Stable accessibility is C^1 dense

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Abstract

We prove that in the space of all C^r $(r \geq 1)$ partially hyperbolic diffeomorphisms, there is a C^1 open and dense set of accessible diffeomorphisms. This settles the C^1 case of a conjecture of Pugh and Shub. The same result holds in the space of volume preserving or symplectic partially hyperbolic diffeomorphisms.

Combining this theorem with results in [Br], [Ar] and [PugSh3], we obtain several corollaries. The first states that in the space of volume preserving or symplectic partially hyperbolic diffeomorphisms, topological transitivity holds on an open and dense set. Further, on a symplectic n-manifold $(n \leq 4)$ the C^1-closure of the stably transitive symplectomorphisms is precisely the closure of the partially hyperbolic symplectomorphisms. Finally, stable ergodicity is C^1 open and dense among the volume preserving, partially hyperbolic diffeomorphisms satisfying the additional technical hypotheses of [PugSh3] .

Introduction

This paper is about the accessibility property of partially hyperbolic diffeomorphisms. We show that accessibility holds for a C^1 open and dense set in the space of all partially hyperbolic diffeomorphisms, thus settling the C^1 version of a conjecture of Pugh and Shub [PugSh1]. Partially hyperbolic diffeomorphisms are similar to Anosov diffeomorphisms, in that they possess invariant hyperbolic directions, but unlike Anosov diffeomorphisms, they can also possess invariant directions of non-hyperbolic behavior. Accessibility means that the hyperbolic directions fill up the manifold on a macroscopic scale. Accessibility often provides enough hyperbolicity for a variety of chaotic properties, such as topological transitivity [Br] and ergodicity [PugSh3], to hold. As a consequence, we derive several density results about stable ergodicity and stable transitivity among partially hyperbolic diffeomorphisms.

Let M be a smooth compact, connected and boundaryless Riemannian manifold. A diffeomorphism $f : M \to M$ is *partially hyperbolic* if the tangent bundle to M splits as a Tf-invariant sum

$$TM = E^u \oplus E^c \oplus E^s,$$

such that Tf uniformly expands all vectors in E^u and uniformly contracts all vectors in E^s, while vectors in E^c are neither contracted as strongly as any vector in E^s nor expanded as strongly as any vector in E^u. More precisely, for each $p \in M$, there exist $0 < a_p < b_p < 1 < B_p < A_p$ such that:

$$\|T_p f|_{E^s}\| \leq a_p < b_p \leq m(T_p f|_{E^c}) \leq \|T_p f|_{E^c}\| \leq B_p < A_p \leq m(T_p f|_{E^u}),$$

where $m(T) = \|T^{-1}\|^{-1}$. Throughout this paper we assume that both subbundles E^u and E^s are nontrivial.

A more stringent condition, often called partial hyperbolicity in the literature (cf. [BrPe], [BuPuShWi]) requires that the constants a_p, b_p, A_p and B_p be chosen independent of p. Since the results in this paper apply to diffeomorphisms satisfying the weaker condition, to avoid excessive terminology, we will use the term partial hyperbolicity in the broader sense.

A partially hyperbolic diffeomorphism f is *accessible* if, for every pair of points $p, q \in M$, there is a C^1 path from p to q whose tangent vector always lies in $E^u \cup E^s$ and vanishes at most finitely many times. We say f is *stably accessible* if every g sufficiently C^1-close to f is accessible. We prove here the following theorem.

Main Theorem: *For any $r \geq 1$, stable accessibility is C^1 dense among the C^r, partially hyperbolic diffeomorphisms of M, volume preserving or not. If M is a symplectic manifold, then stable accessibility is C^1 dense among C^r, symplectic partially hyperbolic diffeomorphisms of M.*

Related to the Main Theorem is the result of Niţică and Török [NiTö] that stable accessibility is C^r-dense among partially hyperbolic diffeomorphisms with 1-dimensional, integrable center bundle E^c. Other results about stable accessibility treat more special classes of diffeomorphisms, such as time-one maps of Anosov flows [BuPuWi], skew products [BuWi1], certain systems where $E^u \oplus E^s$ is integrable [ShWi], and systems whose partially hyperbolic splitting is C^1 [PugSh2].

The Main Theorem has several corollaries. The first corollary concerns the topological transitivity of partially hyperbolic diffeomorphisms and follows immediately from a theorem of Brin [Br]. Denote by $\mathcal{P}H^r(M)$ the set of C^r partially hyperbolic diffeomorphisms of M. If μ and ω are, respectively, Riemannian volume and a symplectic form on M, then set

$$\begin{aligned} \mathcal{P}H^r_\mu(M) &= \{f \in \mathcal{P}H^r(M) \mid f_*(\mu) = \mu\}, \text{ and} \\ \mathcal{P}H^r_\omega(M) &= \{f \in \mathcal{P}H^r(M) \mid f^*(\omega) = \omega\}. \end{aligned}$$

Corollary 0.1 *For $r \geq 1$, there is a C^1-open and dense set of topologically transitive diffeomorphisms in $\mathcal{P}H^r_\mu(M)$. If M has a symplectic form ω, then there is a C^1-open and dense set of transitive diffeomorphisms in $\mathcal{P}H^r_\omega(M)$.*

This corollary is false without the volume preservation assumption. Niţică and Török have shown in [NiTö] that there is an open set of accessible non-transitive diffeomorphisms. While it is plausible that for a C^1 open and dense set of diffeomorphisms in the space $\mathcal{P}H^r(M)$, there are only finitely many transitivity components, it is not a direct corollary of the Main Theorem.

M.-C. Arnaud has shown in [Ar] that if M is a symplectic 4-manifold, then the stably transitive diffeomorphisms in $\mathrm{Diff}^r_\omega(M)$ are partially hyperbolic. (The same result has been announced by J. Xia in arbitrary dimension). Hence there is a complete picture in dimension 4 of the stably transitive diffeomorphisms, which we summarize in the next corollary.

Corollary 0.2 *Let M be a symplectic manifold with $dim(M) \leq 4$. The C^1 closure of the stably transitive diffeomorphisms in $Diff^r_\omega(M)$ coincides with the C^1 closure of the partially hyperbolic ones.*

In other words, invariant tori are essentially the only obstacle for topological transitivity in the symplectic category, at least if $\dim(M) \leq 4$. We conjecture that the same is true in the volume preserving case.

Conjecture 0.3 *In the space of volume preserving diffeomorphisms, the C^1-closure of the stably transitive diffeomorphims coincides with the closure of the diffeomorphisms admitting a dominated splitting.*

For a discussion of the dominated splitting condition and some results related to Conjecture 0.3 see [Vi]. Even though the results of this paper could be useful in attacking this conjecture some other ideas (possibly ones from the paper [BonDi]) are necessary to solve this problem. Here we note only that in [BV] a volume preserving example is presented which is stably transitive yet not partially hyperbolic. A. Tahzhibi has announced a proof that these example are in fact stably ergodic.

Another corollary of the Main Theorem concerns ergodicity of $f \in \mathcal{PH}^r_\mu(M)$. Pugh and Shub proved the following theorem:

Theorem 0.4 [PugSh3, Theorem A] *Let $f \in \mathcal{PH}^2_\mu(M)$. If f is center bunched, dynamically coherent, and essentially accessible, then f is ergodic.*

Thus we also have the corollary:

Corollary 0.5 *Among the center bunched, stably dynamically coherent diffeomorphisms in $\mathcal{PH}^2_\mu(M)$, stable ergodicity is C^1 open and dense.*

Theorem 0.4 refers to partially hyperbolic diffeomorphisms in the stronger sense described earlier, but recently Burns and Wilkinson [BuWi2] have shown that these results extend to the larger class of partially hyperbolic diffeomorphisms described in this paper (satisfying additional center bunching conditions). For a description of examples of diffeomorphisms satisfying the conditions "center bunched" and "stably dynamically coherent" see the survey paper [BuPuShWi]. In particular, the corollary implies that there is a C^1-open neighborhood $\mathcal{U} \subset \mathcal{PH}^2_\mu(M)$ of f in which stable ergodicity is C^1-open and dense, where f is the time-t map of an Anosov flow, a compact group extension of an Anosov diffeomorphism, an ergodic automorphism of a torus or nilmanifold, or a partially hyperbolic translation on a compact homogeneous space.

This paper arose out of an attempt to prove the following conjecture of Pugh and Shub.

Conjecture 0.6 [PugSh2, Conjecture 4] and [PugSh3, Conjecture 2] *Stable accessibility is C^r - dense in both $\mathcal{PH}^r(M)$ and $\mathcal{PH}^r_\mu(M)$.*

In the spirit of Theorem 0.4, Pugh and Shub also conjectured:

Conjecture 0.7 [PugSh3, Conjecture 3] *A partially hyperbolic C^2 volume preserving diffeomorphism with the essential accessibility property is ergodic.*

Finally, combining Conjectures 0.6 and 0.7, they conjectured:

Conjecture 0.8 [PugSh3, Conjecture 4] *Stable ergodicity is C^r - dense in $\mathcal{PH}^r_\mu(M)$.*

As with Theorem 0.4, these conjectures refer to the narrower class of partially hyperbolic diffeomorphisms described above, but in light of the results in [BuWi2] and this paper, it seems reasonable to extend them to the class under consideration here.

The question of accessibility is closely related to problems in control theory (see, e.g. [Lo]). In fact, analogous density theorems in control theory initially suggested the Conjectures 0.6 and 0.7. The sole reason that the results in control theory cannot be directly transported to this setting is that we do not perturb the bundles E^u and E^s directly, but rather the diffeomorphism f. We'd like to be able to say that a specific perturbation of f has a specific effect on E^u and E^s. What makes this difficult is that $E^s(p)$ and $E^u(p)$ are determined by the entire forward and backward orbit of p, respectively; a perturbation will have various effects along the length of this orbit, some desirable and others not.

The key observation that permits a measure of control is that the effects of the perturbation are greatest along the first few iterates of p. To maximize our control over the bundles E^u and E^s, we isolate regions where we need local accessibility and localize the perturbation to these regions. Choosing the support of the perturbation to be highly non-recurrent then minimizes undesirable "noisy" effects of the perturbations. The trade-off is that the desirable effects of the pertubations are necessarily quite small. Nonetheless, with the right C^1-small perturbation, the desirable effects outweigh the undesirable ones and we obtain accessibility. Similar perturbations are found in [PP].

It appears that a localized C^2-small perturbation cannot achieve this, and so the techniques in this paper do not extend to the C^2 setting. New techniques would be required to prove Conjectures 0.6 and 0.7.

Here is how the proof of the Main Theorem goes. Let $f \in \mathcal{PH}^r(M)$ have partially hyperbolic splitting $TM = E^u \oplus E^c \oplus E^s$. By [BrPe, HiPuSh] E^u and E^s are tangent to the leaves of continuous foliations which are denoted $\mathcal{W}^u$ and $\mathcal{W}^s$ and are called the unstable and stable foliations respectively. A *us-path for f* is a path $\gamma : [0,1] \to M$ consisting of a finite number of consecutive arcs — called *legs* — each of which is a curve that lies in a single leaf of $\mathcal{W}^u$ or $\mathcal{W}^s$. It is easy to see that f is accessible if and only if for all $p, q \in M$, there is a us-path for f from p to q.

