals when σ_v passes close to q, K is bounded from above by a negative function of the distance from σ_v to γ. On the complements of these intervals we only assume that K is nonpositive. The following lemma gives an upper bound for solutions to the Riccati equation which will apply in this situation.

Lemma 4.1 *(Estimate for intervals of alternating curvatures.) Let A, B, K_1 be positive constants. Let n be a positive integer and let $I_0, I_0', I_1, I_1', \ldots, I_n, I_n'$*

be closed intervals (arranged in the natural order from left to right) that partition $[-T, 0]$, where $T = \sum_{i=0}^{n}(|I_i| + |I_i'|)$. Assume:

1) All intervals I_i, I_i' are non–empty, except possibly I_0 and I_n'.

2) $|I_i| \geq A$ for $i = 1, \ldots, n-1$, and $|I_n| \geq A$ if $|I_n'| > 0$.

3) In the case $n = 1$, $\quad |I_i| \geq A$ for $i = 0$ or 1.

4) $|I_i'| \leq B$ for $i = 0, \ldots, n$.

Let K_0 be a constant such that $0 < K_0 < K_1$ and let u be a solution to the Riccati equation $u' = -u^2 - K$, where $K \leq -K_0$ on I_i, and $K \leq 0$ on I_i', for $i = 0, \ldots, n$.

Then there exists a positive constant C which depends only on A, B and K_1 such that:

i) If $T \geq K_0^{-1/2}$ and $u(-T) \geq 0$, then $u(0) \geq C\sqrt{K_0}$ and

ii) If $u(-T) \geq \sqrt{K_0}$, then $u(0) \geq C\sqrt{K_0}$.

(The assertion ii) is still true if we delete the assumption 3).)

Proof. Let $y_1(t) = 1/t$ for $t > 0$ and let $y_2(t) = \sqrt{K_0}\tanh(\sqrt{K_0}t)$ for $t \geq 0$. Any solution to the Riccati equation with $K \equiv 0$ and positive initial value follows a horizontal translate of the graph of y_1. Similarly, any solution to the Riccati equation with $K \equiv -K_0$ and nonnegative initial value less than $\sqrt{K_0}$ follows a horizontal translate of the graph of y_2. (If the initial value were greater than or equal to $\sqrt{K_0}$, the solution would remain greater than or equal to $\sqrt{K_0}$ for all future time.)

Note that $y_1'(t) < 0$ and $|y_1'(t)|$ is decreasing for $t > 0$, while $y_2'(t) > 0$ and $y_2'(t)$ is decreasing for $t > 0$. We now compare $|y_1'|$ with y_2'. We have $|y_1'| = t^{-2} = y_1^2$ and $y_2' = K_0(1 - \tanh^2(\sqrt{K_0}t)) = K_0 - y_2^2$. Let C_1 be a constant satisfying $0 < C_1 < 1$ that will be specified below. If $y_1(t) \leq C_1\sqrt{K_0}$ for some $t > 0$, then $|y_1'(t)| \leq C_1^2 K_0$; and if $y_2(t) \leq C_1\sqrt{K_0}$ for some $t \geq 0$, then $y_2'(t) \geq K_0(1 - C_1^2)$.

Let $\mathcal{A} = \sum_{i=0}^{n}|I_i|$ and $\mathcal{B} = \sum_{i=0}^{n}|I_i'|$. It follows from assumptions 1) - 4) that there is a positive constant C_2, depending only on A and B, such that $\mathcal{A} > C_2 T$. Let $C_1 = \min(\sqrt{C_2/2}, \sqrt{A/(A+B)})$.

Let u be a solution to the Riccati equation $u' = -u^2 - K$, where $K \leq -K_0$ on I_i, and $K \leq 0$ on I'_i, for $i = 0, \ldots, n$. Let w be the piecewise smooth solution to the Riccati equation $w' = -w^2 - \tilde{K}$ with $w(-T) = u(-T), \tilde{K} \equiv -K_0$ on int (I_i) and $\tilde{K} \equiv 0$ on int (I'_i), for $i = 0, \ldots, n$. By the Comparison Lemma 3.1(iii), if $u(-T) \geq 0$, then $u(0) \geq w(0)$.

Proof of i). Assume $T \geq K_0^{-1/2}$ and $u(-T) \geq 0$.

Case 1. Suppose that $w(t) \leq C_1\sqrt{K_0}$ for all $t \in [-T, 0]$. By the above estimates on y'_1 and y'_2, we have

$$
\begin{aligned}
w(0) &\geq K_0[\mathcal{A}(1 - C_1^2) - \mathcal{B}C_1^2] + w(-T) \\
&\geq K_0 T[C_2(1 - C_1^2) - (1 - C_2)C_1^2] \\
&\geq \sqrt{K_0}(C_2 - C_1^2) \\
&\geq (C_2/2)\sqrt{K_0}.
\end{aligned}
$$

Case 2. Suppose $w(t) \geq C_1\sqrt{K_0}$ for some $t \in [-T, 0]$. Let t_0 be the largest $t \in [-T, 0]$ such that this inequality holds. Then there is an $i \in \{0, 1, \ldots, n\}$ such that $t_0 \in I'_i = [a'_i, b'_i]$. Moreover, by Lemma 3.1,

$$
w(b'_i) \geq \frac{w(t_0)}{(b'_i - t_0)w(t_0) + 1} \geq \frac{C_1\sqrt{K_0}}{BC_1\sqrt{K_1} + 1}.
$$

If $i = n$, then $b'_i = 0$ and $w(0) = w(b'_i)$. If $i < n$, then we consider j, $i < j \leq n$, such that $|I_j| \geq A$. Let $I_j = [a_j, b_j]$ and $I'_j = [a'_j, b'_j]$, where $b_j = a'_j$. By the same type of estimate as in Case 1,

$$
\begin{aligned}
w(b'_j) &\geq K_0[A(1 - C_1^2) - BC_1^2] + w(a_j) \\
&\geq w(a_j),
\end{aligned}
$$

by the choice of C_1. Repeated application of this inequality gives $w(0) = w(b'_n) \geq w(b'_i)$, if $|I_n| \geq A$, and it gives $w(b'_{n-1}) \geq w(b'_i)$, if $|I_n| < A$. But in the latter case, $|I'_n| = 0$ and $w(0) = w(b_n) \geq w(b'_{n-1}) \geq w(b'_i)$.

Thus, if we let $C_3 = \min(C_2/2, C_1/[BC_1\sqrt{K_1}+1])$ and combine the results of Cases 1 and 2, we have

$$
u(0) \geq w(0) \geq C_3\sqrt{K_0}.
$$

Proof of ii). Assume $u(-T) \geq \sqrt{K_0}$. Since $C_1 < 1, w(-T) = u(-T) \geq C_1\sqrt{K_0}$. Then by the argument in Case 2 of the proof of i), $u(0) \geq w(0) \geq C_4\sqrt{K_0}$, where $C_4 = C_1/(BC_1\sqrt{K_1} + 1)$.□

The next lemma is due to Keith Burns. This lemma will enable us to estimate the lengths of time intervals that geodesics σ_v in S (or in $\tilde{S}$) spend in certain regions, by making the analogous estimates for a surface of constant negative curvature. Note that there is no assumption on the sign of the curvatures.

