PARTIAL DIFFERENTIABILITY OF INVARIANT SPLITTINGS

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Abstract. A key feature of a general nonlinear partially hyperbolic dynamical system is the absence of differentiability of its invariant splitting. In this paper, we show that often partial derivatives of the splitting exist and the splitting depends smoothly on the dynamical system itself.

Dedicated to David Ruelle on his 65th birthday.

October 15, 2002

Shub's work was partly funded by NSF Grant #DMS-9988809.
Wilkinson's work was partly funded by NSF Grant #DMS-0100314.

1. Introduction

One of the major technical barriers to the understanding of Anosov diffeomorphisms is the fact that unstable bundles are not in general differentiable along stable bundles. This situation persists for partially hyperbolic diffeomorphisms, where there are also center bundles present. Under mild bunching conditions, however, the unstable bundles are differentiable along the center bundles, see Theorem A below. This fact has already been observed and exploited in several special situations. First, for Anosov diffeomorphisms themselves, the unstable bundles are differentiable with respect the diffeomorphism, as long as partial derivatives are taken in certain dynamically defined directions given by conjugating maps [7]. Consequently entropy and SRB states also vary differentiably with parameters for Anosov diffeomorphisms and flows [5, 6, 11]. Differentiability of the unstable bundle along the center was a crucial ingredient in proving stable ergodicity for many partially hyperbolic diffeomorphisms [3, 14, 9, 10]. It was also an ingredient in the construction of nonuniformly hyperbolic diffeomorphisms with pathological foliations [13, 2, 12]. While we have similar applications in mind for Theorem A, we will content ourselves here with some general theorems. We describe the main results of this paper in the following section; the proofs occupy the remaining sections.

2. Statements of Results

Suppose that $f : M \to M$ is a partially hyperbolic diffeomorphism. The tangent bundle splits as $E^u \oplus E^c \oplus E^s$. In general the splitting is continuous but not C^1. Here we show that under some mild pointwise bunching conditions, E^u is continuously differentiable in the E^c direction, i.e.,

$$\frac{\partial E^u(p)}{\partial E^c}$$

exists and is a continuous function of $p \in M$. More precisely, we prove:

Theorem A. *Suppose $f : M \to M$ is C^2 and partially hyperbolic with splitting $TM = E^u \oplus E^c \oplus E^s$. Then, under the pointwise bunching condition*

$$\sup_p \frac{\|T_p^c f\|}{\boldsymbol{m}(T_p^u f)\boldsymbol{m}(T_p^c f)} < 1, \tag{1}$$

E^u is continuously differentiable with respect to E^c.

Theorem A is a corollary of a more general result about partial differentiability of dominated splittings – see Theorem 5.1 in Section 5.

Next we show, under the same bunching hypothesis, that in a family $t \mapsto f_t$ of partially hyperbolic diffeomorphisms, the unstable bundle $E^u(f_t)$ is always continuously differentiable along "dynamically defined" curves in M. Roughly speaking, a dynamically defined curve $t \mapsto \varphi_p(t)$ through $p \in M$ is a C^1 curve along which the hyperbolic component of the dynamics of f_t varies as little as possible. For example, if f_t is Anosov and $h_t : M \to M$ is the conjugacy from f_t to f_0, so that $h_t f_0 = f_t h_t$, then $t \mapsto h_t(p)$ is a dynamically defined curve. In the language of Section 7, a dynamically defined curve is the M-component of an integral curve of the center distribution $\mathbb{E}^c$ of the evaluation map $Eval : M \times I \to M \times I$:

$$(p, t) \mapsto (f_t p, t).$$

We prove that dynamically defined curves always exist, and unstable bundles, subject to a bunching condition, vary in a C^1 way along them.

Theorem B. *Let $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ be a C^2 family of C^2, partially hyperbolic diffeomorphisms having, for each $t \in (-\epsilon, \epsilon)$, a Tf_t-invariant splitting:*

$$TM = E^u(f_t) \oplus E^c(f_t) \oplus E^s(f_t).$$

Then there exists $\epsilon_0 > 0$ so that, for every $p \in M$ there exists a C^1 path

$$\varphi_p : (-\epsilon_0, \epsilon_0) \to M$$

with $\varphi_p(0) = p$, and with the following property.

If the pointwise bunching condition

$$\sup_p \frac{\|T_p^c f_0\|}{\boldsymbol{m}(T_p^u f_0)\boldsymbol{m}(T_p^c f_0)} < 1 \tag{2}$$

holds, then $t \mapsto E^u_{\varphi_p(t)}(f_t)$ is C^1.

Theorem B follows from a more general result, Theorem 7.4, which states that any invariant, dominated subbundle of a partially hyperbolic diffeomorphism is continuously differentiable along dynamically defined paths, subject to a bunching condition on the bundle. In addition, Theorem 7.4 produces, for any $v \in E^c_p(f_0)$, a dynamically defined path $\varphi_{p,v}$ so that $\dot\varphi_{p,v}(0) \in v + E^u \oplus E^s(f_0)$.

The machinery behind the proofs of Theorems A and B is Theorem 3.1, a refinement of the C^1 Section Theorem from [4] that handles partial derivatives of a section.

In Section 8, we address the question of when $t \mapsto E^u_p(f_t)$ is differentiable at $t = 0$. The issue here is of a slightly different nature than that in Theorems A and B. While $t \mapsto E^u(f_t)$ is always continuously differentiable along dynamically defined paths, the requirement that the constant path $t \mapsto p$ be dynamically defined for all p is a stringent one, satisfied only for very special families.

If, instead of requiring that $t \mapsto E^u_p(f_t)$ be C^1 in a given family, we just ask that it be differentiable at $t = 0$ *but for all families through* f_0, then the actual dynamics of f_0 becomes irrelevant. It is easy to see that $p \mapsto E^u_p(f_0)$ must be C^1 for this property to hold. What is interesting is that nonsmoothness of $p \mapsto E^u_p(f_0)$ is the *only* obstruction to the differentiability of $t \mapsto E^u_p(f_t)$ at $t = 0$ in every family. Building on Theorem B, one can show:

Theorem C. *Let $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ be a C^1 family of C^2, partially hyperbolic diffeomorphisms having, for each $t \in (-\epsilon, \epsilon)$, a Tf_t-invariant splitting:*

$$TM = E^u(f_t) \oplus E^c(f_t) \oplus E^s(f_t).$$

Assume that the pointwise bunching condition

$$\sup_p \frac{\|T_p^c f_0\|}{\boldsymbol{m}(T_p^u f_0)\boldsymbol{m}(T_p^c f_0)} < 1 \tag{3}$$

holds. Assume also that $E^u(f_0)$ is a $C^{2-\epsilon}$ subbundle of TM, for all $\epsilon > 0$.

Then for all $p \in M$,

$$t \mapsto E^u_p(f_t)$$

is differentiable at $t = 0$.

If φ_p is any dynamically defined path through $p \in M$ given by Theorem B, then:

$$E^u_p(f_t) - E^u_p(f_0) = \left(\frac{d}{dt}E^u_{\varphi_p t}(f_t)\,|_{t=0} - D_pE^u(f_0)(\frac{d}{dt}\varphi_p t\,|_{t=0})\right) t + O(t^{1+\eta}),$$

for some $\eta > 0$.

Subsequent to proving Theorem C, we learned of a more general result, due to Dolgopyat:

Theorem D (Dolgopyat, [2], Theorem 3). *Let $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ be a C^1 family of C^2 diffeomorphisms having, for each $t \in (-\epsilon, \epsilon)$, a Tf_t-invariant dominated splitting:*

$$TM = R(f_t) \oplus S(f_t) \oplus T(f_t).$$

Suppose that $p \mapsto S_p(f_0)$ is C^1. Then, for every $p \in M$, $t \mapsto S_p(f_t)$ is differentiable at $t = 0$.

Dominated splittings are defined in Section 5. In particular, Theorem D applies when S is E^u, E^c, E^s, E^{cu}, or E^{cs}. We present an exposition of Dolgopyat's proof of Theorem D in Section 8.

3. Partial Derivatives of an Invariant Section

Let

$$\begin{array}{ccc} V & \xrightarrow{\quad F \quad} & V \\ \pi\Big\downarrow & & \Big\downarrow\pi \\ M & \xrightarrow{\quad f \quad} & M \end{array}$$

be a C^1 fiber preserving map, where V is a smooth, finite dimensional fiber bundle over the compact manifold M, and f is a diffeomorphism. In addition assume that there is a section $\sigma : M \to V$, invariant under F in the sense that

$$F(\sigma(p)) = \sigma(f(p))$$

for all $p \in M$.

