Better center bunching

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Introduction

Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a compact manifold M. In this paper, partially hyperbolic means the following. There is a nontrivial splitting of the tangent bundle, $TM = E^u \oplus E^c \oplus E^s$, that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, and $\hat{\gamma}$ with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma \leq \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{1}$$

such that, for a unit vector $v \in T_pM$,

$$\|Tfv\| \leq \nu(p), \qquad \text{if } v \in E^s(p), \tag{2}$$
$$\gamma(p) \leq \|Tfv\| \leq \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{3}$$
$$\hat{\nu}(p)^{-1} \leq \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{4}$$

We say that f is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\max\{\nu, \hat{\nu}\} < \gamma\hat{\gamma}. \tag{5}$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Tf on the center. Inequality (5) always holds when $Tf|_{E^c}$ is conformal, since we can choose

$$\gamma(p) = \hat{\gamma}(p)^{-1} = \|Tfv(p)\|,$$

where $v(p)$ is any unit vector in $E^c(p)$. In particular, center bunching holds whenever E^c is one-dimensional.

For any partially hyperbolic diffeomorphism, the stable and unstable subbundles E^s and E^u are tangent to foliations, which we denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively [BP]. These foliations induce an equivalence relation on M: we say that $p \sim_{us} q$ if there is a sequence of points $p = p_0, \ldots, p_k = q$ such that any two consecutive points in the sequence lie in the same $\mathcal{W}^s$-leaf or the same $\mathcal{W}^u$-leaf. A partially hyperbolic diffeomorphism has the *accessibility property* if there is only one $\sim_{us}$-class, i.e. if $p \sim_{us} q$ for any $p, q \in M$. It has the *essential accessibility property* if a set that is measurable (with respect to the volume) and is a union of $\sim_{us}$-classes must have 0 or full measure.

A partially hyperbolic diffeomorphism is *dynamically coherent* if there are foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively. In this case there is also a foliation $\mathcal{W}^c$ tangent to E^c whose leaves are the intersections of the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. Each leaf of $\mathcal{W}^{cs}$ is foliated by leaves of $\mathcal{W}^c$ and $\mathcal{W}^s$; leaves of $\mathcal{W}^{cu}$ have the analogous property. Not all partially hyperbolic diffeomorphisms are dynamically coherent; in Section 5 we describe examples, which go back to Smale.

Our main result is:

Theorem 0.1 *Let f be C^2, volume preserving, partially hyperbolic, dynamically coherent, and center bunched. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

If any of the three bundles E^u, E^c, and E^s is trivial, then the theorem is true, even without the hypotheses of dynamical coherence and center bunching. If E^c is trivial, f is an Anosov diffeomorphism and the result is due to Anosov [A]. If E^s (resp. E^u) is trivial, essential accessibility means that the foliation $\mathcal{W}^u$ (resp. $\mathcal{W}^s$) is ergodic (any set that is a union of unstable leaves must have zero or full measure) and the result follows easily from the Hopf argument; see the discussion after Theorem A in [PS2]. Henceforth we shall assume that all three bundles are nontrivial.

Theorem 0.1 generalizes a well-known result of Pugh and Shub (Theorem A of [PS2]). Their statement is identical to ours, but our definitions of partial hyperbolicity and center bunching are more general.

Pugh and Shub's definition of partial hyperbolicity in [PS2] requires that the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ be constant. This latter definition, which we call *strong partial hyperbolicity*, has been widely used in the literature proving ergodic properties of partially hyperbolic systems (e.g., [BP, PS2]). By contrast, the weaker definition of partial hyperbolicity used in this paper appears in the literature more often as a conclusion, rather than a hypothesis. To give a recent example, Horita-Tahzibi and independently Saghin have proved that every stably ergodic symplectomorphism is partially hyperbolic, in the weaker sense (this was earlier proved in dimension 4 case by Arnaud [Ar]).

Pugh and Shub's definition of center bunching assumes the symmetry condition $\nu = \hat{\nu}$ and $\gamma = \hat{\gamma}$. With this additional assumption, our center bunching condition becomes $\nu < \gamma^2$. Pugh and Shub's center bunching condition is:

$$\nu < \gamma^{2+2/\theta}, \tag{6}$$

where $\theta \leq 1$ is a Hölder exponent for the partially hyperbolic splitting.

It is possible to construct diffeomorphisms that stably satisfy all of the conditions of Theorem 0.1 but are not center bunched in the Pugh-Shub sense; in fact, in these examples we can simultaneously have θ arbitrarily close to 0 and γ arbitrarily close to 1. Pick a volume-preserving Anosov diffeomorphism $f : \mathbf{T}^k \to \mathbf{T}^k$ where the maximum Hölder exponent θ of the Anosov splitting is very small; suitable examples are constructed in [A, HW]. Regarding f as a partially hyperbolic map with trivial center, we obtain a constant $\nu = \hat{\nu} < 1$ for f satisfying inequalities (2) and (4). Let $g : \mathbf{T}^2 \to \mathbf{T}^2$ be an area-preserving diffeomorphism, chosen so that $\gamma = \min\{\|Tg\|^{-1}, \|Tg^{-1}\|^{-1}\}$ satisfies $\gamma^{2+2/\theta} < \nu < \gamma^2$. Then the volume-preserving diffeomorphism $f \times g$ is partially hyperbolic, and the maximal Hölder exponent for the partially hyperbolic splitting is the same as the exponent θ for the Anosov diffeomorphism f. The inequality $\nu < \gamma^2$ implies that center bunching holds and the inequality $\gamma^{2+2/\theta} < \nu$ implies that (6) fails to hold.