Lemma 4.2 *Let S_0 and S_1 be complete surfaces with curvatures K_0 and K_1, respectively. For $i \in \{0,1\}$, let γ_i be a unit-speed geodesic in S_i, and let (s_i, x_i) be Fermi coordinates along γ_i. Let I_0, I_1, and J be intervals in $\mathbf{R}$, with $0 \in J$, such that, for $i \in \{0,1\}$, the map*

$$p \mapsto (s_i(p), x_i(p))$$

is a diffeomorphism from a neighborhood $\mathcal{N}_i$ of $\gamma_i(I_i)$ onto $I_i \times J$.

Assume the following condition on the curvatures K_0 and K_1:

1) For all $p_0 \in \mathcal{N}_0$ and $p_1 \in \mathcal{N}_1$,

$$x_0(p_0) = x_1(p_1) \Rightarrow K_0(p_0) \geq K_1(p_1),$$

For $i \in \{0,1\}$, let $\sigma_i : [0, T_i] \to \mathcal{N}_i$ be a unit-speed geodesic segment. For $t \in [0, T_i]$, let $\phi_i(t)$ be the signed angle between $\sigma_i'(t)$ and the curve $x_i = x_i(\sigma_i(t))$, chosen to lie in the interval $[-\pi/2, \pi/2]$, and consistent with $\sphericalangle(\partial/\partial s, \partial/\partial x) = \pi/2$. Let $d_i(t) = x_i(\sigma_i(t))$ be the signed distance from $\sigma_i(t)$ to γ_i. Suppose further that

2) For $i \in \{0,1\}$, and for $t \in (0, T_i)$, $d_i(t) \geq 0$, and $\phi_i(t) \neq 0$,

3) $d_0(0) = d_1(0)$,

4) $\phi_0(0) \leq \phi_1(0)$.

Then, for all $t_0 \in [0, T_0]$ and $t_1 \in [0, T_1]$ satisfying $d_0(t_0) = d_1(t_1)$, we have

i) $\phi_0(t_0) \leq \phi_1(t_1)$, and

ii) If $\phi_0(t_0) > 0$, then $t_0 \geq t_1$, and if $\phi_0(t_0) < 0$, then $t_0 \leq t_1$.

Proof. It is a straightforward exercise to verify that the conclusions of the lemma hold if either ϕ_0 or ϕ_1 takes one of the values $\pm\pi/2$, so we may assume that $\phi_i[0,T_i] \subseteq (-\pi/2, \pi/2)$, for $i \in \{0,1\}$.

If ϕ_0 and ϕ_1 have opposite sign on the intervals $(0,T_0)$ and $(0,T_1)$, respectively, then d_0 and d_1 are equal only at $t=0$, and the lemma is trivially true. Suppose that ϕ_0 and ϕ_1 have the same sign on $(0,T_0)$ and $(0,T_1)$, respectively, and that $\phi_0(0) = \phi_1(0) = 0$. Let $\sigma_{1,r}$ be a continuous variation of geodesics in S_1 with $\sigma_{1,r}(0) = \sigma_1(0)$ and $\sphericalangle(\sigma'_{1,r}(0), \sigma'_1(0)) = r$. Suppose that the conclusions of the lemma hold for the pair of geodesics σ_0 and $\sigma_{1,r}$ for all $r \neq 0$ sufficiently small. Since the geodesics $\sigma_{1,r}$ converge uniformly on compact time intervals to σ_1, as $|r| \to 0$, it follows that the conclusions of the lemma will also hold for the pair σ_0 and σ_1.

It suffices, then, to prove the lemma in the case $\phi_1(0) \neq 0$. By considering a variation of geodesics in S_0 that contains σ_0, we see that it suffices to prove the lemma in the case $\phi_0(0) \neq 0$. Assume that $\phi_i[0,T_i) \subseteq (0,\pi/2)$, for $i \in \{0,1\}$; the case where the ϕ_i take values in $(-\pi/2, 0)$ is treated similarly.

For $i \in \{0,1\}$, let $\mathcal{S}_i$ and $\mathcal{X}_i$ be the unit-speed vector fields in the directions of $\partial/\partial s_i$ and $\partial/\partial x_i$, respectively. For $p_i \in \mathcal{N}_i$, let $A_i(p_i)$ be the geodesic curvature at the point p_i of the curve $x_i = x_i(p_i)$ in $\mathcal{N}_i$, so that

$$\nabla_{\mathcal{S}_i}\mathcal{X}_i = A_i\mathcal{S}_i.$$

Observe that if $\beta_i : J \to \mathcal{N}_i$ is the geodesic segment tangent to $\partial/\partial x_i$ with $x_i(\beta_i(0)) = 0$, then for $x \geq 0$, the function $w_i(x) = A_i(\beta_i(x))$ is the solution to the Riccati equation

$$w_i'(x) = -w_i(x)^2 - K_i(\beta_i(x)),$$

with initial condition $w_i(0) = 0$. Lemma 3.1, part (iii), combined with assumption 1), gives that

$$A_0(\beta_0(x)) \leq A_1(\beta_1(x)),$$

for all nonnegative $x \in J$. Thus, for $p_0 \in \mathcal{N}_0$, $p_1 \in \mathcal{N}_1$,

$$x_0(p_0) = x_1(p_1) \geq 0 \quad \Rightarrow \quad A_0(p_0) \leq A_1(p_1). \tag{4.1}$$