To prove the Main Theorem, we first find a collection of disjoint disks in M. Each disk is approximately tangent to the center direction E^c. We choose this collection large enough so that f is accessible, modulo these disks. More precisely, for every $p, q \in M$, there is a finite sequence of us-paths for f, the first path originating at p and ending in one of the disks, the last path originating in a disk and ending at q. The intermediate paths all begin and end in disks, each path beginning in the disk where the previous path ends. We then perturb f in a small neighborhood of these disks. We can arrange that if this neighborhood and the C^0-size of the pertubation are both sufficiently small, then the perturbed system will still be accessible, modulo the same collection of disks. It is not hard to see that any additional C^1-small perturbation will preserve this property.

Under the C^r-dense assumption that the fixed points of f^k are isolated, for all $k \geq 1$, we can choose these disks to be very small and their union highly non-recurrent. This is Lemma 1.2. We show in Lemma 1.1 that it is then possible to perturb f in a neighborhood of these disks by a C^1-small perturbation to obtain a stably accessible g. We prove stable accessibility by showing that any two points in a given disk can be connected by a us-path for g, and for any small perturbation of g. Since any small perturbation of g is already accessible modulo these disks, this gives stable accessibility.

Lemma 1.1 is the only place where it is essential that the perturbation be only C^1-small. When we examine the effect of perturbing f on E^u and E^s, we find that in C^1, the contribution to $E^u(p)$ and $E^s(p)$ of the perturbation *near* p is larger than the *combined* contributions along the rest of the orbit of p; this is not true in C^2. Therefore, a more complicated analysis, taking into account the first several returns, is needed to establish the analogue of our result in the C^2-setting.

1 Proof of the Main Theorem

Proof: We first prove the Main Theorem in the case where f preserves a smooth volume μ. The proof is easily modified to the non volume preserving case. In the final section we describe how to modify the proof for the symplectic case.

Let $f \in \mathcal{PH}^r_\mu(M)$ and $\delta > 0$ be given. Our goal is to find a stably accessible $g \in \mathrm{Diff}^r_\mu(M)$ with $d_{C^1}(f,g) < \delta$. We give some preliminary definitions and notation.

Let $\mathcal{P}(M)$ be the collection of all subsets of M. We say that f is *accessible on* $X \in P(M)$ if, for every $p, q \in X$, there is a us-path for f from p to q. The diffeomorphism f is *accessible modulo* $\mathcal{X} \subset \mathcal{P}(M)$ if, for every $p, q \in M$, there exist $X_1, \ldots, X_r \in \mathcal{X}$ and us-paths for f: from p to X_1, from q to X_r, and from X_m to X_{m+1}, for $m = 1, \ldots, r-1$. We say that f is *uniformly accessible modulo* $\mathcal{X}$ if f is accessible modulo $\mathcal{X}$ and, further, there is a number N such that the us-paths in the previous definition can be chosen to have length less than N and to have fewer than N legs.

A property of a diffeomorphism f is said to hold *stably* if it also holds for all $g \in \mathrm{Diff}^1(M)$ sufficiently C^1-close to f. Clearly if f is stably accessible modulo $\{X_1, \ldots, X_k\}$ and f is stably accessible on each X_i, then f is stably accessible.

Define a function $R : \mathcal{P}(M) \to \mathbf{N} \cup \{\infty\}$ as follows. For $X \in \mathcal{P}(M)$, let $R(X)$ be the smallest $J \in \mathbf{N} \cup \{\infty\}$ satisfying:

$$f^i(X) \cap X \neq \emptyset, \quad \text{with} \quad |i| = J + 1. \tag{1}$$

Note that $R(B_\rho(p)) \to \mathrm{per}(p)$, as $\rho \to 0$, where we set $\mathrm{per}(p) = \infty$ if p is not periodic.

We next fix a system of local charts on M. We will on several occasions refer to the orthogonal splitting $\mathbf{R}^n = T_v\mathbf{R}^n = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$, where $a = \dim(E^a)$.

Let $B^n(v, \rho)$ denote the ball of radius ρ about $v \in \mathbf{R}^n$ with respect to the sup norm on coordinates. More generally, we will use the notation $B^a(v, \rho)$, where $a = u, c, s, c+u, c+s$, or $u+s$, to denote the sup-norm ball of radius ρ about v in the affine space $v + \mathbf{R}^a$.

Applying Moser's theorem on the equivalence of volume forms [Mo] we obtain, for any $p \in M$, a C^∞ map

$$\varphi_p : B^n(0,1) \to M$$

such that

1. $\varphi_p(0) = p$,

2. $T_0\varphi_p$ sends the splitting $T_0\mathbf{R}^n = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$ to the splitting $T_pM = E^u \oplus E^c \oplus E^s$,

3. φ_p sends divergence-free vector fields to divergence-free vector fields.

4. $p \mapsto \varphi_p$ is a uniformly continuous map from M to $C^1(B^n(0,1), M)$. The dependence of φ_p on f is also continuous.

([Mo] gives maps satisfying 1, 3 and 4. 2 can be achived by precomposing with a linear map.)

Since we do not assume that E^c is tangent to a foliation, we will work with approximate center manifolds. For $\rho < 1$ and $p \in M$, let

$$V_\rho(p) = \varphi_p(B^c(0,\rho)).$$

We refer to $V_\rho(p)$ as a *c-admissible disk* with *center* p and *radius* ρ and write $r(V_\rho(p)) = \rho$. If D is a c-admissible disk with center p and radius ρ, then for $\beta \in (0,1)$, we denote by βD the c-admissible disk with center p and radius $\beta\rho$.

A *c-admissible family* is a finite collection of pairwise disjoint, c-admissible disks. If $\mathcal{D}$ is a c-admissible family, and $\beta < 1$, then let

$$\begin{aligned} \beta\mathcal{D} &= \{\beta D \mid D \in \mathcal{D}\}, \\ |\mathcal{D}| &= \bigcup_{D\in\mathcal{D}} D, \\ r(\mathcal{D}) &= \sup_{D\in\mathcal{D}} r(D), \text{ and} \\ R(\mathcal{D}) &= R(|\mathcal{D}|). \end{aligned}$$

We have the following lemma.

Lemma 1.1 (Accessibility on central disks) *Let $f \in \mathcal{P}H^r_\mu(M)$ and $\delta > 0$ be given. Then there exists $J > 0$ with the following property.*

If $\mathcal{D}$ is a c-admissible family with $r(\mathcal{D}) < J^{-1}$ and $R(\mathcal{D}) > J$, then for all $\sigma > 0$ and $\beta \in (0,1)$, there exists $g \in \mathit{Diff}^r_\mu(M)$ such that:

1. $d_{C^1}(f,g) < \delta$,

2. $d_{C^0}(f,g) < \sigma$,

3. For each $D \in \mathcal{D}$, g is stably accessible on βD.

We may assume that the fixed points of f^k are isolated, for all $k \geq 1$; this property is C^r-dense in $\mathrm{Diff}^r_\mu(M)$. Under this additional assumption we have the following lemma.

Lemma 1.2 (Accessibility modulo central disks) *Let $f \in \mathcal{P}H^r_\mu(M)$ be given. Assume that the fixed points of f^k are isolated, for all $k \geq 1$. Then for every $J > 0$ there exists a c-admissible family $\mathcal{D}$ such that:*

1. *$r(\mathcal{D}) < J^{-1}$,*

2. *$R(\mathcal{D}) > J$,*

3. *f is uniformly accessible modulo $\frac{1}{2}\mathcal{D}$.*

Lemma 1.3 (Persistence of accessibility modulo $\mathcal{D}$) *There exists $\delta_0 > 0$ so that, given $\delta < \delta_0$, a c-admissible family $\mathcal{D}$ with f uniformly accessible modulo $\frac{1}{2}\mathcal{D}$, and $\beta \in (\frac{1}{2}, 1)$, the following holds.*

There exists $\sigma > 0$ such that any g satisfying

1. *$d_{C^1}(f, g) < \delta$,*

2. *$d_{C^0}(f, g) < \sigma$,*

is accessible modulo $\beta\mathcal{D}$ (and hence g is stably accessible modulo $\beta\mathcal{D}$, since 1. and 2. are open conditions).

The proof of the Main Theorem now follows from Lemmas 1.1, 1.2 and 1.3. Let f and δ be given. After a C^r-small perturbation, we may assume that the fixed points of f^k are isolated, for all k. We may assume that $\delta < \delta_0$, where δ_0 is given by Lemma 1.3

Choose J according to Lemma 1.1. By Lemma 1.2, there exists a c-admissible family $\mathcal{D}$, with $R(\mathcal{D}) > J$ and $r(\mathcal{D}) < J^{-1}$, such that f is accessible modulo $\frac{1}{2}\mathcal{D}$.

Now fix $\beta \in (\frac{1}{2}, 1)$, and choose σ according to Lemma 1.3. Applying Lemma 1.1 we obtain a diffeomorphism $g \in \mathrm{Diff}^r_\mu(M)$, with $d_{C^1}(f, g) < \delta$ and $d_{C^0}(f, g) < \sigma$, such that g is stably accessible on βD, for each $D \in \mathcal{D}$. By Lemma 1.3, g is also stably accessible modulo $\beta\mathcal{D}$. Thus, g is stably accessible.$\diamondsuit$

The proofs of Lemmas 1.2 and 1.3 are given in the next section, and the proof of Lemma 1.1 is given in Section 3.

The arguments of Section 2 become simpler if $E^c(f)$ is integrable. In that case, one can work with central disks instead of c-admissible ones. To construct a family of central disks satsifying the conditions of Lemma 1.2 one should take a small $\varepsilon > 0$, choose an ε–net $\{p_j\}$ and let $\mathcal{D}$ be the union of unit central disks centered at p_j. If there are some $i, j < \mathrm{Card}(\mathcal{D})$ and $n < J$ such that $f^n D(p_i) \bigcap D(p_j) \neq \emptyset$ then one can remove this intersection by arbitrary small shift of p_i in an unstable direction (this is possible even if $i = j$ since the periodic leaves of f are isolated by partial hyperbolicity). In case E^c is not integrable the proof becomes significantly more complicated since one has to work with disks which are almost tangent to E^c, but the idea remains the same. As it is mentioned in the introduction the most difficult part of the proof is Section 3 where the abundance of C^1-perturbations is crucial. Therefore many readers would find it helpful to skip Section 2 during the first reading returning to it after mastering Section 3.

2 Global Accessibility

In this section we prove Lemmas 1.2 and 1.3.

Proof of Lemma 1.2: Let J be given. Let $A = \{p \in M \mid \mathrm{per}(p) \geq J+2\}$. Since the fixed points of f^{J+2} are isolated, there exist $x_1, \ldots, x_m \in M$ such that $A = M \setminus \{x_1, \ldots, x_m\}$.

For $\rho > 0$, let $U_\rho(p)$ be the image of the ball $B^n(0, \rho)$ under φ_p. The proof of the following lemma is straightforward.

Lemma 2.1 *If $r > 0$ is sufficiently small, then every $p \in \bigcup U_r(x_i)$ can be connected to a point in $M \setminus \bigcup U_r(x_i)$ by a us-path with one leg.*

Choose r according to this lemma, and let $A_r = M \setminus \bigcup U_r(x_i)$. Assume r is small enough that A_r is connected. Since A_r is compact and contained in A, there exists $\rho_0 < 1/(2J)$ such that $R(U_{2\rho_0}(q)) > J$, for every $q \in A_r$.

The next lemma follows from the uniform continuity of the invariant splitting for f, and we omit its proof.