In general there is no reason that σ is smooth, or even continuous. For example, if F is the identity map, every section of V is F-invariant. In [4], we showed that if V is a Banach bundle and F is a fiber contraction then σ is unique and continuous, and furthermore, if the fiber contraction dominates the base contraction sufficiently then the σ is of class C^r.

Since F preserves fibers, TF preserves the "vertical" subbundle, $\mathrm{Vert} \subset TV$ whose fiber at $v \in V$ is $\mathrm{kernel}\, T_v\pi$. We write $T^{\mathrm{Vert}}_v F$ for the restriction of T_vF to Vert_v,

$$T^{\mathrm{Vert}}_v F : \mathrm{Vert}_v \to \mathrm{Vert}_{Fv}\,.$$

We assume that TV carries a Finsler structure and that $k_p = \|T^{\mathrm{Vert}}_{\sigma p} F\|$ has

$$\sup_{p\in M} k_p < 1,$$

which means that F is a fiber contraction in the neighborhood of σM.

Theorem 3.1. *Suppose that $E \subset TM$ is a continuous Tf-invariant subbundle such that*

$$\sup_{p \in M} k_p \|(T_p^E f)^{-1}\| < 1$$

where $T^E f$ is the restriction of Tf to E. Then σ is continuously differentiable in the E-direction in the sense that there is a continuous map $H : E \to TV$ such that

(a) *For each $p \in M$, $H : E_p \to T_{\sigma p} V$ is linear.*
(b) $T\pi \circ H = \mathrm{Id} : E \to E$.
(c) *If γ is a C^1 arc in M that is everywhere tangent to E then*

$$(\sigma \circ \gamma)'(t) = H(\gamma'(t)).$$

In particular, if E is integrable then the restriction of σ to each E-leaf is C^1.

We refer to H as the partial derivative of σ in the E-direction

$$H = \frac{\partial \sigma}{\partial E}.$$

Remark. If, in addition, there exist C^r submanifolds everywhere tangent to E, for some $r \in (0, \infty)$, then C^r smoothness of σ along E (i.e., along these manifolds) can be assured by assuming that

$$\sup_p k_p \|(T_p^E f)^{-1}\|^r < 1.$$

Remark. When E is integrable, the proof of Theorem 3.1 is a fairly simple application of the Invariant Section Theorem of [4]. It is the non-integrable case that requires some new ideas.

Remark. There is a uniformity about $\partial\sigma/\partial E$. (In the integrable case, this uniformity is automatic.) Fix $p \in M$ and extend each $w \in E_p$ with $|w| \leq 1$ to a continuous vector field X_w everywhere subordinate to E, and do so in a way that depends continuously on w. Let γ_w be an integral curve of X_w through p. Since E is only continuous, the integral curve γ_w need not be uniquely determined by X_w. Nevertheless, for all p in any fixed C^1 chart, as $t \to 0$ we have

$$\frac{\sigma \circ \gamma_w(t) - \sigma p}{t} \to H(w)$$

uniformly.

Remark. Since M is finite dimensional, Peano's Existence Theorem implies that there exist C^1 arcs everywhere tangent to a continuous plane field, and thus the hypothesis of assertion (c) in Theorem 3.1 is satisfied. In the infinite dimensional case, however, Peano's Theorem fails and (c) could become vacuous.

Proof of Theorem 3.1. We proceed by the graph transform techniques in [4]. Choose a continuous subbundle $\mathrm{Hor} \subset TV$, complementary to Vert,

$$\mathrm{Hor} \oplus \mathrm{Vert} = TV.$$

For example, we could introduce a Riemann structure on TV and take Hor_v as the orthogonal complement to Vert_v. Note that $T\pi$ sends each subspace Hor_v isomorphically onto $T_p M$, $p = \pi v$. With respect to the horizontal / vertical splitting we write

$$T_v F = \begin{bmatrix} A_v & 0 \\ C_v & K_v \end{bmatrix} = \begin{bmatrix} A_v : \mathrm{Hor}_v \to \mathrm{Hor}_{Fv} & 0 \\ C_v : \mathrm{Hor}_v \to \mathrm{Vert}_{Fv} & K_v : \mathrm{Vert}_v \to \mathrm{Vert}_{Fv} \end{bmatrix}.$$

Let $\overline{E} \subset \mathrm{Hor}$ be the lift of E. That is, $T\pi$ sends the plane $\overline{E}_v$ isomorphically to E_p, $p = \pi v$. Since E is Tf-invariant and F covers f, $\overline{E}$ is A-invariant in the sense that

$$\begin{tikzcd} \overline{E}_v \arrow[r, "A_v"] \arrow[d, "T\pi"'] & \overline{E}_{Fv} \arrow[d, "T\pi"] \\ E_p \arrow[r, "Tf"] & E_{fp} \end{tikzcd}$$

commutes.

Let L be the bundle over M whose fiber at p is

$$L_p = L(\overline{E}_{\sigma p}, \mathrm{Vert}_{\sigma p}).$$

An element in L_p is a linear transformation $P : \overline{E}_{\sigma p} \to \mathrm{Vert}_{\sigma p}$. Let LF be the **graph transform** on L that sends $P \in L_p$ to

$$P' = (C_{\sigma p} + K_{\sigma p}P)(A_{\sigma p}|_{\overline{E}_{\sigma p}})^{-1} \in L_{\sigma p}.$$

Then TF sends the graph of P to the graph of P' and LF is an affine fiber contraction

$$\begin{tikzcd} L \arrow[r, "LF"] \arrow[d, "\pi"'] & L \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M. \end{tikzcd}$$

By [4], L has a unique LF-invariant section $\Lambda : M \to L$, and Λ is continuous. Define $H_p : E_p \to T_{\sigma p}V$ by commutativity of

$$\begin{tikzcd} \overline{E}_{\sigma p} \arrow[r, "\mathrm{Id}_p \oplus \Lambda_p"] \arrow[d, "T\pi"'] & \overline{E}_{\sigma p} \oplus \mathrm{Vert}_{\sigma p} \arrow[d, "\text{Inclusion}"] \\ E_p \arrow[r, "H_p"] & T_{\sigma p}V \end{tikzcd}$$

where Id_p is the identity map $\overline{E}_{\sigma p} \to \overline{E}_{\sigma p}$. Then $H : E \to TV$ is the unique bundle map such that HE is a TF-invariant subbundle of $T_{\sigma M}V$,

$$\begin{tikzcd} HE \arrow[r, "TF"] \arrow[d, "T\pi"'] & HE \arrow[d, "T\pi"] \\ E \arrow[r, "Tf"] & E \end{tikzcd}$$

commutes, and $T\pi \circ H = \mathrm{Id}_E$. We claim that H is the partial derivative of σ in the E-direction.

Let $\gamma : (a, b) \to M$ be a C^1 arc such that γ is everywhere tangent to E. To complete the proof of the theorem, we must show that

$$(\sigma \circ \gamma)'(t) = H(\gamma'(t)).$$

For $n \in \mathbb{Z}$, set $\gamma_n = f^n \circ \gamma$ and

$$\Gamma = \bigsqcup_{n \in \mathbb{Z}} \gamma_n.$$

This means that we consider the disjoint union of the arcs γ_n, so if two of them cross in M, we ignore the crossing in Γ. The one dimensional manifold Γ is noncompact; it has countably many components γ_n. In the same way, we discretize V as

$$V_\Gamma = \bigsqcup_n V|_{\gamma_n}.$$

We equip V_Γ and $T\Gamma$ with the Finslers they inherit from V and M. Then $F_\Gamma = F|_{V_\Gamma}$ is a fiber contraction

$$\begin{tikzcd} V_\Gamma \arrow[r, "F_\Gamma"] \arrow[d, "\pi"'] & V_\Gamma \arrow[d, "\pi"] \\ \Gamma \arrow[r, "f"] & \Gamma, \end{tikzcd}$$

and the fiber contraction dominates the base contraction since

$$\sup_p k_p \|T_p^E f\| < 1$$

and $T\Gamma \subset E$. Furthermore, F_Γ is uniformly C^1 bounded since M is compact. The Invariant Section Theorem of [4] then implies that V_Γ has a unique F_Γ-invariant section σ_Γ, and σ_Γ is of class C^1. Furthermore the tangent bundle of $\sigma_\Gamma(\Gamma)$ is the unique nowhere vertical TF_Γ-invariant line field in TV_Γ.