We now calculate $d_i'(t)$ and $\phi_i'(t)$, for $t \in [0,T_i)$. In the following calculation, we drop the subscript i for ease of reading. Notice that

$$\sigma'(t) = \cos\phi(t)\mathcal{S}(\sigma(t)) + \sin\phi(t)\mathcal{X}(\sigma(t)),$$

and so

$$\begin{aligned}
\phi' \cos\phi &= \frac{d}{dt}\sin\phi \\
&= \frac{d}{dt}\langle \mathcal{X}(\sigma), \sigma' \rangle \\
&= \langle \nabla_{\sigma'} \mathcal{X}(\sigma), \sigma' \rangle \\
&= \langle \nabla_{\cos\phi \mathcal{S}(\sigma) + \sin\phi \mathcal{X}(\sigma)} \mathcal{X}(\sigma), \sigma' \rangle \\
&= \cos\phi \langle \nabla_{\mathcal{S}(\sigma)} \mathcal{X}(\sigma), \sigma' \rangle \ + \ \sin\phi \langle \nabla_{\mathcal{X}(\sigma)} \mathcal{X}(\sigma), \sigma' \rangle \\
&= \cos\phi \langle \nabla_{\mathcal{S}(\sigma)} \mathcal{X}(\sigma), \sigma' \rangle \\
&= \cos\phi \langle A(\sigma) \mathcal{S}(\sigma), \sigma' \rangle \\
&= \cos\phi A(\sigma) \cos\phi,
\end{aligned}$$

where the equations are evaluated at t. Replacing the subscript i, we divide both sides of this equation by $\cos\phi_i(t)$, for $t \in [0, T_i)$, to obtain:

$$\phi_i'(t) = A_i(\sigma_i(t)) \cos\phi_i(t). \tag{4.2}$$

Also,

$$\begin{aligned}
d_i'(t) &= \langle \sigma_i'(t), \mathcal{X}_i(\sigma_i(t)) \rangle \\
&= \sin\phi_i(t).
\end{aligned} \tag{4.3}$$

Since $\phi_i(t) \in (0, \pi/2)$ for $t \in [0, T_i)$, it follows from equation (4.3) that d_i is monotone increasing on $[0, T_i]$. Let $a = d_0(0) = d_1(0)$ and $b = \min\{d_0(T_0), d_1(T_1)\}$. For $x \in [a, b]$, let $t_i(x) = d_i^{-1}(x)$, and let $\Phi_i(x) = \phi_i(t_i(x))$. By equations (4.2) and (4.3) it follows that Φ_i and t_i are differentiable on $[a, b)$, and for $x \in [a, b)$,

$$\begin{aligned}
t_i'(x) &= (d_i'(d_i^{-1}(x)))^{-1} \\
&= \csc\Phi_i(x),
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\Phi_i'(x) &= \phi_i'(t_i(x)) t_i'(x) \\
&= A_i(p_i(x)) \cos\Phi_i(x) (\csc\Phi_i(x)) \\
&= A_i(p_i(x)) \cot\Phi_i(x),
\end{aligned} \tag{4.5}$$

where $p_i(x) = \sigma_i(t_i(x))$. (We needed that $\phi_i'(0) \neq 0$ to get differentiability of these functions at a.) Clearly $d_0(p_0(x)) = x = d_1(p_1(x))$ and $\cot\Phi_i(x)$ is positive, for $x \in [a,b)$ and $i \in \{0,1\}$. Combining (4.1) with (4.5), we thereby obtain:

$$\Phi_0(x) = \Phi_1(x) \quad \Rightarrow \quad A_0(p_0(x)) \cot \Phi_0(x) \leq A_1(p_1(x)) \cot \Phi_1(x),$$

for all $x \in [a,b)$. By assumption 4) of the lemma, $\Phi_0(a) \leq \Phi_1(a)$, and so a standard differential inequality (Corollary 4.2 in [7], p. 27) implies that $\Phi_0(x) \leq \Phi_1(x)$, for all $x \in [a,b]$. In other words, for all t_0, t_1 with $d_0(t_0) = d_1(t_1)$,

$$\phi_0(t_0) \leq \phi_1(t_1),$$

proving inequality i) of the lemma.

To prove inequality ii) (in the present case where ϕ_0 is positive), observe that, by inequality i), and equation (4.4),

$$t_0'(x) = \csc \Phi_0(x) \geq \csc \Phi_1(x) = t_1'(x),$$

for all $x \in [a,b)$. Since $t_0(a) = t_1(a) = 0$, it follows that $t_0(x) \geq t_1(x)$, for all $x \in [a,b]$. The case where $\phi_i(0) < 0$ is dealt with similarly. □

The following lemma, together with its analog for stable horocycles, provides the last step in the proof of Lemma 3.3, thereby completing the proofs of Theorems I and II.

Lemma 4.3 *Under the hypothesis of Lemma 3.3, there exists a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ such that for any $v \in \mathcal{U}$ with footpoint having second Fermi coordinate $x = a$, the curvature $k_+(v)$ of the unstable horocycle satisfies*

$$k_+(v) \geq C \max(|a|^{m/2}, |\phi|^{m/(m+2)}),$$

where $\phi = \sphericalangle(v, x = a)$.

Proof. Let s_0 be the length of γ, and let $P : \mathbf{R} \to [0, s_0]$ be the covering map with $P(0) = 0$. For a set $A \subseteq [0, s_0]$, let $\tilde{A}$ denote $P^{-1}(A)$. Since K vanishes to order $m-1$ on $\tilde{\gamma}$, but does not vanish to order m at $\tilde{\gamma}(0)$, there exist positive constants C_1, C_2 and ϵ and an interval $L = [0, s_1]$ for some $s_1 \in (0, s_0)$ such that

$$-C_1 x^m \leq K(s,x), \text{ for } |x| < \epsilon, \text{ for all } s,$$

and

$$-C_1 x^m \leq K(s,x) \leq -C_2 x^m, \text{ for } |x| < \epsilon, \text{ for } s \in \tilde{L}.$$

Let L' be the closure of $[0, s_0] \setminus L$.

In this proof we modify our previous convention and let σ_v denote the maximal geodesic segment (possibly of infinite length) in $\tilde{S}$ with initial tangent vector v (with footpoint in the region where $|x| < \epsilon$) which remains in the region where $|x| < \epsilon$.

The first part of our proof is concerned with the choice of a neighborhood $\mathcal{U}$ such that for $v \in \mathcal{U}$, σ_v will assume all s values in one component of $\tilde{L}$, while taking x values in the interval $[\delta, 2\delta]$ for some suitably chosen $\delta > 0$. We can accomplish this by controlling the angles σ_v makes with the curves $x = c$. The central estimates needed to control these angles are summarized in (4.8) and (4.9). These inequalities will also be used later, in Case 1 and Case 2b. Once the neighborhood $\mathcal{U}$ is chosen, the argument divides into two cases, according to whether the angle v makes with $x = a$ is less than or equal to a constant multiple of $a^{(m+2)/2}$ (Case 1) or greater than or equal to a constant multiple of $a^{(m+2)/2}$ (Case 2). In Case 1, we rely on σ_v spending a long time in the region where the x coordinate has magnitude on the order of a. In Case 2, we use the facts that σ_v attains all s values in a component of $\tilde{L}$ during the time σ_v is in the region where $|x| > \delta$ and afterwards σ_v doesn't spend too much time in the region where the curvature is very close to 0.