Lemma 2.2 *There exists $K > 1$ so that, for ρ_0 sufficiently small, for all $p \in M$, and for every $q_1, q_2 \in U_{\rho_0/K}(p)$, there is a us-path with ≤ 2 legs from q_1 to some point in $V_{\rho_0}(q_2)$.*

The next lemma is key.

Lemma 2.3 *Let $K > 1$. If ρ_0 is sufficiently small, then there exist a cover of A_r by finitely many neighborhoods $U_1, \ldots, U_k$ of the form $U_i = U_{\rho_0/K}(q_i)$, and, for $i = 1, \ldots, k$, points $p_i \in U_i$ such that*

$$V_{2\rho_0}(p_i) \cap V_{2\rho_0}(p_j) = \emptyset,$$

for $i \neq j$, *and*

$$V_{2\rho_0}(p_i) \cap f^m\left(V_{2\rho_0}(p_j)\right) = \emptyset,$$

for all i, j *and* $0 < |m| \leq J$.

Before proving Lemma 2.3, we finish the proof of Lemma 1.2.

Let $U_1, \ldots, U_k$ and $p_1, \ldots, p_k$ be given by Lemma 2.3. For $i = 1, \ldots, c$, let $D_i = V_{2\rho_0}(p_i)$. Note that the collection $\mathcal{D} = \{D_1, \ldots, D_k\}$ satisfies conclusions 1. and 2. of Lemma 1.2, if ρ_0 is sufficiently small.

Lemma 2.2 implies that, for every $p \in B_i$, there is a us-path with ≤ 2 legs from p to some point in $\frac{1}{2}D_i$. It follows that, whenever $B_i \cap B_j \neq \emptyset$, there is a us-path with ≤ 4 legs from some point on $\frac{1}{2}D_i$ to some point in $\frac{1}{2}D_j$. Since A_r is connected and the balls $B_i, \ldots, B_k$ cover A_r, we obtain, for *any* i, j, a sequence of disks $D_{a_0} = D_i, D_{a_1}, \ldots, D_{a_l} = D_j$, such that $\frac{1}{2}D_{a_m}$ is connected to $\frac{1}{2}D_{a_{m+1}}$ by a us-path, for $m = 0, \ldots, l-1$. Then for any $p, q \in A_r$, there are a sequence of disks $D_{b_0}, \ldots, D_{b_s}$ and us-paths: from p to $\frac{1}{2}D_{b_0}$, from q to $\frac{1}{2}D_{b_s}$, and from $\frac{1}{2}D_{b_r}$ to $\frac{1}{2}D_{b_{m+1}}$, for $m = 0, \ldots, s-1$. The length and number of legs of these paths is clearly bounded. Since any point in $M \setminus A_r = \bigcup B_r(x_i)$ can be connected to a point in A_r by a us-path with one leg, it follows that f is uniformly accessible modulo $\{\frac{1}{2}D_1, \ldots, \frac{1}{2}D_k\}$. This proves 3., completing the proof of Lemma 1.2.$\diamondsuit$

Proof of Lemma 2.3: We start with a simple Besicovich-type covering lemma.

Lemma 2.4 (Covering lemma) *For any* $C > 0$ *there exists an integer* $N > 0$ *such that, for any compact set* $A \subseteq M$, *and for* $\rho > 0$ *sufficiently small, there exist* $q_1, \ldots, q_k \in A$ *with the following properties, for* $i = 1, \ldots, k$*:*

- $A \subseteq B_\rho(q_1) \cup \cdots \cup B_\rho(q_k)$, *and*
- $\#\{j \mid B_{C\rho}(q_i) \cap B_{C\rho}(q_j) \neq \emptyset\} \leq N$.

Proof of Lemma 2.4: On the manifold M, there exists a constant $K > 0$ such that for every $\rho < 1$ and every $p \in M$, the volume of the ball $B_\rho(p)$ lies between ρ^n/K and $K\rho^n$. Let $N = (4C+2)^n K^2$; this is an upper bound on the number of disjoint balls of radius $\rho/2$ that can fit inside a ball of radius $(2C+1)\rho$.

Let A and ρ be given. Let $S_\rho \subset M$ be a maximal ρ-separated subset of A. Such a set exists by compactness of A. We claim that S_ρ is also ρ-spanning.

If not, then there exists $x \in A$ such that $d(x,y) > \rho$, for all $y \in A$. But this contradicts maximality of S_ρ. Hence, if $q_1, \ldots, q_k$ are the elements of S_ρ, then

$$A \subseteq B_\rho(q_1) \cup \cdots \cup B_\rho(q_k).$$

For $p \in S_\rho$, let $N(p)$ be the set of $q \in S_\rho$ such that

$$B_{C\rho}(p) \cap B_{C\rho}(q) \neq \emptyset.$$

For each $q \in N(p)$, the distance from p to q is less than $2C\rho$, and so the ball $B_{\rho/2}(q)$ is contained in $B_{(2C+1)\rho}(p)$. Since S_ρ is ρ-separated, the balls $B_{\rho/2}(q)$ and $B_{\rho/2}(q')$ are disjoint, for distinct $q, q' \in N(p)$. The cardinality of $N(p)$ is thus bounded by N, which completes the proof.$\diamondsuit$

The sets $U_\rho(p)$ are uniformly comparable to round balls $B_\rho(p)$, and the maps $\{f^m, \, |m| \leq J\}$ distort distances by a bounded factor. Thus Lemma 2.4 implies the following.

Corollary 2.5 (Strengthened covering lemma) *Let $C, J > 0$ be given. There exists an integer $N > 0$ such that, for any compact set $A \subseteq M$, and for any $\rho > 0$, there exist $q_1, \ldots, q_k \in A$ with the following properties, for $i = 1, \ldots, k$:*

- *$A \subseteq U_\rho(q_1) \cup \cdots \cup U_\rho(q_k)$, and*
- *$\#\{j \mid U_{C\rho}(q_i) \cap f^m(U_{C\rho}(q_j)) \neq \emptyset, \text{ for some } |m| < J\} \leq N$.*

We now return to the proof of Lemma 2.3. To simplify notation, let $\rho_1 = \rho_0/K$ and let $\rho_2 = 4\rho_0$, where ρ_0 is defined after the statement of Lemma 2.1. Thus $\rho_1 < \rho_0 < \rho_2$ In this notation, we have that $R(U_{\rho_2}(p)) > J$, for all p in A_r.

For $p \in M$ and $\rho > 0$, let $T_\rho(p)$ be the connected component of p in $\varphi_p(B^{u+s}(0,\rho))$. For $d(p,q)$ small enough, the maps $\varphi_q^{-1}\varphi_p$ distort the Euclidean structure by a factor ≤ 1.5. Assume that ρ_2 is small enough that this distortion bound holds whenever $p, q \in U_{\rho_2}(z)$, for any z. From this we obtain that for all $p \in M$ and all $q \in T_{\rho_1}(p)$,

$$V_{2\rho_0}(q) \subset U_{\rho_1 + 3\rho_0}(p) \subset U_{\rho_2}(p). \tag{2}$$

We now apply Corollary 2.5 with $C = 4K$, $A = A_r$, and $\rho = \rho_1$. By Corollary 2.5, there exists $N > 0$ and $q_1, \ldots, q_k \in A_r$ such that

- $A_r = U_{\rho_1}(q_1) \cup \cdots \cup U_{\rho_1}(q_k)$, and
- $\#\{j \mid U_{\rho_2}(q_i) \cap f^m\left(U_{\rho_2}(q_j)\right) \neq \emptyset, \text{ for some } |m| < J\} \leq N$.

For $i = 1, \ldots k$, let $U_i = U_{\rho_1}(q_i)$. The neighborhoods $U_1, \ldots, U_k$ cover A_r.

We choose $p_1, \ldots p_k$ inductively. Let $p_1 = q_1$. Since $V_{2\rho_0}(p_1) \subset U_{\rho_2}(p_1)$, and $R(U_{\rho_2}(p_1)) > J$, we have that

$$V_{2\rho_0}(p_1) \cap f^m(V_{2\rho_0}(p_1)) = \emptyset,$$

for $0 < |m| \leq J$.

Fix $i > 1$, and suppose that the points $p_1, \ldots, p_{i-1}$ have already been chosen. We want to choose p_i so that

$$V_{2\rho_0}(p_i) \cap f^m(V_{2\rho_0}(p_i)) = \emptyset,$$

for $0 < |m| \leq J$, and

$$V_{2\rho_0}(p_i) \cap f^m(V_{2\rho_0}(p_j)) = \emptyset,$$

for $0 \leq |m| \leq J$ and $j < i$. The first of these two properties is satisfied if we choose p_i so that $V_{2\rho_0}(p_i) \subseteq U_{\rho_2}(q_i)$. By (2), this is turn will hold if we choose $p_i \in T_{\rho_1}(q_i)$.

Hence, we would like to find $p_i \in T_{\rho_1}(q_i)$ such that

$$V_{2\rho_0}(p_i) \cap f^m(V_{2\rho_0}(p_j)) = \emptyset, \tag{3}$$

for $0 \leq |m| \leq J$ and $j < i$. The neighborhood $U_{\rho_2}(q_i)$ meets at most N sets of the form $f^m\left(U_{\rho_2}(q_j)\right)$, for m between $-J$ and J. Thus, $\#\mathcal{J}_i \leq N$, where $\mathcal{J}_i$ is the collection of all (j, m) with $j < i$, $|m| < J$, and

$$U_{\rho_2}(q_i) \cap f^m(U_{\rho_2}(q_j)) \neq \emptyset.$$

For $q \in M$ and $|m| \leq J$, let $V_\rho^m(q)$ be the connected component of $f^m(q)$ in $U_\rho(f^m(q)) \cap f^m(V_1(q))$. There exists $C_0 \geq 1$ such that, for all $p, q \in M$,

- $V_{C_0\rho_2}^m(q) \supseteq f^m(V_{2\rho_0}(q))$, and
- if $U_{\rho_2}(p) \cap f^m(U_{\rho_2}(q)) \neq \emptyset$, and ρ_0 is sufficiently small, then $V_{C_0\rho_2}^m(q)$ intersects $T_{C_0\rho_1}(p)$ in exactly one point.

(It is not hard to see that C_0 can be chosen to depend only on J, on the Riemannian structure, and on $\|Tf|_{E^c}\|$.) For $(j,m) \in \mathcal{J}_i$, let $p'_{j,m}$ be the point of intersection of $V^m_{C_0\rho_2}(p_j)$ and $T_{C_0\rho_1}(q_i)$:

$$\{p'_{j,m}\} = V^m_{C_0\rho_2}(p_j) \cap T_{C_0\rho_1}(q_i).$$

Consider the collection of these points

$$\mathcal{P}_i = \{p'_{j,m} \mid (j,m) \in \mathcal{J}_i\} \subset T_{C_0\rho_1}(q_i).$$

The points in $\varphi_{q_i}^{-1}(\mathcal{P}_i)$ lie on $B^{u+s}(0, C_0\rho_1) \subset \mathbf{R}^n$. By elementary Euclidean geometry, there exists $C_1 > 0$ such that, for any $\rho > 0$, and any finite collection of points $\mathcal{Q} \subset B^{u+s}(0, C_0\rho)$, there is a point $v \in B^{u+s}(0, C_0\rho)$ whose distance to the points in $\mathcal{Q}$ is at least $\rho/(C_1\#\mathcal{Q})$.