The restriction of σ to $\Gamma = \bigsqcup_n \gamma_n$ is F_Γ-invariant, so by uniqueness

$$\sigma_\Gamma = \bigsqcup_n \sigma|_{\gamma_n}.$$

We claim that

$$H(T\Gamma) = T(\sigma_\Gamma \Gamma).$$

Again the reason is uniqueness. We know that $T(\sigma_\Gamma\Gamma)$ is the unique TF_Γ-invariant, nowhere vertical line field defined over $\sigma_\Gamma\Gamma$. But commutativity of

$$\begin{tikzcd} HE \arrow[r, "TF_\Gamma"] & HE \\ E \arrow[r, "Tf"] \arrow[u, "H"] & E \arrow[u, "H"'] \\ T\Gamma \arrow[r, "Tf"] \arrow[u, "\text{Inclusion}"] & T\Gamma \arrow[u, "\text{Inclusion}"'] \end{tikzcd}$$

implies that $H(T\Gamma)$ is a second such line field. By uniqueness they are equal.

To complete the proof, we show that the line field equality implies the vector equality

$$\frac{d}{dt}\sigma \circ \gamma(t) = H(\gamma'(t)),$$

as the theorem asserts. Differentiating $\gamma(t) = \pi \circ \sigma_\Gamma \circ \gamma(t)$ gives

$$\gamma'(t) = T\pi \circ T\sigma_\Gamma(\gamma'(t)).$$

The vector $T\sigma_\Gamma(\gamma'(t))$ lies in the span of $H(\gamma'(t))$, so there is a real number $c(t)$ such that $T\sigma_\Gamma(\gamma'(t)) = H(c(t)\gamma'(t))$. This gives

$$\gamma'(t) = T\pi \circ H(c(t)\gamma'(t)).$$

Since $T\pi \circ H = \mathrm{Id}_E$ we have

$$\gamma'(t) = c(t)\gamma'(t)$$

and $c(t) = 1$. Thus

$$\frac{d}{dt}\,\sigma_\Gamma \circ \gamma(t) = T\sigma_\Gamma(\gamma'(t)) = H(c(t)\gamma'(t)) = H(\gamma'(t)).$$

□

Remark. Above, it is assumed that γ is everywhere tangent to E. One might expect that tangency of γ to E at $p = \gamma(0)$ suffices to prove that $(\sigma \circ \gamma)'(0) = H(\gamma'(0))$. This is not so. For example E can be the flow direction of an Anosov flow. The bundle E^u can be Hölder, but not C^1. Say its Hölder exponent is $\theta < 1$. One can construct a C^1 curve $\gamma(t)$ which is tangent to E at $p = \gamma(0)$, but which diverges from E at a rate $t^{1+\epsilon}$. The difference between $E^u_{\gamma(t)}$ and E^u_p is then on the order of $t^{\theta+\epsilon\theta}$. If ϵ is small this exponent is < 1, and the map $t \mapsto E^u_{\gamma(t)}$ fails to be differentiable at $t = 0$.

4. A Series Expression for $\partial\sigma/\partial E$

As above σ is the unique F-invariant section and $H = \mathrm{Id}_E \oplus \Lambda$ is its partial derivative in the E-direction. Naturally, $\partial\sigma/\partial E$ depends on the choice of horizontal subbundle $\mathrm{Hor} \subset TV$. We use the isomorphism $T\pi : \mathrm{Hor}_{\sigma p} \to T_pM$ to identify the linear map $A_{\sigma p} : \mathrm{Hor}_{\sigma p} \to \mathrm{Hor}_{\sigma(fp)}$ with its $T\pi$-conjugate T_pf. Then, using the canonical isomorphism $\mathrm{Vert}_{\sigma p} \approx V_p$, we can express $TF = \begin{bmatrix} A & 0 \\ C & K \end{bmatrix}$ as

$$T_{\sigma p}F = \begin{bmatrix} T_pf : T_pM \to T_{fp}M & 0 \\ C_p : T_pM \to V_{fp} & K_p : V_p \to V_{fp} \end{bmatrix}.$$

Thus, the bundle map $LF : L \to L$ becomes

$$P \mapsto (C_p + K_pP) \circ (T^E_{fp} f^{-1}).$$

Denote by Λ_0 the zero section of L, and call its N^{th} iterate in L,

$$\Lambda_N = (LF)^N(\Lambda_0).$$

We know that $\Lambda_N \to \Lambda$ uniformly as $N \to \infty$. Also, we claim that

$$\Lambda_N(p) = \sum_{n=0}^{N-1} K^n_p \circ C_{f^{-n-1}(p)} \circ (T^E_p f^{-n-1})$$

where $K^0 = \mathrm{Id}$ and for $n \geq 1$,

$$K^n_p = K_{f^{-1}(p)} \circ \cdots \circ K_{f^{-n}(p)} : V_{f^{-n}(p)} \to V_p.$$

If $N = 1$ we have

$$\Lambda_1(p) = (C_{f^{-1}(p)} + K_{f^{-1}(p)}P_0)(T^E_p f^{-1}) = C_{f^{-1}(p)} T^E_p f^{-1}$$

because $\Lambda_0 = 0$ implies that $P_0 = 0$. Thus, the assertion holds with $N = 1$; the proof is completed by induction.

Since the partial sums of the infinite series $\sum_{n=0}^{\infty} K^n_p C_{f^{-n-1}(p)} T^E_p f^{-n-1}$ converge uniformly to Λ, we are justified in writing

$$\frac{\partial\sigma}{\partial E} = H(p) = \mathrm{Id}_E \oplus \sum_{n=0}^{\infty} K^n_p C_{f^{-n-1}(p)} T^E_p f^{-n-1}.$$

5. The Dominated and Partially Hyperbolic Cases: Proof of Theorem A

Let $f : M \to M$ be a diffeomorphism of a compact Riemannian (or Finslered) manifold. In this section, we consider Tf-invariant dominated and partially hyperbolic splittings of the tangent bundle TM. We recall the definitions. The **conorm** of a linear transformation $T : X \to Y$ is

$$\boldsymbol{m}(T) = \inf_x \frac{|Tx|}{|x|}.$$

Suppose that TM splits as a Tf-invariant sum of two bundles:

$$TM = R \oplus S.$$

This splitting is *dominated* if the following condition holds:

$$\inf_p \frac{\boldsymbol{m}(T_p^R f)}{\|T_p^S f\|} > 1$$

where the notation $T^X f$ is used for the restriction of Tf to the bundle X. We say that f has a *dominated decomposition* if there is a Tf-invariant dominated splitting of TM.

Then f is *partially hyperbolic* if it has a Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ such that:

(1) $E^u \oplus (E^c \oplus E^s)$ and $(E^u \oplus E^c) \oplus E^s$ are both dominated splittings of TM,
(2)

$$\inf_p \boldsymbol{m}(T_p^u f) > 1 \qquad \sup_p \|(T_p^s f)\| < 1$$

In other words, Tf expands vectors in E^u, contracts vectors in E^s, and is relatively neutral on vectors in E^c.

Theorem 3.1 can be applied in the dominated decomposition and partially hyperbolic contexts, as follows.

Theorem 5.1. *Suppose that the C^2 diffeomorphism $f : M \to M$ has a dominated decomposition:*

$$TM = R \oplus S,$$

And let $E \subset TM$ be a Tf-invariant subbundle. Then, under the pointwise bunching condition

$$\sup_p \frac{\|T_p^S f\|}{\boldsymbol{m}(T_p^R f)\boldsymbol{m}(T_p^E f)} < 1, \tag{4}$$

R is continuously differentiable with respect to E.

Theorem A is an immediate corollary of Theorem 5.1 , where we set $R = E^u$, $S = E^c \oplus E^s$, and $E = E^c$.

In [4] we defined the "bolicity" of a linear transformation to be the ratio of its norm to its conorm, so the poinwise bunching condition in Theorem A can be re-stated as

$$\sup_p \frac{\mathrm{bol}(T_p^c f)}{\boldsymbol{m}(T_p^u f)} < 1.$$

Similarly, to show that E^s is differentiable along E^c, we assume

$$\sup_p \|T_p^s f\| \,\mathrm{bol}(T_p^c f) < 1. \tag{5}$$

Proof of Theorem 5.1. Let d be the fiber dimension of R in the dominated decomposition

$$TM = R \oplus S.$$

Tf acts naturally on the Grassmann $G = G(d, TM)$ of all d-planes in TM,

$$Gf : G \to G.$$

If Π is a d-plane in T_pM then $Gf(\Pi) = T_pf(\Pi)$. Since f is C^2, Gf is a C^1 fiber preserving map,

$$\begin{array}{ccc} G & \xrightarrow{\quad Gf \quad} & G \\ \pi \big\downarrow & & \big\downarrow \pi \\ M & \xrightarrow{\quad f \quad} & M \end{array}$$

where π sends $\Pi \subset T_pM$ to p. Besides, $p \mapsto R_p$ is a Gf-invariant section of G. We show that, at the invariant section R, the fiber contraction rate dominates the contraction rate along E as follows.