Let $\ell = \max\{||(\partial/\partial s)_p|| : |x(p)| \leq \epsilon\}$. If $0 < \delta < \min(\epsilon/2, \ell s_0 \pi)$, then a geodesic segment σ in the region of $\tilde{S}$ where $\delta \leq x \leq 2\delta$ that starts at $x = \delta$, ends at $x = 2\delta$, and satisfies

$$\sphericalangle(\sigma', x = c) \leq \delta/(4\ell s_0), \quad \text{for } \delta \leq c \leq 2\delta, \tag{4.6}$$

must assume all s values in some interval of length at least $2s_0$. (Reason: If we let $\nu(c) = \sphericalangle(\sigma', x = c)$, then $0 < \nu(c) < \pi/4$ and $|(ds/dx)_{x=c}| = (\cot \nu(c))/||\partial/\partial s|| \geq (2||\partial/\partial s|| \sin \nu(c))^{-1} \geq (2\ell\nu(c))^{-1} \geq 2s_0/\delta$.) In particular, the interval of s values so obtained would include at least one component of $\tilde{L}$. We now show that there is a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ such that if $\delta > 0$ is sufficiently small and $v \in \mathcal{U}$, then any geodesic segment contained in σ_v which goes from $x = \delta$ to $x = 2\delta$ satisfies (4.6). By the Gauss-Bonnet Theorem, the angle such a geodesic segment makes with $x = c$ is monotone increasing in c for $\delta \leq c \leq 2\delta$. Thus it suffices to check that the angle with $x = 2\delta$ is less than or equal to $\delta/(4\ell s_0)$.

Fix $0 < b < \epsilon$ and let $K_1 = K_1(b) = -C_1 b^m$. We will apply Lemma 4.2 in the case $S_0 = \tilde{S}$ and S_1 is a simply connected surface of constant curvature K_1. Let γ_1 be a geodesic in S_1 and let (s_1, x_1) be Fermi coordinates about γ_1.

Suppose σ $[\sigma_1]$ is a geodesic segment on $\tilde{S}$ $[S_1]$ which lies in the region where $0 \leq x < \epsilon$ $[0 \leq x_1 < \epsilon]$ and which makes angle θ, $0 < \theta < \pi/2$, with $\tilde{\gamma}$ $[\gamma_1]$. Let θ_0 $[\theta_1]$ be the angle σ $[\sigma_1]$ makes with $x = b$ $[x_1 = b]$. By Lemma 4.2, we have $\theta_0 \leq \theta_1$. A formula from hyperbolic trigonometry gives

$$\cos\,\theta = \cosh\,\left(b\sqrt{-K_1(b)}\right)\cos\,\theta_1. \tag{4.7}$$

We may assume that ϵ was chosen sufficiently small so that there exists an $\eta > 0$ such that (4.7) implies that $\theta_1 < \pi/4$, whenever $0 < \theta < \eta$ (for any choice of b with $0 < b < \epsilon$).

We will use the estimates $\cos\, z \geq 1 - z^2/2$ for $0 \leq z \leq \pi/2$ and $\cos\, z \leq 1 - z^2/4$ for $0 \leq z \leq \pi/4$. Also, there is a constant $C_3 > 0$ such that if $0 \leq z \leq \epsilon^{(m+2)/2}\sqrt{C_1}$, then $\cosh\, z \leq 1 + C_3 z^2$. (Note that $\epsilon^{(m+2)/2}\sqrt{C_1} > b\sqrt{-K_1(b)}$, if $0 < b < \epsilon$.) Then, for $0 < \theta < \eta$, (4.7) implies

$$1 - \frac{\theta^2}{2} \leq \left(1 + C_3 b^2 |K_1|\right)\left(1 - \frac{\theta_1^2}{4}\right).$$

Thus

$$\frac{\theta_1^2}{4} \leq \frac{\theta^2}{2} + C_3 b^2 |K_1|,$$

and we obtain

$$\theta_0 \leq \theta_1 \leq \sqrt{2}\theta + 2\sqrt{C_1 C_3}\, b^{(m+2)/2}. \tag{4.8}$$

It also follows from Lemma 4.2 that if $\tilde{\sigma}$ is a geodesic segment in $\tilde{S}$ which is tangent to $x = c$ and goes from $x = c$ to $x = b$, for some $0 < c < b$, and makes angle θ_0 with $x = b$, then

$$\theta_0 \leq 2\sqrt{C_1 C_3}\, b^{(m+2)/2}. \tag{4.9}$$

This can be seen by comparing $\tilde{\sigma}$ with geodesic segments in S_1 making angle θ with γ_1 and letting θ go to 0.

Fix δ such that $0 < \delta < \min(\epsilon/2, \ell s_0 \pi)$ and $2\sqrt{C_1 C_3}(2\delta)^{(m+2)/2} < \delta/(8\ell s_0)$. This is possible, since m is at least 2. If $\theta < \min(\eta, \delta/(8\sqrt{2}\ell))$ and $b = 2\delta$, then $\theta_0 \leq \theta_1 \leq \delta/(4\ell s_0)$. Choose a neighborhood $\mathcal{U}$ of $T^1\tilde{\gamma}$ in $T^1\tilde{S}$ such that the footpoints of vectors in $\mathcal{U}$ are in the region where $-\delta < x < \delta$ and for each $v \in \mathcal{U}$, σ_v intersects γ with angle of absolute value less than $\min(\eta, \delta/(8\sqrt{2}\ell))$ or does not intersect γ at all. If σ is a geodesic segment contained in σ_v, for $v \in \mathcal{U}$, that goes from $x = \delta$ to $x = 2\delta$, then (4.6) is satisfied, and, by the above discussion σ takes all s values in at least one component of $\tilde{L}$. This property will be used in Case 2a below. For Case 1 we will only need the fact that for $v \in \mathcal{U}$, σ_v makes an angle of absolute value at most $\pi/4$ with any curve $x = c$, where $|c| < \epsilon$.

For the rest of the proof fix a choice of $v \in \mathcal{U}$, let $x = a$ be the second Fermi coordinate of the footpoint of v, and let $\phi = \sphericalangle(v, x = a)$. We may assume that $a \geq 0$.

Case 1. Suppose $|\phi| \leq C_0 a^{(m+2)/2}$ for some $C_0 > 0$. (In Case 1, C_0 may be *any* positive constant.) If $a = 0$, then $\phi = 0$ and $k_+(v) = 0$; so assume $a > 0$.

Case 1a. Suppose $\phi \leq 0$. Then σ_v intersects $\tilde{\gamma}$ with angle θ satisfying $|\theta| \leq |\phi|$ or does not intersect $\tilde{\gamma}$ at all. (See Figure 4.2.) In either case, it follows from (4.8) that

$$\begin{aligned} |\sphericalangle(\sigma_v, x = 2a)| &\leq \sqrt{2}|\phi| + 2\sqrt{C_1 C_3}(2a)^{(m+2)/2} \\ &\leq C_4 a^{(m+2)/2} \end{aligned}$$

for some constant $C_4 > 0$. Therefore σ_v is in the region $a \leq x \leq 2a$ during a time interval $[-T, 0]$ where $T \geq (2a - a)/(\sin(C_4 a^{(m+2)/2})) \geq C_4^{-1} a^{-m/2}$.