Applying this fact to the points in $\varphi_{q_1}^{-1}(\mathcal{P}_i)$, we find that there exists a point $p_i \in T_{\rho_1}(q_i) \subset U_i$, such that, for all $p \in \mathcal{P}_i$,

$$\|\varphi_{q_i}^{-1}(p_i) - \varphi_{q_i}^{-1}(p)\| \geq \rho_1/(C_1\#\mathcal{P}_i) \geq \rho_1/(C_1 N). \tag{4}$$

We claim that if ρ_0 is sufficiently small, then p_i satisfies (3); that is, for all $j < i$ and $|m| \leq J$,

$$V_{2\rho_0}(p_i) \cap f^m\left(V_{2\rho_0}(p_j)\right) = \emptyset.$$

Clearly the claim is true for those (m,j) such that $U_{\rho_2}(q_i) \cap f^m\left(U_{\rho_2}(q_j)\right) = \emptyset$, so suppose that $(m,j) \in \mathcal{J}_i$. We show that

$$V_{2\rho_0}(p_i) \cap V^m_{C_0\rho_2}(p_j) = \emptyset,$$

which clearly implies the result.

We shall view everything in $\mathbf{R}^n$. Under the map $\varphi_{p_i}^{-1}$, the sets $V_{2\rho_0}(p_i)$ and $V^m_{C_0\rho_2}(p_j)$ lift, respectively, to $B^c(0, 2\rho_0)$ and a set we'll call W. We show that $B^c(0, 2\rho_0)$ and W are disjoint, for ρ_0 sufficiently small.

The set $V^m_{C_0\rho_2}(p_j)$ is a C^1 disk, tangent at $f^m(p_j)$ to $E^c(f^m(p_j))$. Thus W is a C^1 disk, tangent at a point $w_1 \in W$ to the uniformly continuous distribution $T\varphi_{p_i}^{-1}(E^c)$. Furthermore, the distribution $T\varphi_{p_i}^{-1}(E^c)$ coincides at p_i with $\mathbf{R}^c$.

By the distortion bounds, the diameter of W is on the order of ρ_2:

$$\text{diam } (W) \quad \leq \quad 3C_0\rho_2. \tag{5}$$

Let $w_2 = \varphi_{p_i}^{-1}(p'_{j,m}) \in W$. Combining the distortion bounds with (4), we obtain that

$$\|w_2\| = \|\varphi_{p_i}^{-1}(p_i) - \varphi_{p_i}^{-1}(p'_{j,m})\| \geq 2\rho_1/(3C_1N). \tag{6}$$

All of these statements – about the C^1-smoothness of W, the continuity of the distribution $T\varphi_{p_i}^{-1}E^c$, etc. – hold uniformly over p_i, ρ_0 and $|m| < J$. Thus, to summarize the preceding remarks, we have a constant $C_2 > 0$, and functions $\theta_1, \theta_2 : \mathbf{R}_+ \to \mathbf{R}_+$, all independent of p_i, ρ_0 and m, such that W is contained in the graph of a C^1 function $F : B^c(0, C_2\rho_0) \to \mathbf{R}^{u+s}$, with:

1. $\|DF(x_1)\| \leq \theta_1(\|x_1\|)$, for some $x_1 \in B^c(0, C_2\rho_0)$, ($x_1$ corresponds to the point $w_1 \in W$),
2. $\|F(x_2)\| \geq \rho_0/C_2$, for some $x_2 \in B^c(0, C_2\rho_0)$, ($x_2$ corresponds to the point w_2),
3. $\|F(y) - F(x) - DF(x)(y-x)\| \leq \theta_2(\|y-x\|)$, for all $x, y \in B^c(0, C_2\rho_0)$,
4. $\lim_{r\to 0}\theta_1(r) = 0$, and $\lim_{r\to 0}\theta_2(r)/r = 0$.

We claim that if ρ_0 is small enough, then $\|F(x)\| > 0$, for all $x \in B^c(0, C_2\rho_0)$. This implies that W is disjoint from $B^c(0, 2\rho_0)$, which is the desired result. By 3., we have that for all $x \in B^c(0, 2\rho_0)$,

$$\begin{aligned} \|F(x) - F(x_1)\| &\leq \|DF(x_1)(x - x_1)\| + \theta_2(\|x - x_1\|) \\ &\leq 2C_2\,\rho_0\theta_1(2C_2\rho_0) + \theta_2(2C_2\rho_0). \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \|F(x)\| &\geq \|F(x_2)\| - \|F(x_2) - F(x_1)\| - \|F(x_1) - F(x)\| \\ &\geq \rho_0/C_2 - 4C_2\rho_0\,\theta_1(2C_2\rho_0) - 2\theta_2(2C_2\rho_0). \end{aligned}$$

If ρ_0 is sufficiently small, then this quantity is positive.$\diamondsuit$

Proof of Lemma 1.3: We want to pass from infinitesimal conditions given in the definition of partial hyperbolicity to local conditions. To this end, let

$$\overline{a}_p(r) = \max_{q\in B_r(p)} \|T_qf|_{E^s(q)}\|,$$

$$\overline{b}_p(r) = \min_{q\in B_r(p)} m(T_qf|_{E^c(q)}),$$

$$\overline{B}_p(r) = \max_{q \in B_r(p)} \|T_q f|_{E^c(q)}\|,$$

$$\overline{A}_p(r) = \min_{q \in B_r(p)} m(T_q f|_{E^u(q)}).$$

By continuity of Tf we can choose $r_0 > 0$ and $\theta < 1$ so that $\overline{a}_p(r_0) < \theta \overline{b}_p(r_0)$ and $\overline{A}_p(r_0) < \theta \overline{B}_p(r_0)$. From now on we will fix this r_0 and write $\overline{a}_p$ etc. instead of $\overline{a}_p(r_0)$ etc.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous foliations with C^1 leaves on M. We say that $\mathcal{F}_2$ is ε (C^0–) close to $\mathcal{F}_1$ if given any p, q on the same leaf of $\mathcal{F}_1$ with the leafwise distance at most 1 apart, the $\mathcal{F}_2$ leaf passing through p intersects the ε ball centered at q.

It is clear that there exists $\varepsilon > 0$ such that g is stably accessible modulo $\beta\mathcal{D}$ provided that $\mathcal{W}^u_g$ is ε close to $\mathcal{W}^u_f$ and $\mathcal{W}^s_g$ is ε close to $\mathcal{W}^s_f$. So we need to show that given $\varepsilon > 0$ there is $\sigma > 0$ such that the conditions of Lemma 1.3 with this σ imply ε closeness of dynamical foliations of g to those of f. Namely we prove that for all $p, q \in B^s_{r_0/2}(f, p)$, the intersection $W^s(g, p) \bigcap B_\varepsilon(q)$ is non-empty. (Thus, 1 in the initial definition is replaced by $r_0/2$ but this is sufficient because the unit ball can be covered by a finite number of $r_0/2$-balls.) Let $\alpha_p(n)$ denote

$$\alpha_p(n) = \alpha_p \alpha_{fp} \dots \alpha_{f^{n-1}p}$$

where α is either of $\overline{a}, \overline{A}, \overline{b}, \overline{B}$.

Partial hyperbolicity implies that given $\eta \in (\theta, 1)$ there is a continuous cone family K^{cu} around $E^u \oplus E^c$ such that

(a) $Tf(K^{cu}(p)) \subset K^{cu}(fp)$,

(b) K^{cu} is uniformly transverse to E^s_f, and

(c) for any $v \in K^{cu}(p)$
$$\|Tf(v)\| > \overline{a}_p \eta \|v\|.$$

For δ_0 sufficiently small, K^{cu} will also satisfy (a)–(c) if f is replaced by any g such that $d_{C^1}(f, g) \le \delta_0$. Let $q \in \mathcal{W}^s_f(p)$ and let $d_{\mathcal{W}^s}(p, q) \le r_0/2$. Then $d(f^N p, f^N q) \le \overline{a}_p(N)$. Let V be a topological disk of dimension $\dim(E^u \oplus E^c)$ passing through q and such that TV belongs to K^{cu} (for example, we could take $V = \varphi_q(B^{u+c}(0, 1))$).

Given n we can find σ so small that $d_{C^0}(f,g) < \sigma$ implies that

$$d(g^n p, g^n V) < 2\overline{a}_p(n).$$

Since $g^n V$ is uniformly transverse to E^s there exists $C = C(f)$ such that $\mathcal{W}^s_g(g^n p) \bigcap g^n V$ contains a point z with $d(g^n q, z) \leq C\overline{a}_p(n)$. Hence $g^{-n}z \in \mathcal{W}^s(p)$ and $d(q, g^n z) \leq C\eta^n$. Thus, if n is large enough, $\mathcal{W}^s_g$ is $\varepsilon-$close to W^s_f. $\diamondsuit$

3 Local accessibility

Proof of Lemma 1.1: Let f and $\delta < \delta_0$ be given. Assume δ_0 is small enough that any g within δ_0 of f in the C^1-metric remains partially hyperbolic, with constants a, b. Since our perturbations are local, it is convenient to adapt the structures we use to a neighborhood of a point p. To each $p \in M$ we shall associate:

1. a neighborhood $U_p = \varphi_p(B^n(0,1))$,
2. a C^∞ Riemann structure g_p on U_p with path metric d_p, isometric under φ_p^{-1} to the Euclidean metric on $B^c(0,1)$,
3. a C^∞ splitting $TU_p = \widetilde{E}^u_p \oplus \widetilde{E}^c_p \oplus \widetilde{E}^s_p = T\varphi_p(\mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s)$, that agrees with $E^u \oplus E^c \oplus E^s$ at p,
4. C^∞ foliations $\widetilde{\mathcal{W}}^u_p, \widetilde{\mathcal{W}}^s_p, \widetilde{\mathcal{W}}^{cu}_p, \widetilde{\mathcal{W}}^{cs}_p$ of U_p, tangent to the corresponding subbundles of the C^∞ splitting in 3.,
5. for $i = 1, \ldots, c$, partial flows $\zeta^i_t : U_p \to U_p$ tangent to the leaves of $\widetilde{\mathcal{W}}^c$,
6. partial flows $\tau^u_t : U_p \to U_p$ and $\tau^s_t : U_p \to U_p$ tangent to the leaves of $\widetilde{\mathcal{W}}^u$, $\widetilde{\mathcal{W}}^s$, respectively.

We describe the construction of 5. and 6. in more detail. Let $\{e_1, \ldots, e_c\}$ be an orthonormal basis for the $\mathbf{R}^c$ factor in the splitting $\mathbf{R}^n = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$. For $i = 1, \ldots c$, define the partial flows $\zeta^i_t : B \to B$ by

$$\zeta^i_t(\varphi_p(v)) = \varphi_p(v + te_i).$$

Similarly, fix unit vectors w^u and w^s tangent to the $\mathbf{R}^u$ and $\mathbf{R}^s$ factors in the splitting $\mathbf{R}^n = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$, and define the partial flows $\tau_t^u, \tau_t^s : B \to B$, by

$$\begin{aligned}\tau_t^u(\varphi_p(v)) &= \varphi_p(v + tw^u), \text{ and}\\ \tau_t^s(\varphi_p(v)) &= \varphi_p(v + tw^s).\end{aligned}$$

Note that τ_t^u (resp. τ_t^s) sends $\widetilde{\mathcal{W}}^{cs}$ leaves (resp. $\widetilde{\mathcal{W}}^{cu}$ leaves) to other $\widetilde{\mathcal{W}}^{cs}$ leaves, and between $\widetilde{\mathcal{W}}^{cs}$ leaves is the exactly the $\widetilde{\mathcal{W}}^u$ (resp. $\widetilde{\mathcal{W}}^s$) holonomy map. Note that, where defined, $\tau_{-t}^s \tau_{-t}^u \tau_t^s \tau_t^u$ is the identity. This expresses the fact that $\widetilde{\mathcal{W}}^u$ and $\widetilde{\mathcal{W}}^s$ are jointly integrable. In Subsection 3.2, we will use the partial flows $\tau^u, \tau^s, \zeta_t^1, \ldots, \zeta_t^c$ to define g.