A compact neighborhood N_p of R_p in G_p consists of d-planes Π such that $\Pi = \operatorname{graph} P$ where $P : R_p \to S_p$ is a linear transformation with $\|P\| \leq 1$. Give G a Finsler which is the operator norm on each N_p and any other Finsler on the rest of G. Then Gf is a fiber preserving map whose fiber contraction rate at R_p is

$$k_p = \frac{\|T_p^S f\|}{\boldsymbol{m}(T_p^R f)} < 1.$$

Since this dominates the contraction rate $\boldsymbol{m}(T_p^E f)$ along E, Theorem 3.1 applies and $p \mapsto R_p$ is seen to be a continuously differentiable function of p in the E direction. □

6. A series formula for $\partial E^u / \partial E^c$

From Section 4 we know that there is a series that expresses $\partial E^u/\partial E^c$. We write this formula out after making a convenient choice of the horizontal bundle.

To do so, we coordinatize G near E_p^u as follows. Fix a smooth Riemann structure on M that exhibits the partial hyperbolicity of f, and let exp be its exponential map. Abusing notation, we denote by $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ the planes $\mathbb{R}^u \times 0$ and $0 \times \mathbb{R}^{cs}$ in $\mathbb{R}^m$. For each $p \in M$, define a linear map $I_p : \mathbb{R}^m \to T_pM$ that carries $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ isometrically to E_p^u and E_p^{cs}. The restriction of $\exp_p \circ I_p$ to a small neighborhood $U = U_p$ of 0 in $\mathbb{R}^m$ is a diffeomorphism of U to a neighborhood $Q = Q_p$ of p in M,

$$\varphi_p : U \to Q$$

and

(a) $\varphi_p(0) = p$
(b) $T_0\varphi_p$ carries $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ isometrically to E_p^u and E_p^{cs}.
(c) If we denote by E_{pq}^u and E_{pq}^{cs} the planes $T_x\varphi_p(\mathbb{R}^u)$ and $T_x\varphi_p(\mathbb{R}^{cs})$, where $q = \varphi_p(x)$, then $q \mapsto E_{pq}^u \oplus E_{pq}^{cs}$ is a smooth splitting of TQ that reduces to $E_p^u \oplus E_p^{cs}$ when $p = q$.

We now coordinatize G near E_p^u. Let $\mathcal{M}$ be the space of $(u \times cs)$-matrices, thought of as linear transformations $X : \mathbb{R}^u \to \mathbb{R}^{cs}$. Given $(x, X) \in U \times \mathcal{M}$, let $q = \varphi_p(x)$ and consider the linear transformation $S : E^u_{pq} \to E^{cs}_{pq}$, defined by commutativity of

$$\begin{array}{ccc} \mathbb{R}^u & \xrightarrow{\quad X \quad} & \mathbb{R}^{cs} \\ T_x\varphi_p \big\downarrow & & \big\downarrow T_x\varphi_p \\ E^u_{pq} & \xrightarrow{\quad S \quad} & E^{cs}_{pq}. \end{array}$$

The graph of S is a plane $\Pi \in G$ near E^u_p, and thus

$$\Phi_p : (x, X) \mapsto \Pi$$

is a local trivialization of G at E^u_p.

Because $U \times \mathcal{M}$ is a product, $T(U \times \mathcal{M})$ carries a natural horizontal structure, the horizontal space at (x, X) being

$$\mathbb{R}^m \times 0 \ \subset \ \mathbb{R}^m \times \mathcal{M} = T_{(x,X)}(U \times \mathcal{M}).$$

We define the horizontal space at $\Pi = \Phi_p(0, X) \in G_p$ to be

$$\mathrm{Hor}_\Pi = T_{(0,X)}\Phi_p(\mathbb{R}^m \times 0).$$

Writing $T(Gf) : TG \to TG$ with respect to the horizontal / vertical splitting of TG gives

$$T_\Pi(Gf) = \begin{bmatrix} A_\Pi : \mathrm{Hor}_\Pi \to \mathrm{Hor}_{Gf(\Pi)} & 0 \\ C_\Pi : \mathrm{Hor}_\Pi \to \mathrm{Vert}_{Gf(\Pi)} & K_\Pi : \mathrm{Vert}_\Pi \to \mathrm{Vert}_{Gf(\Pi)} \end{bmatrix}$$

Take $\Pi = E^u_p$ and identify

$$T_{E^u_p} G_p = \mathrm{Vert}_{E^u_p} \approx L(E^u_p, E^{cs}_p).$$

Fix $v \in E^c_p$. Then $C_{f^{-n-1}p} \circ T^c f^{-n-1}(v)$ is a linear transformation $Y_n(v) : E^u_{f^{-n}p} \to E^{cs}_{f^{-n}p}$, and $Y_n(v)$ is susceptibe to the n^{th} power of the linear graph transform, which converts it to a linear transformation $E^u_p \to E^{cs}_p$ defined by

$$T^{cs}_{f^{-n}p} f^n \circ (Y_n(v)) \circ T^u_p f^{-n}.$$

This is the same as the repeated action of K. (That is, the graph transform of Tf^n is the same as the n^{th} power of the graph transform of Tf.) Thus, by the formula in Section 4,

$$\frac{\partial E^u}{\partial E^c}(v) = \sum_{n=0}^{\infty} T^{cs}_{f^{-n}p} f^n \circ (C_{f^{-n-1}p} \circ T^c f^{-n-1}(v)) \circ T^u_p f^{-n}.$$

We also can express this in charts as follows. Writing f in the φ-charts gives

$$f_p = \varphi^{-1}_{fp} \circ f \circ \varphi_p$$

and

$$(Df_p)_x = \begin{bmatrix} D^{u,u}_x f_p & D^{cs,u}_x f_p \\ D^{u,cs}_x f_p & D^{cs,cs}_x f_p \end{bmatrix}$$

where the $D^{u,u}_x f_p$ block consists of the partial derivatives of the u-components of f_p with respect to the u-variables, evaluated at the point x, etc. At $x = 0$, the off-diagonal blocks are zero, while the diagonal blocks are $T\varphi$-conjugate to $T^u_p f$ and $T^{cs}_p f$. Thus, the coordinate expression of Gf becomes

$$(Gf)_p : (x, X) \mapsto (f_p x, (D^{u,cs}_x f_p + (D^{cs,cs}_x f_p)X)(D^{u,u}_x f_p + (D^{cs,u}_x f_p)X)^{-1}.$$

Differentiating this with respect to x and X at the origin $(0,0) \in \mathbb{R}^m \times \mathcal{M}$ yields

$$(D((Gf)_p))_{(0,0)} = \begin{bmatrix} A_p : \mathbb{R}^m \to \mathbb{R}^m & 0 \\ C_p : \mathbb{R}^m \to \mathcal{M} & K_p : \mathcal{M} \to \mathcal{M} \end{bmatrix},$$

where A_p is $T\varphi$-conjugate to $T_p f$,

$$A_p = (T_0 \varphi_{fp})^{-1} \circ T_p f \circ T_0 \varphi_p,$$

C_p represents the second derivatives of f in the φ-charts,

$$\begin{aligned} C_p &= \frac{\partial}{\partial x}\Big|_{x=0} (D_x^{u,cs} f_p)(D_x^{u,u} f_p)^{-1} \\ &= (D_x(D_x^{u,cs} f_p))(D_x^{u,u} f_p)^{-1} \\ &\quad - (D_x^{u,cs} f_p)(D_x^{u,u} f_p)^{-1}(D_x(D_x^{u,u} f_p))(D_x^{u,u} f_p)^{-1}, \end{aligned}$$

and, because the off-diagonal blocks vanish at the origin, K_p is $T\Phi$-conjugate to the graph transform of Tf,

$$P \mapsto T_p^{cs} f \circ P \circ (T_p^u f)^{-1}.$$

It is worth noting that the norm of C is uniformly bounded on a neighborhood of E^u in G because f is C^2 and M is compact. Also, this is clear from the formula expressing C in the φ-charts.