Case 1b. Suppose $\phi > 0$ and σ_v intersects $x = a/2$. (See Figure 4.3.) Then the angle that σ_v makes with $x = a/2$ is less than or equal to ϕ, and it follows from Case 1a, applied at $x = a/2$ instead of $x = a$, that there is a constant $C_5 > 0$ such that σ_v is in the region $a/2 \leq x \leq a$ during a time interval $[-T, 0]$ with $T \geq C_5 a^{-m/2}$.

Case 1c. Suppose $\phi > 0$ and σ_v does not intersect $x = a/2$. (See Figure 4.4.) Then σ_v is tangent to $x = a_1$ for some a_1 satisfying $a/2 \leq a_1 \leq a$. It follows from Case 1a applied at $x = a_1$ with angle 0 that σ_v is in the region $a_1 \leq x \leq 2a_1$ during a time interval $[-T, 0]$ with $T \geq C_6 a^{-m/2}$ for some constant $C_6 > 0$.

Conclusion of Case 1. In all three subcases there is a constant $C_7 > 0$ and an interval $[-T, 0]$, where $T \geq C_7 a^{-m/2}$, such that $\sigma_v(t)$ is in the region where $a/2 \leq x \leq 2a$ for $t \in [-T, 0]$.

Let $I_0', I_0, I_1', I_1, \ldots, I_n', I_n$ be a partition of $[-T, 0]$ such that for $t \in I_i$ [$t \in I_i'$] the s coordinate of σ_v is in $\tilde{L}$ [$\tilde{L'}$]. Then for $t \in I_i$, $K(\sigma_v(t) \leq -C_2(a/2)^m$. Since $|\sphericalangle(\sigma_v, x = \text{constant})| \leq \pi/4$ for $v \in \mathcal{U}$, there exist positive constants A, B such that the hypothesis of Lemma 4.1 holds with K_0 a constant multiple of a^m. From part i) of this lemma we conclude that there is a constant $C_8 > 0$ such that $k_+(v) \geq C_8 a^{m/2}$. Since $\phi \leq C_0|a|^{(m+2)/2}$, it follows that there exists $C > 0$ such that $k_+(v) \geq C \max(|a|^{m/2}, |\phi|^{m/(m+2)})$.

Case 2. $|\phi| \geq C_0 a^{(m+2)/2}$. (Here C_0 is a sufficiently large positive constant, as described in Case 2b. This constant depends only on C_1 and C_3.) For starters require that $C_0 > 1$. Since ϕ cannot be 0 except in the trivial case, when a is also 0, we will assume $\phi \neq 0$.

Case 2a. Suppose $\phi < 0$. Let $T_0 > 0$ be such that σ_v crosses $x = \delta$ at time $-T_0$. By the choice of $\mathcal{U}$, there exist T_1, $T_2 > 0$ such that $T_0 \leq T_1 < T_2$ and during the time interval $[-T_2, -T_1]$ the s values taken by σ_v lie in a component of $\tilde{L}$ and cover this component. (See Figure 4.5.) Assume that T_1 and T_2 are chosen as small as possible, while satisying these requirements. The interval of s values assumed by σ_v in the time interval $[-T_1, -T_0]$ has length less than s_0. Since $|\sphericalangle(\sigma_v, x = c)| \leq \pi/4$ for $\delta \leq c \leq 2\delta$, this implies that $|T_1 - T_0| \leq \ell/\sqrt{2}$. Also, we have $|T_2 - T_1| \geq |L|$. (This inequality depends on the fact that $||\partial/\partial s|| \geq 1$, which follows from the nonpositive curvature assumption.) If $t \in [-T_2, -T_1]$, then $K(\sigma_v(t)) \leq -C_2\delta^m$. It follows that there is a positive constant C_9 (depending on δ, but not on ϕ) such that the unstable Riccati solution along σ_v is at least C_9 at $t = -T_0$. (This could be viewed as a simple case of Lemma 4.1.) By reducing the size of the neighborhood $\mathcal{U}$, if necessary, we may assume that ϕ satisfies $\sqrt{C_2}|\phi|^{m/(m+2)} < C_9$. Then the hypothesis of part (ii) of the Lemma 4.1 holds for the time that σ_v is in the region $|\phi|^{2/(m+2)} \leq x \leq \delta$ with $K_0 = C_2|\phi|^{2m/(m+2)}$. From this lemma we conclude that the value of the unstable Riccati solution along σ_v is at least $C_{10}|\phi|^{m/(m+2)}$, for some $C_{10} > 0$, at the time σ_v crosses $x = |\phi|^{2/(m+2)}$. Since σ_v makes angle of absolute value at least $|\phi|$ with $x = c$ for $a \leq c \leq |\phi|^{2/(m+2)}$, the length of time σ_v is in the region where $a \leq x \leq |\phi|^{2/(m+2)}$ is less than or equal to

$$\frac{|\phi|^{2/(m+2)} - a}{\sin|\phi|} \leq \frac{|\phi|^{2/(m+2)}(1 - C_0^{-2/(m+2)})}{|\phi|/2} \leq C_{11}|\phi|^{-m/(m+2)}$$

for some $C_{11} > 0$. Then, by Lemma 3.1, $k_+(v)$, the value of the unstable Riccati solution along σ_v at time 0, satisfies

$$k_+(v) \geq \frac{C_{10}|\phi|^{m/(m+2)}}{C_{11}|\phi|^{-m/(m+2)}C_{10}|\phi|^{m/(m+2)}+1} \geq C_{12}|\phi|^{m/(m+2)}$$

for some $C_{12} > 0$.

Case 2b. Suppose $\phi > 0$. It follows from (4.8) and (4.9) that if $\phi \geq \sqrt{2C_1C_3}a^{(m+2)/2}$, then σ_v crosses $\tilde{\gamma}$ with angle θ satisfying

$$\begin{aligned} \theta &\geq \frac{1}{\sqrt{2}}\Big[\phi - 2\sqrt{C_1C_3}a^{(m+2)/2}\Big] \\ &\geq \frac{\phi}{\sqrt{2}}\Big[1 - \frac{2\sqrt{C_1C_3}}{C_0}\Big] \end{aligned}$$

Choose C_0 sufficiently large so that $C_0 > 1$ and $2\sqrt{C_1C_3}/C_0 < 1 - 1/\sqrt{2}$. Then, if $\phi \geq C_0a^{(m+2)/2}$, σ_v crosses $\tilde{\gamma}$ with angle θ satisfying $\phi \geq \theta \geq \phi/2$. By Case 2a, the unstable Riccati solution u along σ_v satisfies $u(-T_3) \geq C_{12}\theta^{m/(m+2)} \geq C_{12}2^{-m/(m+2)}\phi^{m/(m+2)}$, where $T_3 \geq 0$ is such that σ_v crosses $\tilde{\gamma}$ at time $-T_3$. Also, we have $T_3 \leq a/\sin\theta \leq 2a/\theta \leq 4a/\phi$. Hence, by Lemma 3.1(iii),

$$k_+(v) = u(0) \geq \frac{C_{12}2^{-m/(m+2)}\phi}{(4a/\phi)(C_{12}2^{-m/(m+2)}\phi)+1} \geq C_{13}\phi,$$

for some $C_{13} > 0$.