The next lemma follows directly from the uniform continuity of φ_p.

Lemma 3.1 *The structures described in 1.-6. are uniform over $p \in M$ and over g sufficiently C^1-close to f. For all $p \in M$, the structure g_p is uniformly comparable to the original Riemann structure on U_p.*

Since all estimates involving the Riemann structure on M in this paper are local, uniform over $p \in M$, any statement about the Riemann structure becomes valid for g_p by introducing a multiplicative constant. We will therefore be deliberately ambiguous in our notation, using d interchangeably for the Riemannian metric and the local metric d_p. Also, when the point is clear from the context, we will drop the subscript p in describing the various structures.

3.1 A criterion for stable accessibility

We describe here a condition on the foliations $\mathcal{W}_g^u$, $\mathcal{W}_g^s$ that implies that g is stably accessible on a c-admissible disk for f.

Let D be a c-admissible disk for f centered at $p \in M$, and let $\rho = r(D)$. Let $m = m(c, \dim(M))$ be the constant given by Lemma 3.10. Suppose that g is partially hyperbolic. We say that g is *θ-accessible on D* if, for each $i = 1, \ldots, c$, there exists a continuous map

$$H^i : [0,1] \times D \to N_{(m-2)\rho}(2D)$$

with $t \mapsto H^i(t,q)$ a 4-legged us-path for g originating at q, and, for some $t_0 \in (0, \rho/2)$, the condition

$$d(H^i(1,q), \zeta_{t_0}^i(q)) \;<\; t_0\theta \tag{7}$$

holds for all $q \in D'$. Here ζ^i, d, etc. are the structures described in the previous section, adapted at p.

The next lemma gives a criterion for central accessibility. A basic element of the proof is the "quadrilateral argument" of Brin, in which 4-legged paths are homotoped to the trivial path along 4-legged paths with endpoint in a fixed c-admissible disk. The reader unfamiliar with this argument may consult the survey paper [BuPuShWi] for a detailed description; the case $c = 1$ is also proved there. The case $c > 1$ is essentially proved in [ShWi], using an index argument.

Lemma 3.2 (Central accessibility criterion) *Suppose $\beta > 1/2$. For every $\beta' \in (\beta, 1)$, there exist $\theta > 0$, $\delta_1 > 0$ and $\rho_0 > 0$ such that, for every c-admissible disk D of radius $r(D) < \rho_0$, if*

- *$d_{C^1}(f, g) < \delta_1$, and*
- *g is θ-accessible on $\beta' D$,*

then g is stably accessible on βD.

Proof of Lemma 3.2: Let β, β' be given. Choose $\theta < (\beta' - \beta)/4\beta' c$. By continuity of the bundles $E^u_g(p)$ and $E^c_g(p)$ in (p, g), there exist $\delta_1 > 0$ and $\rho_0 > 0$ such that, if $d_{C^1}(f, g) < \delta_1$, if D is any c-admissible disk of radius $r(D) = \rho \leq \rho_0$, and if $s : [0, 1] \to N_{(m-2)\rho}(D)$ is any 4-legged us-path for g with $s(0), s(1) \in D$, then

$$d(s(0), s(1)) \quad \leq \quad \rho(\beta' - \beta)/4c. \tag{8}$$

Let g, D be any such diffeomorphism and c-admissible disk. Suppose that g is θ-accessible on $\beta' D$. For $i = 1, \ldots c$, we have maps H^i satisfying (7) with $D' = \beta' D$. We show g is accessible on βD. Since the existence of such H^i is a C^1-open condition, this implies that g is stably accessible on βD.

By varying the lengths of the last 2 legs in the path $t \mapsto H^i(t, q)$, we may arrange that that $H^i(1, q) \in D$, for all $q \in \beta' D$. The reader may verify that it is possible to do so while preserving property (7). (If necessary, the value of θ can be reduced a little).

By a standard argument, the path $t \mapsto H^i(t, q)$, for $q \in \beta' D$, can be homotoped through 4-legged us-paths originating at q to the trivial path so that the endpoints stay in D during the homotopy. The trace of these endpoints along the homotopy gives a curve in D from q to $H^i(1,q)$. More precisely, for $i = 1,\dots,c$, we obtain

$$\Psi^i : [0,1] \times [0,1] \times \beta' D \to N_\theta(D)$$

such that, for all $s \in [0,1]$, $t \mapsto \Psi^i(s,t,q)$ is a us-path for g with $\Psi^i(s,1,q) \in D$, $\Psi^i(0,t,q) = q$ and $\Psi^i(1,t,q) = H^i(t,q)$. Thus, $s \mapsto \Psi^i(s,1,q) =: \Phi^i_s(q)$ is a curve in D, from q to $H^i(1,q)$. Every point on this curve is the endpoint of a us-path originating at q.

By (8), we have, for $q \in \beta' D$:

$$\mathrm{diam}(\Phi^i([0,1] \times \{q\})) \quad < \quad \rho(\beta' - \beta)/4c. \tag{9}$$

For $q \in \beta' D$, we then extend the definition of $\Phi^i_s(q)$ to values of $s > 1$ by the inductive formula

$$\Phi^i_{t+m}(q) = \Phi^i_t(\Phi^i_m(q)),$$

for $t \in (0,1]$ and $m \in \mathbf{N}$. How far $\Phi^i_s(q)$ can be extended in s depends of course on q. Note, however, that (7) gives, for $m \in \mathbf{N}$,

$$\begin{aligned}
d(\Phi^i_m(q), \zeta^i_{mt_0}(q)) &= d(H^i(1, H^i(1, \cdots, H^i(1,q)\cdots)), \zeta^i_{mt_0}(q)) \\
&= d(H^i(1, q_{m-1}), \zeta^i_{t_0}(q'_{m-1})) \\
&\leq d(H^i(1, q_{m-1}), \zeta^i_{t_0}(q_{m-1})) + d(\zeta^i_{t_0}(q_{m-1}), \zeta^i_{t_0}(q'_{m-1})) \\
&\leq t_0(\beta' - \beta)/4\beta' c + d(q_{m-1}, q'_{m-1}) \\
&\quad \cdots \\
&\leq mt_0(\beta' - \beta)/4\beta' c.
\end{aligned} \tag{10}$$

As before, every point in the image of $\Phi^i_{\cdot}(q)$ is the endpoint of a us-path originating at q, although this path can have more than 4 legs.

Let $q_0 = \varphi(-\beta'\rho/2(e_1 + \cdots + e_c))$, and define a map $Z : [0, \beta'\rho]^c \to M$ by:

$$Z(a_1, \dots, a_c) = \zeta^1_{a_1} \cdots \zeta^c_{a_c}(q_0).$$

Note that Z is a homeomorphism onto $\beta' D$.

Next, consider the the map $P : [0, \rho\beta']^c \to D$ defined by:

$$P(a_1, \cdots, a_c) \quad = \quad \Phi^1_{a_1/t_0} \Phi^2_{a_2/t_0} \cdots, \Phi^c_{a_c/r_1 t_0}(q_0).$$

Each point in the image of P is the endpoint of a us-path for g originating at q_0. We claim that D is in the interior of its image. Since Z is a homeomorphism onto $\beta' D$, it suffices to show that $d_{C^0}(P, Z) < d(\partial\beta D, \partial\beta' D) = \rho(\beta' - \beta)/2$.

If $a = (a_1, \dots, a_c) \in [0, \beta'\rho]^c$, with $a_i = t_0(m_i + s_i)$, $m_i \in \mathbf{N}$ and $s_i \in (0, 1]$, then, by (9), and (10),

$$\begin{aligned} d(P(a), Z(a)) &= d(\Phi^1_{a_1/t_0}\Phi^2_{a_2/t_0}\cdots\Phi^c_{a_c/t_0}(q_0), \zeta^1_{a_1}\zeta^2_{a_2}\cdots\zeta^c_{a_c}(q_0)) \\ &\leq \sum_{i=1}^{c} d(\Phi^i_{a_i/t_0}\Phi^{i+1}_{a_{i+1}/t_0}\cdots\Phi^c_{a_c/t_0}(q_0), \zeta^i_{a_1}\Phi^{i+1}_{a_{i+1}/t_0}\cdots\Phi^c_{a_c/t_0}(q_0)) \\ &\leq \sum_{i=1}^{c} \mathrm{diam}(\Phi([0,1]\times\{\Phi^{i+1}_{a_{i+1}/t_0}\cdots\Phi^c_{a_c/t_0}(q_0)\}) + \\ &\quad \sum_{i=1}^{c} d(\Phi^i_{m_i}\Phi^{i+1}_{a_{i+1}/t_0}\cdots\Phi^c_{a_c/t_0}(q_0), \zeta^i_{m_i t_0}\Phi^{i+1}_{a_{i+1}/t_0}\cdots\Phi^c_{a_c/t_0}(q_0)) \\ &\leq c(\beta - \beta')/4c + \sum_{i=1}^{c} m_i t_0(\beta' - \beta)/4\beta' c \\ &< \rho(\beta' - \beta)/2, \end{aligned}$$

since $t_0 m_i < \rho\beta'$. Then g is accessible on βD. $\diamondsuit$

3.2 Constructing the perturbation

Fix $\beta' \in (\beta, 1)$. The next lemma completes the proof of Lemma 1.1.

Lemma 3.3 *For every $\delta, \theta > 0$, if $r(\mathcal{D})$ is sufficiently small and $R(\mathcal{D})$ is sufficiently large, then there exists $g \in \mathit{Diff}^r_\mu(M)$ such that*

1. *$d_{C^1}(f, g) < \delta$,*
2. *$d_{C^0}(f, g) < \theta$*
3. *g is θ-accessible on $\beta' D$, for each $D \in \mathcal{D}$.*

The proof of Lemma 1.1 now follows. Choose $\theta < \sigma$ satisfying the hypotheses of Lemma 3.2. Let g be given by Lemma 3.3. Since g is θ-accessible on $\beta' D$, g is stably accessible on D.$\diamondsuit$.

Proof of Lemma 3.3: Let $\delta, \theta > 0$ be given. We will perturb f by composing with a C^∞, volume preserving diffeomorphism $\psi : M \to M$. We first estimate the effect of the composition $\psi \circ f$ on the partially hyperbolic splitting.

Say that $\psi : M \to M$ is *supported on* $X \subset M$ if $\psi = id$ outside of $X \subset M$. The next lemma states that if $R(X)$ is sufficiently large, and $p, q \in X$ are sufficiently close, then for any g, with gf^{-1} supported on X, the subspaces $T\psi^{-1}(E^u_g)(q)$ and $E^s_g(q)$ are very close to $\widetilde{E}^u_p(q)$ and $\widetilde{E}^s_p(q)$, respectively.

Lemma 3.4 (Bundle Perturbation Lemma) *There exists $\delta_0 > 0$ such that the following is true. For every $\gamma > 0$, there exists $J > 0$ such that, if $\psi = g \circ f^{-1}$ is supported on a set X with $R(X) > J$, and $d_{C^1}(\psi, id) < \delta_0$, then for any $p, q \in X$ with $d(p,q) < J^{-1}$, we have:*

1. $\angle_q(E^s_g, \widetilde{E}^s_p) \leq \gamma$, *and*
2. $\angle_q(T\psi^{-1}(E^u_g), \widetilde{E}^u_p) \leq \gamma$.