7. Dependence of E^u, E^s on f: Proof of Theorem B

As has been highlighted in the Katok-Milnor examples [8], the conjugacy between an Anosov diffeomorphism and its perturbations is a smooth function of the perturbation, even though the conjugacies themselves are only continuous. For example, consider a 1-parameter family of Anosov diffeomorphisms $g_t : M \to M$. The map g_0 is conjugate to g_t by a homeomorphism $h_t : M \to M$, and h_t is uniquely determined by the requirements that $h_0 = \mathrm{Id}$ and $t \mapsto h_t$ is continuous. The map

$$\begin{gathered} (-\epsilon, \epsilon) \times M \to (-\epsilon, \epsilon) \times M \\ (t, p) \mapsto (t, g_t p) \end{gathered}$$

is smooth, partially hyperbolic, and supports a center foliation $\mathcal{W}^c$ whose leaf through $(0,p)$ is $\{(t, h_t(p)) : t \in (-\epsilon, \epsilon)\}$. As a foliation $\mathcal{W}^c$ is only continuous, but its leaves are smooth. Theorem 3.1 applies perfectly well to this situation, and we conclude that $E^u_{h_t p}$, $E^s_{h_t p}$ are C^1 functions of t. In this section we replace the Anosov condition by partial hyperbolicity, and derive an analogous result.

We assume that $f_0 : M \to M$ is a C^2, partially hyperbolic diffeomorphism with splitting

$$TM = E^u \oplus E^c \oplus E^s$$

and that $\mathcal{F}$ is a small neighborhood of f_0 in $\mathrm{Diffeo}^2 M$. By the usual linear graph transform techniques, all $f \in \mathcal{F}$ are partially hyperbolic and their splittings

$$TM = E^u(f) \oplus E^c(f) \oplus E^s(f)$$

depend continuously on f.

The space $\mathrm{Diff}^2(M)$ is a Banach manifold, see [1] for details. For $f \in \mathrm{Diff}^2(M)$, the tangent space to $\mathrm{Diff}^2(M)$ at f has a natural description. Let X_f be the Banach space of C^2 sections of the pullback bundle f^*TM, that is, the bundle whose fiber over $p \in M$ is $T_{fp}M$. We write $f+g$ to indicate the diffeomorphism $\exp_f \circ g$. That is, if g is a small vector field in X_f, then

$$(f+g)(p) = \exp_f(g(p))$$

is in $\mathrm{Diff}^2(M)$ and is close to f. So a small disk in X_f is a chart for a small neighborhood of f, and X_f is thereby identified with the tangent space $T_f\mathrm{Diff}^2(M)$.

Define the map

$$\begin{aligned} \mathrm{Eval} : \mathcal{F} \times M &\to \mathcal{F} \times M \\ (f,p) &\mapsto (f, fp). \end{aligned}$$

Eval is C^2 because left-composition is a smooth operation on functions.

Lemma 7.1. *If the diameter of $\mathcal{F}$ is sufficiently small, then Eval has a partially hyperbolic splitting:*

$$T(\mathcal{F} \times M) = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s,$$

where

$$\begin{aligned} \mathbb{E}^u_{f,p} &= 0 \times E^u_p(f) \subset 0 \times TM \\ \mathbb{E}^s_{f,p} &= 0 \times E^s_p(f) \subset 0 \times TM, \end{aligned}$$

and $\mathbb{E}^c_{f,p}$ is the graph of a linear map:

$$P_{f,p} : X_f \oplus E^c_p(f) \to E^u_p(f) \oplus E^s_p(f).$$

Proof. The tangent to Eval at (f,p) acts on a vector $\begin{bmatrix} g \\ v \end{bmatrix} \in T_{f,p}(\mathcal{F} \times M)$ as

$$T_{f,p}\,\mathrm{Eval} \begin{bmatrix} g \\ v \end{bmatrix} = \begin{bmatrix} g \\ g(p) + T_p f(v) \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_{\mathcal{F}} & 0 \\ \mathrm{ev}_p & T_p f \end{bmatrix} \begin{bmatrix} g \\ v \end{bmatrix},$$

where ev_p evaluates the section of f^*TM at p. In particular, this implies that the subbundles $\mathbb{E}^u = 0 \times E^u$, $\mathbb{E}^s = 0 \times E^s$ cited above are $T\,\mathrm{Eval}$-invariant. (The bundle $0 \times E^c$ is also $T\,\mathrm{Eval}$-invariant, but it is too small to be the $\mathbb{E}^c$ we want.)

Note that the subbundle $T\mathcal{F} \oplus 0$ is not $T\,\mathrm{Eval}$-invariant, nor is the subbundle $T\mathcal{F} \oplus E^c$ whose fiber at (f,p) is $X_f \oplus E^c_p(f)$. For if $v \in E^c_p(f)$ then the $T\,\mathrm{Eval}$-image of $\begin{bmatrix} g \\ v \end{bmatrix}$ is $\begin{bmatrix} g \\ g(p) + T^c f_p(v) \end{bmatrix}$, and this vector need not lie in $T\mathcal{F} \oplus E^c$. Nevertheless, by the domination hypotheses, the $T\,\mathrm{Eval}$ graph transform defines a fiber contraction of the bundle whose fiber at (f,p) is

$$L(X_f \oplus (E^u \oplus E^c)_p(f), E^s(f)).$$

The resulting invariant section is the unique $T\,\mathrm{Eval}$-invariant subbundle $\mathbb{E}^c \oplus \mathbb{E}^s \subset T(\mathcal{F} \times M)$ whose fiber at (f,p) projects isomorphically onto $X_f \oplus (E^c \oplus E^s)_p(f)$.

Similarly, we find the unique $T\,\mathrm{Eval}^{-1}$-invariant subbundle $\mathbb{E}^c \oplus \mathbb{E}^u \subset T(\mathcal{F} \times M)$ whose fiber at (f,p) projects isomorphically onto $X_f \oplus (E^c \oplus E^u)_p(f)$. Intersecting these bundles, we obtain the $T\,\mathrm{Eval}^{-1}$-invariant subbundle $\mathbb{E}^c$. □

Remark. At the end of this section, we give a series expression for $\mathbb{E}^{cu}$.

Corollary 7.2. *Suppose that $f : M \to M$ is C^2 and partially hyperbolic, with splitting:*

$$TM = E^u \oplus E^c \oplus E^s.$$

If f_0 satisfies the pointwise bunching condition:

$$\sup_p \frac{\|T_p^c f_0\|}{\boldsymbol{m}(T_p^u f_0)\boldsymbol{m}(T_p^c f_0)} < 1, \tag{6}$$

then, for all $p \in M$, E^u is continuously differentiable at (f_0, p) with respect to $\mathbb{E}^c$, where $\mathbb{E}^c$ is given by Lemma 7.1.

In fact this corollary can be stated in a more general form that can be useful in applications. Not only is it possible to differentiate E^u along $\mathbb{E}^c$, but in fact bundles in dominated decompositions can be differentiated along $\mathbb{E}^c$ as well. If f_0 has a dominated decomposition

$$TM = R \oplus S,$$

then standard graph-transform arguments apply to show that for f sufficiently C^1-close to f_0, this decomposition has a unique continuation

$$TM = R(f) \oplus S(f)$$

that is dominated for Tf. Under appropriate bunching hypotheses, we can differentiate $R(f)$ in the $\mathbb{E}^c$ direction:

Corollary 7.3. *Suppose that $f : M \to M$ is C^2 and partially hyperbolic, with splitting:*

$$TM = E^u \oplus E^c \oplus E^s.$$

Suppose also that

$$TM = R \oplus S$$

is a dominated decomposition for f_0. If f_0 satisfies the pointwise bunching condition:

$$\sup_p \frac{\|T_p^S f_0\|}{\boldsymbol{m}(T_p^R f_0)\boldsymbol{m}(T_p^c f_0)} < 1, \tag{7}$$

then, for all $p \in M$, R is continuously differentiable at (f_0, p) with respect to $\mathbb{E}^c$, where $\mathbb{E}^c$ is given by Lemma 7.1.

Proof of Corollary 7.3. We first construct the bundle over $\mathcal{F} \times M$ whose fiber over (f, p) is the space of linear maps $L(R_p(f), S_p(f))$. Since Eval preserves the factors $\{f\} \times M$, its tangent map $T\,\mathrm{Eval}$ induces a graph transform map on this bundle, covering Eval, which is a fiber contraction, with:

$$k_{f,p} = \frac{\|T_p^S f\|}{\boldsymbol{m}(T_p^R f)}.$$

The unique invariant section of this graph transform is $\mathbb{R} = 0 \oplus R \subset T\mathcal{F} \times TM$. (note that the bundle $\mathbb{R}$ is not to be confused with the real numbers $\mathbf{R}$).