Conclusion of Case 2. In both subcases $k_+(v) \geq C_{14}|\phi|$ for some $C_{14} > 0$. Since $|\phi| \geq C_0|a|^{(m+2)/2}$, this implies that there is a constant $C > 0$ such that $k_+(v) \geq C\max(|a|^{m/2}, |\phi|^{m/(m+2)})$, which completes the proof. □

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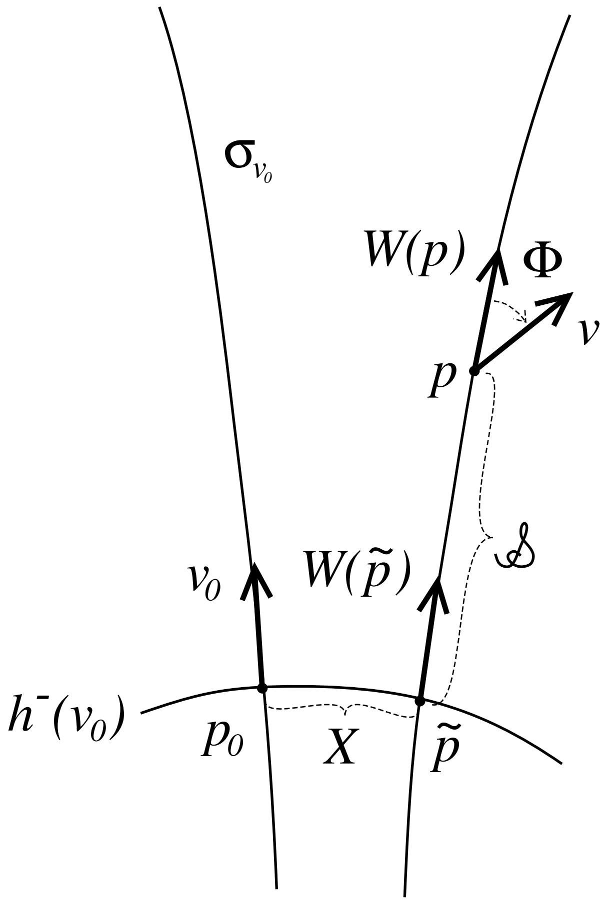