Proof of Lemma 3.4: Let $\gamma > 0$ be given. Recall that the splittings $TU_p = E^u \oplus E^c \oplus E^s$ and $TU_p = \widetilde{E}^u_p \oplus \widetilde{E}^c_p \oplus \widetilde{E}^s_p$ coincide at p.

By uniform continuity of the splitting $E^u \oplus E^c \oplus E^s$, uniformity of φ_p, and smoothness of ψ, there exists a continuous function $\theta_1 : \mathbf{R}_+ \to \mathbf{R}_+$, with $\theta_1(0) = 0$, such that, for all $p, q \in M$,

$$\begin{aligned} \angle_q(T\psi(E^u), T\psi(\widetilde{E}^u_p)) &\leq \theta_1(d(p,q)) && (11)\\ \angle_q(E^s, \widetilde{E}^s_p) &\leq \theta_1(d(p,q)), && (12) \end{aligned}$$

provided $d_{C^1}(\psi, id)$ is small enough.

Let

$$\lambda \ = \max_p \left(\max\left(\tfrac{a_p}{b_p}\right), \left(\tfrac{B_p}{A_p}\right)\right)$$

and note that $\lambda < 1$, because f is partially hyperbolic. There exist $C_1, \theta_0 > 0$ such that, for all all subspaces F^u, F^s with

$$\max\{\angle(F^u, E^u),\ \angle(F^s, E^s)\} \leq \theta_0$$

we have:

$$\begin{aligned} \angle(Tf^{-j}(F^s), Tf^{-j}(E^s)) &\leq C_1\lambda^j, and\\ \angle(Tf^j(F^u), Tf^j(E^u)) &\leq C_1\lambda^j, \end{aligned}$$

for all $j \geq 0$. The splitting $E^u_g \oplus E^c_g \oplus E^s_g$ depends continuously on g, and so

$$\max\{\angle(E^u_g, E^u),\ \angle(E^s_g, E^s)\} \ \leq\ \theta_0,$$

if $d_{C^1}(\psi, id)$ (and so $d_{C^1}(f, g)$) is sufficiently small.

Fix positive $R < R(X)$. If $q \in X$, then $g^i(q) = f^i(q)$, for all i between 0 and R. For these q, we have

$$\begin{aligned}
\angle_q(E^s_g, E^s) &= \angle_q(Tg^{-R}E^s_g, Tf^{-R}E^s)\\
&= \angle_q(Tf^{-R}E^s_g, Tf^{-R}E^s)\\
&\leq C_1\lambda^R.
\end{aligned}$$

Similarly, for $q \in X$, $g^{-i}(q) = f^{-i}\psi^{-1}(q) = f^{-i+1}g^{-1}(q)$, for all i between 1 and $R-1$, and so

$$\begin{aligned}
\angle_q(E^u_g, T\psi(E^u)) &= \angle_q(Tg(E^u_g), Tg(E^u))\\
&\leq C_2\angle_{g^{-1}(q)}(E^u_g, E^u)\\
&= C_2\angle_{g^{-1}(q)}(Tf^{R-1}E^u_g, Tf^{R-1}E^u)\\
&\leq C_1C_2\lambda^{R-1}.
\end{aligned}$$

Combining these inequalities with (11), we have shown: there exist λ, $\theta_1 : \mathbf{R}_+ \to \mathbf{R}_+$, and $C > 0$ such that, for any ψ sufficiently close to the identity and supported on X, for all $R < R(X)$, and all $q \in X$, $p \in M$, we have

- $\angle_q(E^s_g, \widetilde{E}^s_p) \leq C(\lambda^R + \theta_1(d(p,q)))$, and
- $\angle_q(T\psi^{-1}(E^u_g), \widetilde{E}^u_p) \leq 2\angle_q(E^u_g, T\psi(\widetilde{E}^u_p)) \leq C(\lambda^R + \theta_1(d(p,q)))$.

Hence if R is sufficiently large and $d(p,q)$ is sufficiently small, these quantities are bounded by γ. $\diamondsuit$

We will also need the following lemma.

Lemma 3.5 *There exists $T > 0$ such that, for $\epsilon > 0$ sufficiently small, for any $p \in M$, and for any c-admissible disk D centered at p, there are C^∞, volume preserving flows $\xi^1_t, \ldots, \xi^c_t : U_p \to U_p$ such that, for each i:*

1. *$\xi_t^i = id$ outside $N_{2\epsilon}(D)$.*

2. *For $q \in N_\epsilon(D)$,*
$$\xi_t^i(q) = \zeta_{\epsilon t}^i(q).$$
(hence ξ_t^i preserves the leaves of $\widetilde{\mathcal{W}}^c \cap N_\epsilon(D)$),

3. *$d_{C^1}(id, \xi_t^i) < T|t|$.*

Proof of Lemma 3.5: Let $G = \varphi_p^{-1}(D) = B^c(0,\rho)$, for some $\rho > 0$. Fix i, and let E be the divergence-free vector field on $N_{2\epsilon}(G) \subset \mathbf{R}^n$ such that, for all v:
$$E(v) = \epsilon e_i.$$
Let ω be the Euclidean volume form on $\mathbf{R}^n$, and let $\phi_0 = i_E\omega$. Since E is divergence-free, the $(n-1)$-form ϕ_0 is closed: $d\phi_0 = di_E\omega = \mathrm{div}(E)\omega = 0$. Since $N_{2\epsilon}(G)$ is contractible, there exists an $(n-2)$-form ν on $N_{2\epsilon}(G)$ such that $d\nu = \phi_0$. We may choose ν so that
$$\|\nu\| \leq 2\epsilon^2, \text{ and } \|\nu\|_{C^1} \leq \epsilon.$$

Let $\sigma : N_{2\epsilon}(G) \to [0,1]$ be a C^∞ bump function, vanishing on a neighborhood of $\partial N_{2\epsilon}(G)$ and identically 1 on $N\epsilon(G)$, such that
$$\|d\sigma\| \leq 2/\epsilon, \text{ and } \|d\sigma\|_{C^1} \leq 2/\epsilon^2.$$
Let $\phi = d(\sigma\nu)$. Then
$$\begin{aligned}\|\phi\|_{C^1} &= \|d\sigma \wedge \nu + \sigma\phi_0\|_{C^1} \\ &\leq \|d\sigma\| \cdot \|\nu\|_{C^1} + \|d\sigma\|_{C^1} \cdot \|\nu\| + \|\sigma\|_{C^1} \cdot \|\phi_0\|_{C^1} \\ &\leq 8\end{aligned}$$

Hence ϕ has the following properties:

- $\|\phi\|_{C^1} \leq T$, where $T = 8$,
- $d\phi = 0$,
- $\phi = \phi_0$ on $N\epsilon(G)$,
- $\phi = 0$ on $\partial N_{2\epsilon}(G)$.

Let X be the vector field on $\mathbf{R}^n$ satsifying $i_X\omega = \varphi$, and let X_t be the flow generated by X. Let $\xi^i_t = \varphi \circ X_t \circ \varphi^{-1}$. Then ξ^i has the desired properties. $\diamondsuit$

Returning to the proof of Lemma 3.3, let T be given by Lemma 3.5. Let $\gamma = \theta\delta/100cT$. Choose $J > 0$ according to Lemma 3.4.

Now suppose that $\mathcal{D} = \{D_1, \ldots, D_k\}$ is any c-admissible family with $R(\mathcal{D}) > J$ and $r(\mathcal{D}) < J^{-1}$. Choose $\eta < r(\mathcal{D})$ so that the η-neighborhoods of any two c-admissible disks in $\mathcal{D}$ are disjoint.

To prove Lemma 3.3, it suffices to show that for any $D \in \mathcal{D}$, there is a C^∞ volume preserving diffeomorphism ψ, supported on the η-neighborhood $N_\eta(D)$, such that

1. $d_{C^1}(\psi, id) < \delta$,
2. $d_{C^0}(\psi, id) < \theta$,
3. if $\overline{f}$ is any diffeomorphism with $f^{-1}\overline{f}$ supported on $N_\eta(|\mathcal{D}|) \setminus N_\eta(D)$, and $d_{C^1}(\overline{f}, f) < \delta$, then $\psi \circ \overline{f}$ is θ-accessible on $\beta' D$.

To construct the final diffeomorphism g, we proceed disk by disk, constructing for each $D_i \in \mathcal{D}$ a diffeomorphism ψ_i suppported on $N_\eta(D_i)$ so that $\psi_i \circ \psi_{i-1} \cdots \psi_i \circ f$ is θ- accessible on $\beta' D_i$. Then $g = \psi_k \cdots \psi_1 \circ f$ will satisfy the conclusions of Lemma 3.3.

Fix $D \in \mathcal{D}$ centered at p and choose $\epsilon < \eta/4c$ small enough to satisfy the hypotheses of Lemma 3.5. Let the flows $\xi^1_t, \ldots, \xi^c_t$ be given by Lemma 3.5.

For $i = 1, \ldots, c$, let $\epsilon_i = 4i\epsilon$, let $Z_i = \tau^u_{\epsilon_i}(D)$, and let

$$N_i = N_{2\epsilon}(Z_i) = \tau^u_{\epsilon_i}(N_{2\epsilon}(D)).$$

The neighborhoods $N_1, \ldots, N_c$ are pairwise disjoint. Define $\psi : M \to M$ by

$$\psi(q) = \begin{cases} \tau^s_{-\epsilon_i}\tau^u_{\epsilon_i}\tau^s_{\epsilon_i}\xi^i_{\delta/T}\tau^u_{-\epsilon_i}(q) & \text{if } q \in N_i, \text{ for some } i, \\ q & \text{otherwise.} \end{cases}$$

Observe that ψ has the following properties:

- ψ preserves μ,
- $\psi = id$ outside $N_1 \cup \cdots \cup N_c \subset N_\eta(D)$,

- ψ preserves the leaves of $\widetilde{\mathcal{W}}^c$ outside of $N_1 \cup \cdots \cup N_c$ and inside of $N_\epsilon(Z_i)$, for $i = 1, \ldots, c$.
- $d_{C^1}(\psi, id) < \delta$.

Let $\overline{f}$ be any diffeomorphism with $d_{C^1}(\overline{f}, f) < \delta$ and $f^{-1}\overline{f}$ supported on $N_\eta(|\mathcal{D}|) \setminus N_\eta(D)$, and let $g = \psi \circ \overline{f}$. It remains to show that g is θ-accessible on $\beta' D$.