Now suppose γ is any curve tangent to $\mathbb{E}^c$. As in the proof of Theorem 3.1, we obtain differentiability of $\mathbb{R}$ (and hence, of R) along γ when $k_{f,p}$ dominates the contraction along γ at (f, p) The contraction along γ at (f, p) is bounded below by the conorm of $T_{f,p}^{\mathbb{E}^c}\,\mathrm{Eval}$, which in turn is approximately given by

$$\boldsymbol{m}(T_{f_0,p}^{\mathbb{E}^c}\,\mathrm{Eval}) = \min\{1, \boldsymbol{m}(T^c f_0)\}.$$

Hence, $\mathbb{R} = 0 \oplus R$ is differentiable along γ if

$$\sup_{f,p} \frac{k_{f,p}}{\min\{\boldsymbol{m}(T_p^c f_0), 1\}} < 1. \tag{8}$$

Since $k_{f,p} < 1$ for all p, and by the bunching hypothesis (7), $k_{f,p} < \boldsymbol{m}(T^c_p f_0)$, the condition in (8) is satisfied.

Now we apply Theorem 3.1 and conclude that $\mathbb{R}$ is continuously differentiable along $\mathbb{E}^c$.

Note that Theorem 3.1 needs to be re-proved in this more general context, but because its original proof relied on uniform estimates (this was the only necessity for the compactness assumption on M), it is not hard to do.

□

We are now ready to prove Theorem B. As mentioned in the introduction, Theorem B is a corollary of the following more general result.

Theorem 7.4. *Let $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ be a C^2 family of C^2, partially hyperbolic diffeomorphisms having, for each $t \in (-\epsilon, \epsilon)$, a Tf_t-invariant splitting:*

$$TM = E^u(f_t) \oplus E^c(f_t) \oplus E^s(f_t).$$

Then there exists $\epsilon_0 > 0$ so that, for every $p \in M$ and every $v \in E^c(p)$, there exists a C^1 path

$$\varphi_p : (-\epsilon_0, \epsilon_0) \to M$$

with the following properties:

(1) $\varphi_{p,v}(0) = p$
(2) $\dot\varphi_{p,v}(0) \in v + E^u \oplus E^s$,
(3) *If*

$$TM = R(f_0) \oplus S(f_0)$$

is any dominated decomposition for f_0 satisfying the pointwise bunching condition

$$\sup_p \frac{\|T^S_p f_0\|}{\boldsymbol{m}(T^R_p f_0)\boldsymbol{m}(T^c_p f_0)} < 1, \tag{9}$$

then $t \mapsto R_{\varphi_{p,v}(t)}(f_t)$ is C^1.

Remark. E^c is allowed to be the trivial bundle in Theorem 7.4, in which case f_0 is Anosov. If f_0 is Anosov, then $\mathbb{E}^c$ is uniquely integrable, $\varphi_{p,0}$ is unique, and $p \mapsto \varphi_{p,0}(t)$ is the homeomorphism conjugating f_0 to f_t.

Remark. Similarly, if E^c is integrable and tangent to a plaque-expansive foliation $\mathcal{W}^c$, then $\mathbb{E}^c$ is also tangent to a foliation $\mathbb{W}^c$. The maps $p \mapsto \varphi_{p,0}(t)$ can be shown to be leaf conjugacies between f_0 and f_t.

If f_0 is r-normally-hyperbolic:

$$\|T^c f\|^r < \boldsymbol{m}(T^u f) \qquad \|T^s f\| < \boldsymbol{m}(T^c f)^r, \tag{10}$$

then the leaves of $\mathbb{W}^c$ are C^r. In this case, $t \mapsto \varphi_{p,v}(t)$ can also be chosen to be C^r.

If, in addition, the stronger center bunching condition:

$$\sup_p \frac{\|T^c_p f_0\|}{\boldsymbol{m}(T^u_p f_0)\boldsymbol{m}(T^c_p f_0)^r} < 1 \tag{11}$$

holds, then the C^r Section Theorem implies that $\mathbb{E}^u$ is C^r along the leaves of $\mathbb{W}^c$ (and so $t \mapsto E^u_{\varphi_{p,v}(t)}(f_t)$ is also C^r).

Remark. A simple refinement of the proof shows that both $t \mapsto R_{\varphi_{p,v}(t)}(f_t)$ and $t \mapsto \varphi_{p,v}(t)$ are $C^{1+\alpha}$, where there is a bound on the α-Hölder norm of the t-derivative that is uniform in p, v. The exponent α is determined by several bunching conditions.

Proof of Theorem 7.4. Let $\mathcal{F}$ and, for $(f,q) \in \mathcal{F} \times M$, the linear map $P_{f,q} : X_f \oplus E^c_q(f) \to E^u_q(f) \oplus E^s_q(f)$ be given by Lemma 7.1, so that

$$\mathbb{E}^c_{f,q} = \operatorname{graph}(P_{f,q}).$$

Choose $\epsilon_0 > 0$ so that $f_t \in \mathcal{F}$, for all $t \in (-\epsilon_0, \epsilon_0)$.

We identify the submanifold

$$\{f_t \times M \mid t \in (-\epsilon_0, \epsilon_0) \subset \mathcal{F} \times M$$

with $(-\epsilon_0, \epsilon_0) \times M$ in the obvious way. In the tangent bundle $\mathbf{R} \times TM$ to this manifold, $P_{f_t,q}$ and $\mathbb{E}^c_{f_t,q}$ have their counterparts $P_{t,q} : \mathbf{R} \oplus E^c_{f_t}(q) \to (E^u \oplus E^s)_{f_t}(q)$ and $\mathbb{E}^c_{t,q}$, where

$$\mathbb{E}^c_{t,q} = \operatorname{graph}(P_{t,q}).$$

Let p and v be given, and let V be any continuous vector field on M with the properties: $V(p) = v$ and $V(q) \in E^c(q)$, for all $q \in M$. Define a vector field Ω on $(-\epsilon_0, \epsilon_0) \times M$ by:

$$\Omega(t,q) = \frac{\partial}{\partial t} + V(q) + P_{t,q}(\frac{\partial}{\partial t} + V(q)).$$

Notice that $\Omega(t,q) \in \mathbb{E}^c_{t,q}$, for all $(t,q) \in (-\epsilon_0, \epsilon_0) \times M$. It follows from the bunching hypothesis and Corollary 7.3 that R is differentiable along the integral curves of Ω.

Let $\hat{\varphi}_{p,v}$ be any integral curve of Ω through $(p, 0)$. Now $\varphi_{p,v}$ is defined to be the M coordinate of $\hat{\varphi}_{p,v}$:

$$\hat{\varphi}_{p,v} = (t, \varphi_{p,v}).$$

It is straightforward to check that $\varphi_{p,v}$ satisfies (1)-(3). $\square$

7.1. **A series expansion for $\mathbb{E}^{cu}$.** We give a series expression for $\mathbb{E}^{cu}$ as follows. Define the linear map $P^{cu}_{f,p} : X \oplus E^{cu}_p(f) \to E^s_p(f)$ as the series

$$P^{cu}_{f,p}(g,v) = \sum_{k=0}^{\infty} T^s_{f^{-k}p} f^k(g^s(f^{-k}p)).$$

Note that the series does not depend on v. The domination conditions imply that the series converges. Under $T\,$Eval, the graph of $P^{cu}_{f,p}$ is sent to the graph of $P^{cu}_{f,fp}$. Hence, by uniqueness,

$$\mathbb{E}^{cu}_{f,p} = \operatorname{graph}(P^{cu}_{f,p}).$$

In the same way we get a unique $T\,$Eval-invariant subbundle $\mathbb{E}^{cs} \subset T(\mathcal{F} \times M)$ whose fiber at (f,p) projects isomorphically onto $X \oplus E^{cs}_p(f)$, and $\mathbb{E}^{cs}_{f,p} = \operatorname{graph}(P^{cs}_{f,p})$ where

$$P^{cs}_{f,p}(g,v) = \sum_{k=0}^{\infty} T^u_{f^k p} f^{-k}(g^u(f^k p)).$$

The intersection of these two subbundles is the center bundle $\mathbb{E}^c$. Namely, at (f,p), the fiber of $\mathbb{E}^c$ is the graph of the map $P^c_{f,p} : X \oplus E^c_p(f) \to (E^u_p(f) \oplus E^s_p(f))$, where

$$P^c_{f,p}(g,v) = \sum_{k=0}^{\infty} T^u_{f^k p} f^{-k}(g^u(f^k p)) + \sum_{k=0}^{\infty} T^s_{f^{-k}p} f^k(g^s(f^{-k}p)).$$

8. WHEN $f \mapsto E^u_p(f)$ ACTUALLY *is* DIFFERENTIABLE

We described in the previous section how $f \mapsto E^u_p(f)$ is generally not differentiable, even if f is Anosov. In fact, if $p \mapsto E^u_p(f_0)$ fails to be differentiable in even one direction at p_0, then $f \mapsto E^u_{p_0}(f)$ is not differentiable at f_0. For in that case, it is easy to construct a smooth 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $t \mapsto E^u_{\varphi_t p_0}(f_0)$ is not differentiable at $t = 0$; but then $E^u_{p_0}(\varphi_t f_0 \varphi_t^{-1}) = T\varphi_t(E^u_{\varphi_t p_0}(f_0))$ is not differentiable at $t = 0$.