Figure 3.1

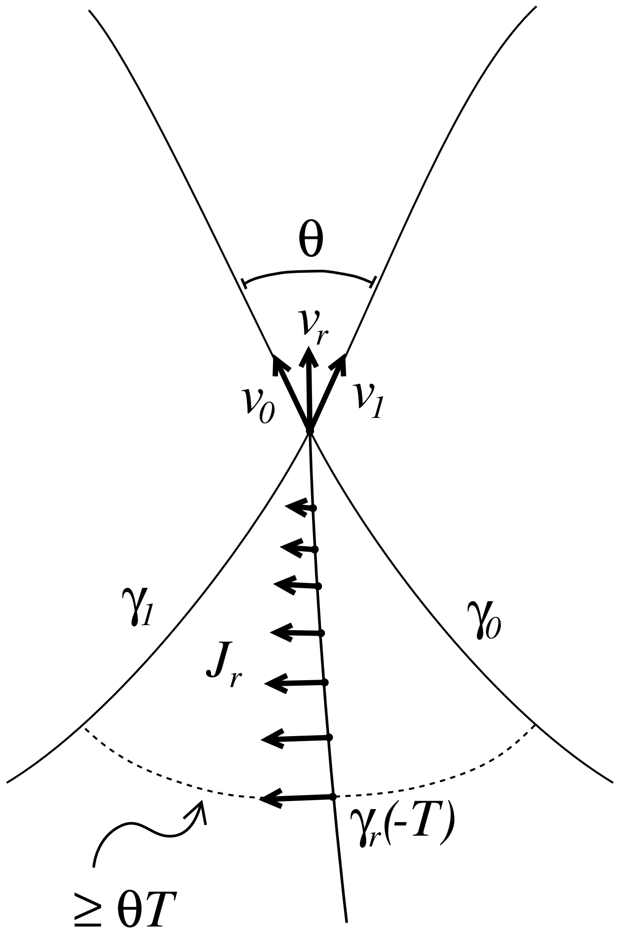

Figure 3.2

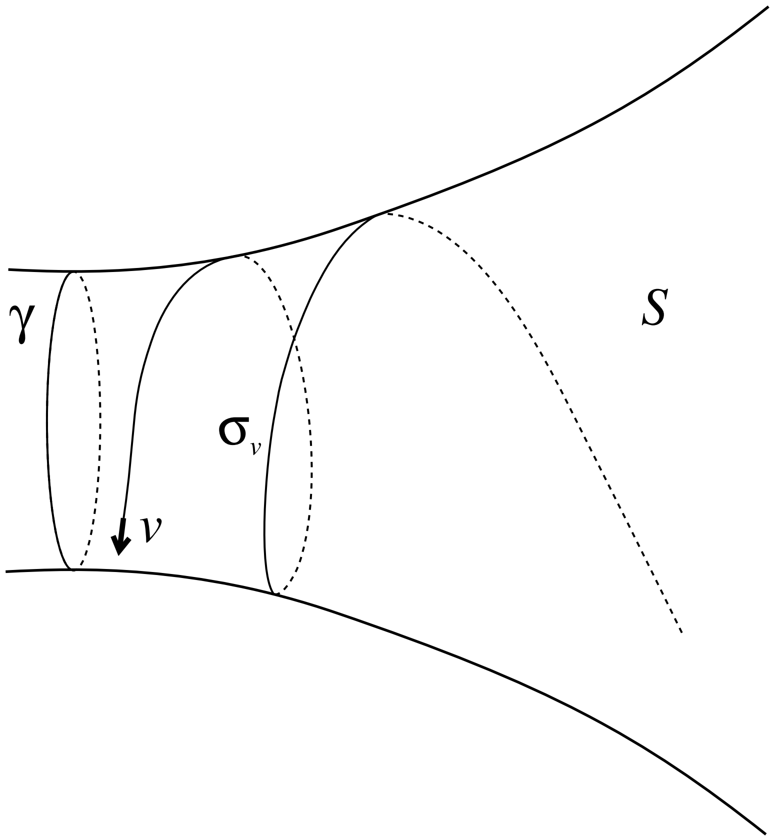

Figure 4.1

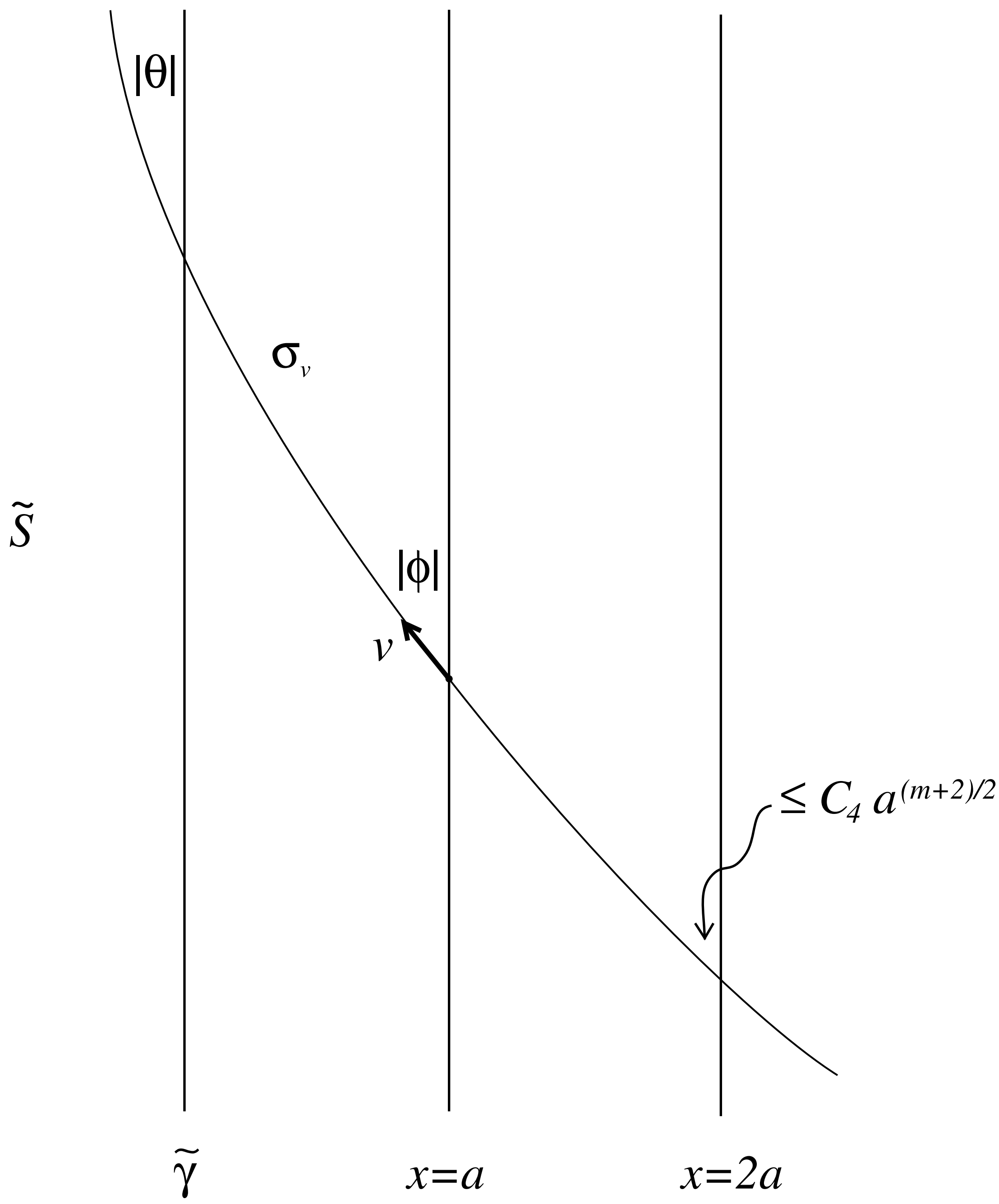

Figure 4.2. Case 1a.

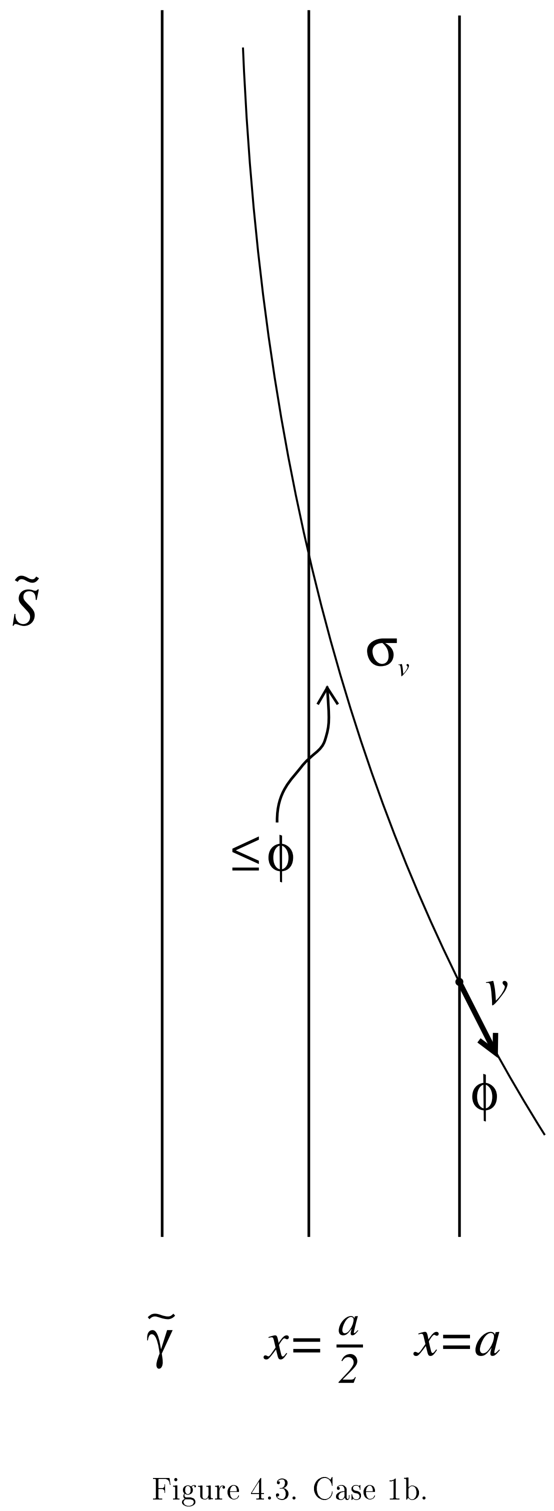

Figure 4.3. Case 1b.