We will now examine the behavior of the holonomy maps for g along us-paths whose corners are near the points:

$$\tau^u_{\epsilon_i}(p), \quad \tau^s_{\epsilon_i}\tau^u_{\epsilon_i}(p), \quad \tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\tau^u_{\epsilon_i}(p).$$

In analogue with τ^u_t and τ^s_t, which translate along leaves of $\widetilde{\mathcal{W}}^u, \widetilde{\mathcal{W}}^s$, respectively, we introduce maps $\pi^u_t, \pi^s_t : N_\eta(\beta' D) \to B$, which translate along $\mathcal{W}^u_g$ and $\mathcal{W}^s_g$ leaves:

$$\begin{aligned} \{\pi^u_t(q)\} &= \mathcal{W}^u_g(q) \cap \widetilde{\mathcal{W}}^{cs}(\tau^u_t(q)) \\ \{\pi^s_t(q)\} &= \mathcal{W}^s_g(q) \cap \widetilde{\mathcal{W}}^{cu}(\tau^s_t(q)). \end{aligned}$$

If $d_{C^1}(f, g)$ is sufficiently small, these maps are well-defined for $|t| \leq \epsilon_c$. Notice that if we were to replace $\mathcal{W}^u_g$ and $\mathcal{W}^s_g$ with $\widetilde{\mathcal{W}}^u$ and $\widetilde{\mathcal{W}}^s$, these equations would instead define τ^u_t and τ^s_t, respectively. Between $\widetilde{\mathcal{W}}^{cs}$ leaves, π^u_t is the $\mathcal{W}^u$-holonomy (and similarly for π^s_t). Lemma 3.4 will allow us to predict the behavior of these maps. The upshot is:

On the appropriate domains, $\pi^u \sim \psi\tau^u$ and $\pi^s \sim \tau^s$,

where we will be precise about $\sim$ later.

For $i = 1, \ldots c$, let

$$h^i = \pi^s_{-\epsilon_i}\pi^u_{-\epsilon_i}\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}.$$

Then h^i is a homeomorphism of $N_\epsilon(\beta' D)$ onto its image. Observe that $h^i(q)$ is the endpoint of a 4-legged us-path for g, originating at q. By construction, these paths depend continuously on q, and so there are continuous maps

$$H^i : [0,1] \times \beta' D \to N_\eta(D)$$

with $t \mapsto H^i(t, q)$ a 4-legged us-path for g, such that $H^i(0, q) = q$ and $H^i(1, q) = h^i(q)$. The rest of the argument goes as follows. We will show that $\pi^u \sim \psi\tau^u$ and $\pi^s \sim \tau^s$, which will imply that $h^i \sim \tau^s_{-\epsilon_i}\psi\tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}$. Since τ^u and τ^s are just translations, $\psi = \zeta^i_{t_0}$ on $\tau_{\epsilon_i}(D)$, and $\psi = id$ on $\tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\tau_{\epsilon_i}(D)$, we find that $h^i \sim \zeta^i_{t_0}$, where $t_0 = \epsilon\delta/T$. The remainder of the argument is devoted to making $\sim$ precise.

Lemma 3.6 (Holonomy Perturbation Lemma) *For ϵ and δ sufficiently small,*

$$d_{C^0}(h^i, \zeta^i_{t_0}) \leq t_0\theta$$

where $t_0 = \epsilon\delta/T$, and the C^0-distance is measured on $\beta' D$.

Proof of Holonomy Perturbation Lemma 3.6: For δ sufficiently small, there exists a neighborhood $Q \subset N_\epsilon(\beta' D)$ of $\beta' D$ such that, for $i = 1, \ldots, c$,

$$\begin{aligned} \pi^u_{\epsilon_i}(Q) &\subseteq \psi\tau^u_{\epsilon_i}(N_\epsilon(\beta' D)), \\ \pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(Q) &\subseteq \tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(N_\epsilon(\beta' D)), \text{ and} \\ \pi^u_{-\epsilon_i}\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(Q) &\subseteq \tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(N_\epsilon(\beta' D)). \end{aligned}$$

We now show that $d(h^i(q), \zeta^i_{t_0}(q)) \leq t_0\theta$, for all $q \in Q$ and $i = 1 \ldots, c$, which implies the conclusion of the lemma.

From the definition of ψ, we write, for $q \in Q$,

$$\begin{aligned} \zeta^i_{t_0}(q) &= \xi^i_{t_0/\epsilon}(q) \\ &= \xi^i_{\delta/T}(q) \\ &= \tau^s_{-\epsilon_i}\tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(q) \\ &= \tau^s_{-\epsilon_i}\psi\tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(q), \end{aligned} \tag{13}$$

the final equality a consequence of the fact that ψ is supported on $N_1 \cup \cdots \cup N_c$, which is disjoint from $\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(Q)$. On the other hand,

$$h^i(q) = \pi^s_{-\epsilon_i}\pi^u_{-\epsilon_i}\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(q). \tag{14}$$

We show that the corresponding factors in the two compositions (13) and (14) satisfy the desired inequality. More specifically, we show that, restricted to the appropriate domains, the distances $d_{C^0}(\pi^u_{\pm\epsilon_i}, \psi\tau^u_{\pm\epsilon_i})$ and $d_{C^0}(\pi^s_{\pm\epsilon_i}, \tau^s_{\pm\epsilon_i})$ are bounded by $\theta t_0/4$.

First, consider the maps $\psi \circ \tau^u_{\epsilon_i}$ and $\pi^u_{\epsilon_i}$ on the domain Q. Recall that, restricted to $\widetilde{\mathcal{W}}^{cs}$ leaves, $\tau^u_{\epsilon_i}$ is the $\widetilde{\mathcal{W}}^u$ holonomy map. But $\psi \circ \tau^u_{\epsilon_i}$ sends $\widetilde{\mathcal{W}}^{cs}$ leaves in $Q \subset N_\epsilon(\beta' D)$ to $\widetilde{\mathcal{W}}^{cs}$ leaves: $\tau^u_{\epsilon_i}$ sends $\widetilde{\mathcal{W}}^c$ leaves to $\widetilde{\mathcal{W}}^c$ leaves, and ψ preserves $\widetilde{\mathcal{W}}^{cs}$ leaves in $\tau^u_{\epsilon_i}(Q) \subset N'_i$. It follows that, restricted to $\widetilde{\mathcal{W}}^{cs}(q) \cap Q$, the map $\psi \circ \tau^u_{\epsilon_i}$ is the $\psi(\widetilde{\mathcal{W}}^u)$-holonomy map to the transversal $\widetilde{\mathcal{W}}^{cs}(\tau^u_{\epsilon_i}(q))$, where $\psi(\widetilde{\mathcal{W}}^u)$ is the image of $\widetilde{\mathcal{W}}^u$ under ψ. Recall that $\pi^u_{\epsilon_i}$ restricts to the $\mathcal{W}^u_g$-holonomy map between $\widetilde{\mathcal{W}}^{cs}(q) \cap N_\epsilon(\beta' D)$ and $\widetilde{\mathcal{W}}^{cs}(\tau^u_{\epsilon_i}(q))$.

Thus, between $\widetilde{\mathcal{W}}^{cs}$ leaves, we are comparing the holonomy maps for the foliations $\mathcal{W}^u_g$ and $\psi(\widetilde{\mathcal{W}}^u)$. To compare the holonomies for $\mathcal{W}^u_g$ and $\psi(\widetilde{\mathcal{W}}^u)$, we first apply the smooth change of coordinates $p \mapsto \psi^{-1}(p)$ and compare the holonomies for $\psi^{-1}(\mathcal{W}^u_g)$ and $\widetilde{\mathcal{W}}^u$. Since $d_{C^1}(\psi, id)$ is small, this change of coordinates distorts distances by a factor very close to 1.

The tangent distributions to $\psi^{-1}(\mathcal{W}^u_g)$ and $\widetilde{\mathcal{W}}^u$ are $T\psi^{-1}E^u_g$ and $\widetilde{E}^u$, respectively. According to the Bundle Perturbation Lemma 3.4, the distributions $T\psi^{-1}(E^u_g)$ and $\widetilde{E}^u$ are close; in particular,

$$\angle_q(T\psi^{-1}(E^u_g), \widetilde{E}^u) \quad \leq \quad \gamma = \theta\delta/100cT, \tag{15}$$

for all $q \in N_\eta(D)$. We now apply the next elementary lemma.

Lemma 3.7 *Let $\mathcal{F}$ be a continuous foliation of $B \subset U_p$ with C^1, u-dimensional leaves, transverse to $\widetilde{E}^s \oplus \widetilde{E}^c$. Let T_1 and T_2 be smooth disks tangent to $\widetilde{E}^s \oplus \widetilde{E}^c$. Assume that both the $\mathcal{F}$- and $\widetilde{\mathcal{W}}^u$-holonomy maps between T_1 and T_2 are defined, and denote them by $h^{\mathcal{F}}$ and $h^{\widetilde{\mathcal{W}}^u}$, respectively. Then, for all $q \in T_1$,*

$$d(h^{\mathcal{F}}(q), h^{\widetilde{\mathcal{W}}^u}(q)) \quad \leq \quad dist(T_1, T_2) \cdot \sup_{q \in B} \angle_q(T\mathcal{F}, \widetilde{E}^u).$$

The analogous statement holds for s-dimensional foliations transverse to $\widetilde{E}^s \oplus \widetilde{E}^c$.

Applying Lemma 3.7 to the foliation $\psi^{-1}\mathcal{W}^u_g$, and using inequality (15), we obtain that, for any two transversals $T_1 \subset Q$ and $T_2 = \tau^u_{\epsilon_i}(T_1)$, and $q \in T_1$,

$$\begin{aligned} d(h^{\psi^{-1}(\mathcal{W}^u_g)}(q), h^{\widetilde{\mathcal{W}}^u}(q)) &\leq \mathrm{dist}(T_1, T_2) \cdot \sup_{q \in N_\eta(D)} \angle_q(T\psi^{-1}(E^u_g), \widetilde{E}^u) \\ &\leq \epsilon_i \gamma \\ &\leq (4c\epsilon)\gamma \\ &\leq (4c\epsilon)(\theta\delta/100cT) \\ &< \theta(\epsilon\delta/8T) \\ &= \theta t_0/8. \end{aligned}$$

But then, for all $q \in Q$,

$$\begin{aligned}
d(\pi^u_{\epsilon_i}(q), \psi\tau^u_{\epsilon_i}(q)) &= d(\psi h^{\psi^{-1}(\mathcal{W}^u_g)}(q), \psi h^{\widetilde{\mathcal{W}}^u}(q)) \\
&\leq \mathrm{Lip}(\psi)\theta t_0/8 \\
&\leq \theta t_0/4.
\end{aligned}$$

Similarly, for $q \in \psi\tau^u_{\epsilon_i}(Q)$,

$$\begin{aligned}
d(\pi^s_{\epsilon_i}(q), \tau^s_{\epsilon_i}(q)) &\leq 2\epsilon_i \angle(E^s_g, \widetilde{E}^s) \\
&< \theta t_0/4.
\end{aligned}$$

Combining these inequalities and using the fact that $\tau^s_{\epsilon_i}$ is an isometry, we have, for all $q \in Q$,

$$\begin{aligned}
d(\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(q), \tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(q)) &\leq d(\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(q), \tau^s_{\epsilon_i}\pi^u_{\epsilon_i}(q)) + d(\tau^s_{\epsilon_i}\pi^u_{\epsilon_i}(q), \tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(q)) \\
&= d(\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(q), \tau^s_{\epsilon_i}\pi^u_{\epsilon_i}(q)) + d(\pi^u_{\epsilon_i}(q), \psi\tau^u_{\epsilon_i}(q)) \\
&< \theta t_0/4 + \theta t_0/4 \\
&= \theta t_0/2.
\end{aligned}$$

Proceeding in this fashion, we obtain that for $q \in Q$,

$$d(\pi^s_{-\epsilon_i}\pi^u_{-\epsilon_i}\pi^s_{\epsilon_i}\pi^u_{\epsilon_i}(q), \tau^s_{-\epsilon_i}\psi\tau^u_{-\epsilon_i}\tau^s_{\epsilon_i}\psi\tau^u_{\epsilon_i}(q)) \quad < \quad \theta t_0,$$

which completes the proof. $\diamondsuit$

This completes the proof of Lemma 3.3. We have now shown that for each i, there exists

$$H^i : [0,1] \times \beta' D \to N_\eta(D)$$

with $t \mapsto H^i(t,q)$ a 4-legged us-path for g, such that $H^i(0,q) = q$ and

$$d(H^i(1,q), \zeta^i_{t_0}(q)) = d(h^i(q), \zeta^i_{t_0}(q)) < \theta t_0,$$

where $t_0 = \epsilon\delta/T$. Hence, g is θ-accessible.$\diamondsuit$.