It turns out that, under the usual center bunching hypothesis, nonsmoothness of $p \mapsto E^u_p(f_0)$ is the *only* obstruction to differentiability of $f \mapsto E^u_p(f)$ at f_0 in all directions.

The results that follow apply to 1-parameter families of C^2 diffeomorphisms $\{f_t\}_{t\in I}$ such that $t \mapsto f_t$ is a C^1 map from I into $\mathrm{Diff}^2(M)$ – a C^1 family of C^2 diffeomorphisms. Since the original proof of Theorem C is somewhat lengthy and the result is subsumed by Theorem D, we omit the proof of Theorem C and present instead a proof of Theorem D, following closely the approach of Dolgopyat in [2].

Assume that for each $t \in I$, f_t is partially hyperbolic with splitting

$$TM = E^u_t \oplus E^c_t \oplus E^s_t.$$

Write

$$E_t = E^c_t \quad H_t = E^u_t \oplus E^s_t.$$

Theorem 8.1 (Theorem D). *If E_0 is a C^1 bundle then the curve of subbundles $t \mapsto E_t$ is differentiable with respect to t at $t = 0$, and the derivative $(dE_t(x)/dt)_{t=0}$ depends continuously on $x \in M$.*

Remark. Theorem D remains valid, and the proof is the same, if the partially hyperbolic splitting is replaced by a dominated triple splitting $R_t \oplus S_t \oplus T_t$. Namely, the middle bundle S_t is differentiable with respect to t at $t = 0$, provided that $S_{x,0}$ is C^1. Similarly, there is nothing special about the one-dimensionality of the parameter t.

The following facts about weak continuity will be used. We assume that W is a Banach space, but that W also carries a weak topology. Of course, if W has finite dimension, the weak and strong topologies coincide. We have in mind the case that W is a space of operators on the the infinite dimensional Banach space of continuous sections of a vector bundle and $\Lambda = \mathbb{R}$.

Definition 8.2. A function $h : \Lambda \to W$ is **weakly continuous** at $\mu \in \Lambda$ if $h(\lambda)$ tends weakly to $h(\mu)$ and $\|h(\lambda)\|$ stays bounded as $\lambda \to \mu$.

Proposition 8.3 (Weak Inversion Rule). *If a curve of invertible operators $t \mapsto A_t$ is weakly continuous at $t = 0$ and if the operators' conorms are uniformly positive then the curve of inverse operators is also weakly continuous at $t = 0$.*

Proof. Let $t \mapsto A_t$ be the curve of operators, and let V be the Banach space on which they operate. Then, as $t \to 0$, A_t converges weakly to A_0 and $\|A_t - A_0\|$ stays bounded. The conorm assumption means that for all small t, $\|A_t^{-1}\| \le M$.

For each $v \in V$,

$$\begin{aligned} |A_t^{-1}(v) - A_0^{-1}(v)| &= |A_t^{-1} \circ (A_0 - A_t) \circ A_0^{-1}(v)| \\ &\le M|v - A_t(A_0^{-1}(v))|. \end{aligned}$$

Since $A_0^{-1}(v)$ is fixed, and A_t converges weakly to A_0, $A_t(A_0^{-1}(v)) \to v$ as $t \to 0$, which completes the proof that A_t^{-1} converges weakly to A_0^{-1} as $t \to 0$. But also,

$$\begin{aligned} |A_t^{-1}(v) - A_0^{-1}(v)| &= |A_t^{-1} \circ (A_0 - A_t) \circ A_0^{-1}(v)| \\ &\leq M\|A_0 - A_t\|M|v| \end{aligned}$$

implies that $\|A_t^{-1} - A_0^{-1}\|$ stays bounded as $t \to 0$, and completes the proof that the inverse curve is weakly continuous. $\square$

Now we return to the splitting $TM = E_t \oplus H_t$, where H_t is the hyperbolic part of the partially hyperbolic splitting for f_t, and E_t is the center part. We are assuming that $E = E_0$ is a C^1 bundle.

Let $\widetilde{H}$ be a smooth approximation to H_0, and express Tf_t with respect to the splitting $TM = E \oplus \widetilde{H}$ as

$$T_x f_t = \begin{bmatrix} A_{x,t} & B_{x,t} \\ C_{x,t} & K_{x,t} \end{bmatrix}.$$

Since f_t is a C^1 curve of C^2 diffeomorphisms, A, B, C, K are C^1 functions of x, t. At $t = 0$ we have

$$C_{x,0} = 0 \qquad \text{and} \qquad A_{x,0} = T_x f_0|_E$$

for all x. Furthermore, when $\widetilde{H}$ closely approximates H, $\|B\|$ is small. Consequently, if $P : E \to \widetilde{H}$ has norm ≤ 1 then $A + BP$ is invertible and the norm of its inverse is uniformly bounded. Uniformity refers to P, x, t.

Let $\mathcal{L}$ be the vector bundle over M whose fiber at x is $\mathcal{L}_x = L(E_x, \widetilde{H}_x)$. Equipping $\mathcal{L}_x$ with the operator norm gives $\mathcal{L}$ a Finsler; let $\mathcal{L}(1)$ be its unit ball bundle. Denote by $\mathrm{Sec}(\mathcal{L})$ the Banach space of continuous sections $X : M \to \mathcal{L}$, equipped with the sup norm $\|\ \ \|$. Its unit ball is $\mathrm{Sec}(\mathcal{L}(1))$.

Tf_t defines a graph transform

$$\begin{array}{ccc} \mathcal{L}(1) & \xrightarrow{(Tf_t)_\#} & \mathcal{L} \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow{f_t} & M \end{array}$$

according to the condition $T_x f_t(\operatorname{graph} P) = \operatorname{graph}((T_x f_t)_\#(P))$. That is,

$$(T_x f_t)_\#(P) = (C_{x,t} + K_{x,t}P) \circ (A_{x,t} + B_{x,t}P)^{-1},$$

which is a linear map $E_{f_t x} \to \widetilde{H}_{f_t x}$. The graph transform naturally induces a nonlinear map on the space of sections,

$$G_t : \mathrm{Sec}(\mathcal{L}(1)) \to \mathrm{Sec}(\mathcal{L})$$

such that

$$G_t(X) = (Tf_t)_\# \circ X \circ f_t^{-1}.$$

Proposition 8.4. *G_t is uniformly analytic.*

Remark. $(Tf_t)_\#$ is not analytic, it is only C^1. Nevertheless, for each fixed t, its action on the space of continuous sections is analytic. The uniformity refers to t.

We prove Proposition 8.4 by factoring G_t into a product of several analytic maps. Let $\mathcal{E}$, $\mathcal{E}_t$ and $\mathcal{E}_t^{-1}$ denote the bundles whose fibers at $x \in M$ are $\mathcal{E}_x = L(E_x, E_x)$, $\mathcal{E}_{x,t} = L(E_x, E_{f_t x})$, and $\mathcal{E}_{x,t}^{-1} = L(E_{f_t x}, E_x)$. Let $\mathcal{A}$, $\mathcal{A}_t$, and $\mathcal{A}_t^{-1}$ denote the invertible elements in $\mathcal{E}$, $\mathcal{E}_t$ and $\mathcal{E}_t^{-1}$, and denote sectional inversion as $\mathrm{Inv} : \mathrm{Sec}(\mathcal{A}) \to \mathrm{Sec}(\mathcal{A})$, $\mathrm{Inv}_t : \mathrm{Sec}(\mathcal{A}_t) \to \mathrm{Sec}(\mathcal{A}_t^{-1})$.