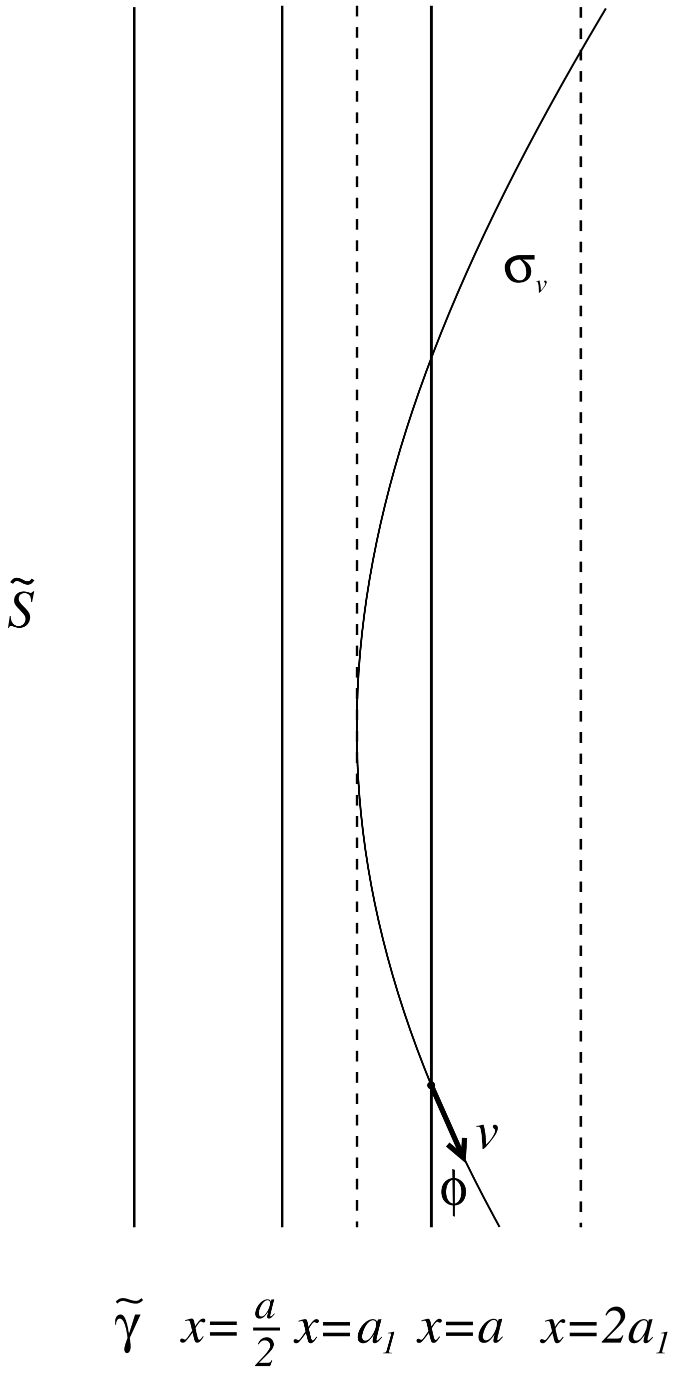

Figure 4.4. Case 1c.

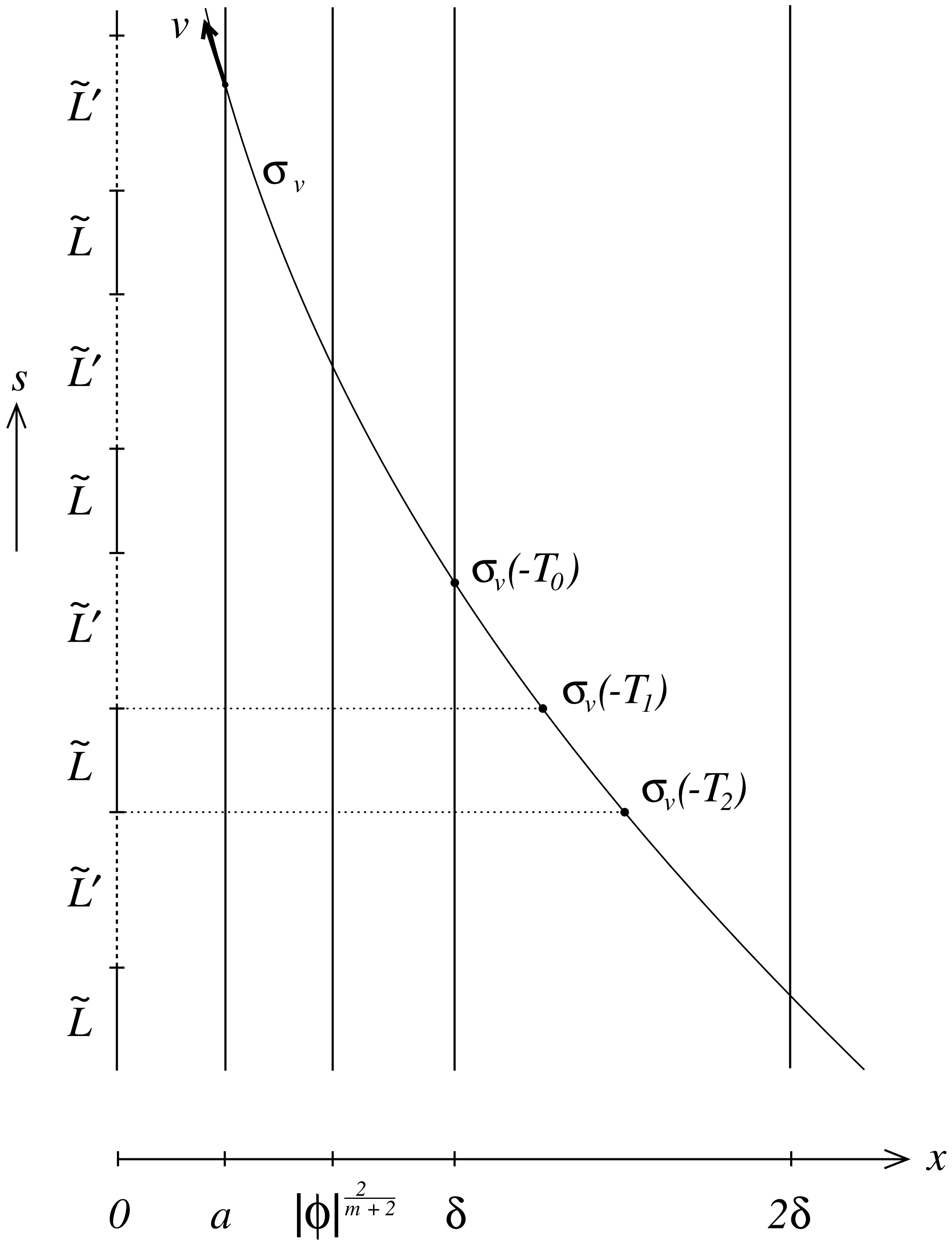

Figure 4.5. Case 2a.