3.3 The symplectic case

If f preserves a symplectic form ω, then the perturbation g can also be made symplectic.

As in the proof for the volume preserving case, we begin with a local system of C^∞ charts $\varphi_p : B^{2m}(0,1) \to M$, defined for each $p \in M$, where $2m = n$. Similar to the volume preserving case, these charts can be chosen to have the following properties:

1. $\varphi_p(0) = p$,
2. $T_0\varphi_p$ sends the splitting $T_0\mathbf{R}^n = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$ to the splitting $T_pM = E^u \oplus E^c \oplus E^s$,
3. the symplectic form $\varphi_p^*\omega$ is a linear pullback of the standard symplectic form on $\mathbf{R}^{2m}$:
$$\varphi_p^*\omega = A_p^*(\sum dp_i \wedge dq_i),$$
for some linear map A_p on $\mathbf{R}^{2m}$,
4. $p \mapsto \varphi_p$ is a uniformly continuous map from M to $C^1(B^n(0,1), M)$. The dependence of φ_p, A_p on f is also continuous.

By Darboux's theorem, for each $p \in M$, there exists a neighborhood U_p of p and coordinates $\kappa_p : U_p \to \mathbf{R}^{2m}$ such that, in these coordinates, ω takes the standard form $\sum dp_i \wedge dq_i$. For each p, $T_p\kappa_p$ sends the splitting $T_pM = E^u(p) \oplus E^c(p) \oplus E^s(p)$ to a splitting $\mathbf{R}^{2m} = \mathbf{R}^u_p \oplus \mathbf{R}^c_p \oplus \mathbf{R}^s_p$. Let $A_p : \mathbf{R}^{2m} \to \mathbf{R}^{2m}$ be a linear map that sends $B^{2m}(0,1)$ into $\kappa_p(U_p)$ and sends the trivial splitting $\mathbf{R}^{2m} = \mathbf{R}^u \oplus \mathbf{R}^c \oplus \mathbf{R}^s$ to $\mathbf{R}^{2m} = \mathbf{R}^u_p \oplus \mathbf{R}^c_p \oplus \mathbf{R}^s_p$, chosen to depend continuously on p, f. Then $\varphi_p = \kappa_p^{-1} \circ A_p$ satisfies properties 1.-4.

With this modification, the proof of the Main Theorem in the symplectic case proceeds exactly as in the volume preserving one, replacing "μ" by "ω", until the proof on Lemma 3.3. Since we will modify slightly the statement and proof of this lemma, we restate it here in the symplectic case.

Lemma 3.8 *For every $\delta, \theta > 0$, if $r(\mathcal{D})$ is sufficiently small and $R(\mathcal{D})$ is sufficiently large, then there exists $g \in \mathit{Diff}_\omega(M)$ such that*

1. $d_{C^1}(f, g) < \delta$,
2. $d_{C^0}(f, g) < \theta$, *and*

3. *each $D \in \mathcal{D}$ is covered by c-admissible disks $\beta V_1, \dots, \beta V_k$ such that g is θ-accessible on V_i, for each i.*

Remark: If θ and δ are sufficiently small, then any $g \in \mathrm{Diff}^r_\omega(M)$ satisfying conditions 1.- 3. in Lemma 3.8 is stably accessible on D; for then Lemma 3.2 implies that g is stably accessible on each βV_i, which implies stabe accessibility on their union, which contains D.

Proof of Lemma 3.8. Let $\mathcal{D}$ be a c-admissible family. Using Lemma 3.10 below, we will cover each $D \in \mathcal{D}$ with c-disks $\beta V_1, \dots, \beta V_k$. The lemma associates to each i an open set (a union of balls) $N(D, i) \subset N_\eta(D)$; for different i, these sets are disjoint. We will then perturb inside of $N(D, i)$ to obtain θ-accessibility on V_i.

Similar to the volume preserving case, we will need to show that if $r(\mathcal{D})$ is sufficiently small and $R(\mathcal{D})$ is sufficiently large, then for each $D \in \mathcal{D}$ and each c-disk V_i in the cover of D, there is a symplectic C^∞ diffeomorphism ψ, supported on $N(D, i)$, with

(a) $d_{C^1}(\psi, id) < \delta$,

(b) $d_{C^0}(\psi, id) < \theta$,

(c) if $\overline{f}$ is any diffeomorphism with $f^{-1}\overline{f}$ supported on $N_\eta(|\mathcal{D}|) \setminus N(D, i)$ and $d_{C^1}(\overline{f}, f) < \delta$, then $\psi \circ \overline{f}$ is θ-accessible on V_i.

Each perturbation ψ is supported on a union of balls (as opposed to a tubular neighborhood); this allows for symplectic perturbations. The next lemma replaces Lemma 3.5 for the symplectic case.

Lemma 3.9 *There exists $T > 0$ such that, for $\epsilon > 0$ sufficiently small, for each $p \in M$ and $q \in B_{1/2}(p)$, there are C^∞, symplectic flows $\xi^i_t = \xi^{i,q}_t : M \to M$, $i = 1, \dots, c$, such that, for each i:*

1. *$\xi^i_t = id$ outside $B_{2\epsilon}(q)$*
2. *For $x \in B_\epsilon(q)$,*
$$\xi^i_t(x) = \zeta^i_{\epsilon t}(x).$$
(hence ξ^i_t preserves the leaves of $\widetilde{\mathcal{W}}^c \cap B_\epsilon(q)$),
3. *$d_{C^1}(id, \xi^i_t) < T|t|$.*

Here, all balls $B_\rho(q)$ are measured in the d_p-metric, and all other invariant structures $\widetilde{\mathcal{W}}^c, \zeta^i$, etc. are adapted at p.

Proof of Lemma 3.9. Let p, q, i be given and let $v = \varphi_p^{-1}(q) \in B^n(0, 1/2)$. We will explain how T is chosen later. Since constant vector fields are locally Hamiltonian with respect to $\varphi_p^*\omega = A_p^*(\sum dp_i \wedge dq_i)$, there exists a Hamiltonian vector field X^i, supported on $B^n(v, 1/2)$, such that $X^i = e_i$ on $B^n(v, 1/4)$. Since the C^1-size of φ_p is uniformly controlled, there exists a $T_0 > 0$, independent of p, q, i, such that $\|X^i\|_{C^1} < T_0$. Given $\epsilon < 1/4$, let

$$Y^i(x) = \epsilon X^i((x - v)/4\epsilon).$$

Then Y^i is Hamiltonian (if X^i has Hamiltonian H, then Y^i has Hamiltonian $4\epsilon^2 H((x - v)/4\epsilon)$), is supported on $B^n(v, 2\epsilon)$, and satisfies $\|Y^i\|_{C^1} \leq 4T_0$. Furthermore, $Y^i = \epsilon e_i$ on $B^n(v, \epsilon)$. The vector field $(\varphi_p)_* Y^i$ generates the desired flow ξ_i. Clearly T can be chosen to depend only on T_0 and other uniform data. $\diamondsuit$

Next, we choose the balls. The proof of the next lemma is an elementary exercise in Euclidean geometry.

Lemma 3.10 *There exists $m > 2$, depending only on c and $dim(M)$, such that, for $\epsilon > 0$ sufficiently small and all $p \in M$, there exist $k > 0$ and points*

$$\{q_{i,j} \,|\, i = 1, \ldots, c,\ j = 1, \ldots, k\} \subset N_{(m-2)\epsilon}(D)$$

with the following properties:

1. *there exist $p_1, \ldots, p_k \in D$ and $\epsilon_{i,j} > 0$ such that $\tau^u_{-\epsilon_{i,j}}(q_{i,j}) = p_j$,*
2. *the balls in the collection*

   $$\{B_{2\epsilon}(q_{i,j}), B_{2\epsilon}(\tau^s_{\epsilon_{i,j}}(q_{i,j})), B_{2\epsilon}(\tau^u_{\epsilon_{i,j}}\tau^s_{\epsilon_{i,j}}(q_{i,j})) \ | \ i = 1, \ldots, c,\ j = 1, \ldots, k\}$$

   are pairwise disjoint,
3. *the balls*

   $$B_{\beta\epsilon}(p_1), \ldots, B_{\beta\epsilon}(p_k)$$

   cover D.

Given δ and θ, let T be given by Lemma 3.9, let $\gamma = \theta\delta/100cT$, and choose J according to Lemma 3.4. Let $\mathcal{D}$ be any c-admissible family with $R(\mathcal{D}) > J$ and $r(\mathcal{D}) < J^{-1}$. Let $D \in \mathcal{D}$ with center p. Proceeding as in the proof of Lemma 3.3, choose $\epsilon < \theta/4mc$ satisfying the hypotheses of Lemma 3.9, where m is given by Lemma 3.10. Fix $1 \leq i \leq c$, and let the points $\{q_{i,j}\}, p_i$ be given by Lemma 3.10. Let $V_i = V_\epsilon(p_i)$; by Lemma 3.10 the disks $\beta V_1, \ldots, \beta V_k$ cover D. Let

$$N(D,i) = \bigcup\{B_{2\epsilon}(q_{i,j}), B_{2\epsilon}(\tau^s_{\epsilon_{i,j}}(q_{i,j})), B_{2\epsilon}(\tau^u_{\epsilon_{i,j}}\tau^s_{\epsilon_{i,j}}(q_{i,j})) \mid j = 1, \ldots c\}.$$

We show that properties (a)-(c) above are satisfied for this i.

In each ball $B_{2\epsilon}(q_{i,j})$, let $\xi_t^{i,j} = \xi_t^{i,q_j}$ be the flow given by Lemma 3.9, with $q = q_j$.

Define $\psi : M \to M$ by

$$\psi(q) = \begin{cases} \xi^{i,j}_{\delta/T} & \text{if } q \in B_{2\epsilon}(q_{i,j}), \text{ for some } j, \\ q & \text{otherwise.} \end{cases}$$

Then ψ has the following properties:

- $\psi^*\omega = \omega$,
- $\psi = id$ outside $N(D,i)$,
- ψ preserves the leaves of $\widetilde{\mathcal{W}}^c$ outside of $\bigcup_j B_{2\epsilon}(q_{i,j})$ and inside of $\bigcup_j B_\epsilon(q_{i,j})$,
- $d_{C^1}(\psi, id) < \delta$.

Let $\overline{f}$ be any diffeomorphism with $f^{-1}\overline{f}$ supported on $N_\eta(|\mathcal{D}|) \setminus N(D,i)$, and let $g = \psi \circ \overline{f}$. Let It remains to show that g is θ-accessible on V_i.

By the same argument as in the proof of the Main Theorem, we obtain that for each $x \in V_i$, there is a 4-legged us-path for g from x to a point $y \in N_{(m-2)\epsilon}(D)$ such that

$$d(y, \zeta^i_{t_0}(x)) < \theta t_0,$$

In other words, g is θ-accessible on V_i.$\diamondsuit$

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