Lemma 8.5. *Sectional inversion is uniformly analytic.*

Proof. Consider the identity section Id of $\mathcal{A}$. Any section near Id is inverted by the power series

$$A^{-1} = \sum_{k=0}^{\infty} (\mathrm{Id} - A)^k,$$

and hence sectional inversion is analytic in a neighborhood of the identity section. For A in a neighborhood of the general section $A_0 : M \to \mathcal{A}$, sectional inversion factors according to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\text{Inv near } A_0"] \arrow[d, "L_{A_0^{-1}}"'] & A^{-1} \\ A_0^{-1}A \arrow[r, "\text{Inv near } Id"] & A^{-1}A_0 \arrow[u, "R_{A_0}"'] \end{tikzcd}$$

where $L_{A_0^{-1}}$ and R_{A_0} are left and right multiplication by the sections A_0^{-1} and A_0. Since $L_{A_0^{-1}}$ and R_{A_0} are continuous linear transformations of the section spaces, they are analytic, which completes the proof of the lemma for sections in a neighborhood of the identity section. The corresponding diagram

$$\begin{tikzcd} \mathrm{Sec}(\mathcal{A}_t) \arrow[r, "\mathrm{Inv}_t \text{ near } A_0"] \arrow[d, "L_{A_0^{-1}}"'] & \mathrm{Sec}(\mathcal{A}_t^{-1}) \\ \mathrm{Sec}(\mathcal{A}) \arrow[r, "\text{Inv near } Id"] & \mathrm{Sec}(\mathcal{A}) \arrow[u, "R_{A_0}"'] \end{tikzcd}$$

applies to sectional inversion in the neighborhood of a section $A_0 : M \to \mathcal{A}_t$, and shows that Inv_t is analytic.

Uniform analyticity means that for any r, the r^{th} derivative of Inv_t is uniformly bounded on sets of sections such that $\|A\|$ and $\|A^{-1}\|$ are uniformly bounded; this is clear from the higher order chain rule and the factorization of sectional inversion given above. $\square$

Proof of Proposition 8.4. We have $G_t(X) = (Tf_t)_\# \circ X \circ f_t^{-1}$, and must show that G_t is a uniformly analytic function of $X \in \mathrm{Sec}(\mathcal{L})$. We factor G_t as the Cartesian product of two affine maps on section spaces, followed by inversion in one of the two spaces, followed by sectional linear composition, all of which is expressed by commutativity of

$$\begin{tikzcd} \mathrm{Sec}(\mathcal{L}) \arrow[r, "G_t"] \arrow[d, "\mathrm{Aff}_1 \times \mathrm{Aff}_2"'] & \mathrm{Sec}(\mathcal{L}) \\ \mathrm{Sec}(\mathcal{L}_t) \times \mathrm{Sec}(\mathcal{A}_t) \arrow[r, "\mathrm{Id} \times \mathrm{Inv}_t"] & \mathrm{Sec}(\mathcal{L}_t) \times \mathrm{Sec}(\mathcal{A}_t^{-1}) \arrow[u, "\text{composition}"'] \end{tikzcd}$$

where $\mathcal{L}_t$ is the bundle over M whose fiber at x is $L(E_x, \widetilde{H}_{f_t x})$, and

$$\mathrm{Aff}_1(X) = C_t + K_t X \qquad \mathrm{Aff}_2(X) = A_t + B_t X.$$

Uniform analyticity of G_t then follows from Lemma 8.5. □

The r^{th}-order Taylor expansion of G_t at the zero section is

$$G_t(X) = Z_t + Q_t(X) + \cdots + \frac{1}{r!}(D^r G_t)_0(X^r) + R_t(X),$$

where $Z_t = G_t(0)$, $Q_t = (DG_t)_0$.

Proposition 8.6. *For small t,*

(a) $t \mapsto Z_t$ *is* C^1.
(b) $t \mapsto (I - Q_t)^{-1}$ *is weakly continuous.*
(c) $\|R_t(X)\|/\|X\|^2$ *is uniformly bounded for all small* $X \in \mathrm{Sec}(\mathcal{L})$.

Proof. At the zero section, the 0^{th} and first derivatives of

$$G_t(X) = (C_t + K_t X)(A_t + B_t X)^{-1} \circ f_t^{-1},$$

with respect to X are computed at once as

$$Z_t = (C_t A_t^{-1}) \circ f_t^{-1}$$
$$Q_t(X) = (K_t X A_t^{-1} + C_t A_t^{-1} B_t X A_t^{-1}) \circ f_t^{-1}$$

Since f_t is a C^1 curve of C^2 diffeomorphisms, and since the splitting $E \oplus \widetilde{H}$ is C^1, the curves $t \mapsto A_t$, $t \mapsto B_t$, $t \mapsto C_t$, $t \mapsto K_t$ in the appropriate bundles are C^1. This makes (a) immediate, and also shows that the curve $t \mapsto Q_t$ in $\mathrm{Sec}(\mathcal{L})$ is weakly continuous.

By inspection, at $t = 0$, Q_t becomes the hyperbolic operator

$$Q_0(X) = (K_0 X A_0^{-1}) \circ f_0^{-1},$$

because $C_{t=0} = 0$. Thus, for all small t, $I - Q_t$ is uniformly invertible, and Proposition 8.3 implies that $t \mapsto (I - Q_t)^{-1}$ is weakly continuous.

Assertion (c) follows from the Mean Value Theorem and the fact that the second derivative of G_t is uniformly bounded near the zero section. □

Proof of Theorem D. Proposition 8.6 implies that

$$G_t(X) = Z_t + Q_t(X) + R_t(X)$$

and $\|R_t(X)\| = O(1)\|X\|^2$ as $\|X\| \to 0$. Let $P_t : x \mapsto P_{x,t}$ be the unique G_t-invariant section of $\mathcal{L}$ with norm ≤ 1. Thus $P_{x,t} : E_x \to \widetilde{H}_x$ and

$$E_{x,t} = \operatorname{graph} P_{x,t} = \{v + P_{x,t}(v) \in T_x M : v \in E_x\}.$$

Theorem D asserts that E_t is differentiable at $t = 0$. That is,

$$\left.\frac{dP_{x,t}}{dt}\right|_{t=0}$$

exists and is continuous with respect to x.

Plugging $X = P_t$ into the Taylor expansion of G_t gives

$$P_t = Z_t + Q_t(P_t) + R_t(P_t),$$

and since $I - Q_t$ is invertible, we get

$$P_t = (I - Q_t)^{-1}(Z_t + R_t(P_t)).$$

Thus

$$\|P_t\| \leq \|(I - Q_t)^{-1}\|(\|Z_t\| + M\|P_t\|^2). \tag{12}$$

(These norms refer to section sup-norms or to operator norms, as appropriate.)

Now we estimate $Z_t = (C_t \circ A_t^{-1}) \circ f_t^{-1}$ as follows. It is differentiable with respect to t, and since $C_{t=0} = 0$, we have $Z_{t=0} = 0$. Thus $\|Z_t\| = O(1)t$ as $t \to 0$. Since P_t is continuous in t, and $P_0 = 0$, we get $\|P_t\|_0^2 \ll \|P_t\|_0$ when t is small, which lets us absorb the squared term into the l.h.s. of the inequality (12), so

$$\|P_t\| = O(1)t$$

as $t \to 0$. Consequently, we get a bootstrap effect on the remainder:

$$\|R_t(P_t)\| = O(1)t^2$$

as $t \to 0$. Combined with the more exact estimate on Z_t,

$$Z_t = tZ_0' + o(1)t$$

where $Z_0' = (d/dt)_{t=0}(Z_t)$, this gives

$$\frac{P_t}{t} = (I - Q_t)^{-1}Z_0' + (I - Q_t)^{-1}(o(1) + O(1)t).$$

Proposition 8.6 implies that $(I - Q_t)^{-1}$ converges weakly to $(I - Q_0)^{-1}$ as $t \to 0$, so

$$\lim_{t \to 0}(I - Q_t)^{-1}Z_0' = (I - Q_0)^{-1}Z_0',$$

while uniform boundedness of $\|(I - Q_t)^{-1}\|$ implies that

$$\lim_{t \to 0}(I - Q_t)^{-1}(o(1)) + O(1)t) = 0.$$

Thus, as $t \to 0$,

$$\frac{P_{x,t} - P_{x,0}}{t} \to (I - Q_0)^{-1}Z_0',$$

uniformly in $x \in M$, which completes the proof that $t \mapsto E_t$ is differentiable at $t = 0$, and that its derivative there, $(I - Q_0)^{-1}Z_0'$, depends continuously on $x \in M$. □

Remark. Suppose that E_0 and Df_t are C^r, $r \geq 2$. We tried to show that E_t is r^{th}-order differentiable at $t = 0$ in the sense that there is an r^{th} order Taylor expansion for E_t at $t = 0$. Many ingredients of the preceding proof of the $r = 1$ case above generalize very nicely to $r \geq 2$. There is a natural notion of weak r^{th}-order differentiability, and it behaves well with respect to operator inversion and operator products. However, we would also need affirmative answers to the following two questions:

(a) Is the curve $t \mapsto (I - Q_t)^{-1}$ in $\mathrm{Sec}(\mathcal{L})$ weakly differentiable at $t = 0$?
(b) Does the operator $(I - Q_0)^{-1}$ send C^1 sections of $\mathcal{L}$ to C^1 sections?

At first, it would be acceptable to assume analyticity of E_0 and f_t.

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