The example $f \times g$ is not accessible, but the techniques in [SW1] can be used to perturb $f \times g$ to obtain a volume-preserving skew product

$$(x, y) \mapsto (f(x), g_x(y))$$

that is stably accessible, is center bunched in the sense of Theorem 0.1, but fails to be center-bunched in the Pugh-Shub sense.

The difference between Theorem 0.1 and Theorem A in [PS2] is most striking when the bundle E^c is one dimensional. Since center bunching is automatic in this case, we immediately obtain:

Corollary 0.2 *Let f be C^2, volume preserving, partially hyperbolic with $\dim(E^c) = 1$, and dynamically coherent. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

Pugh and Shub have conjectured ([PS2], Conjecture 3) that essential accessibility alone should imply ergodicity for C^2, volume preserving, partially hyperbolic diffeomorphisms. Theorem 0.1 presents the closest attempt so far to proving this conjecture. The next step toward proving this conjecture might be to eliminate the hypothesis of dynamical coherence from Theorem 0.1 and Corollary 0.2. This would completely prove the conjecture in the case where E^c is 1-dimensional.

It is possible that center bunching implies dynamical coherence. We discuss this question further in Section 5. The Appendix to [BPSW] contains a proof of Theorem 0.1 in the (very) special case that $\mathcal{W}^c$ is absolutely continuous. As far as we know, the following question is still open:

Question: Assume that f is dynamically coherent and $\mathcal{W}^c$ is absolutely continuous. If f is essentially accessible, then is f ergodic?

Theorem 0.1 is proved in Section 2 as a consequence of Theorem 2.1, which is really the central result of the paper. Theorem 2.1 is proved in Sections 3 and 4. We thank Marcelo Viana for telling us about his proof of absolute continuity of stable foliations in the pointwise partially hyperbolic setting [V]. We thank Charles Pugh and Mike Shub for useful comments. Keith Burns was supported by NSF grant DMS-0100416, and Amie Wilkinson by NSF grant DMS-0100314.

1 Prelimaries and Notation

We assume that the Riemannian metric on M is chosen so that the inequalities (1)–(5) involving $\nu, \gamma, \hat{\nu}, \hat{\gamma}$ in the Introduction hold. Such a metric will be called adapted. Note that a rescaling of an adapted metric is still adapted. It will be convenient to assume that the metric is scaled so that the geodesic balls of radius 1 are very small neighborhoods of their centers. Distance with respect to the metric will be denoted by d.

The optimal choice of the function γ is:

$$\gamma_0(p) = \inf\{\|Tfv\| : v \in E^c(p),\ \|v\| = 1\}.$$

This function γ_0 is always Hölder continuous because the splitting $TM = E^u \oplus E^c \oplus E^s$, and in particular, the bundle E^c, is Hölder continuous (see Theorem A in [PSW]). Similarly, there are optimal functions $\nu_0, \hat{\gamma}_0$, and $\hat{\nu}_0$, which are Hölder continuous. There is no harm in increasing ν and $\hat{\nu}$ and decreasing $\gamma, \hat{\gamma}$ slightly, provided that the inequalities (1)–(5) still hold. By doing so, we may, if we wish, assume that these functions are smooth, though Hölder continuity will suffice for our purposes. We will have to make such adjustments for other reasons.

1.1 Foliation boxes and local leaves

Let $\mathcal{F}$ be a foliation of an n-manifold M with d-dimensional smooth leaves. For $r > 0$, we denote by $\mathcal{F}(x, r)$ the connected component of x in the intersection of $\mathcal{F}(x)$ with the ball $B(x, r)$. For B any subset of M, we set:

$$\mathcal{F}(B, r) = \bigcup_{x \in B} \mathcal{F}(x, r).$$

A *foliation box for* $\mathcal{F}$ is the image of $\mathbf{R}^{n-d} \times \mathbf{R}^d$ under a homeomorphism that sends each vertical $\mathbf{R}^d$-slice into a leaf of $\mathcal{F}$. The images of the vertical $\mathbf{R}^d$-slices will be called *local leaves of* $\mathcal{F}$ *in* U. A *complete transversal* to $\mathcal{F}$ in U is a smooth codimension-d disk in U that intersects each local leaf in U exactly once. If τ_1 and τ_2 are two complete transversals to $\mathcal{F}$ in U, we have the *holonomy map* $h_{\mathcal{F}} : \tau_1 \to \tau_2$, which takes a point in τ_1 to the intersection of its local leaf in U with τ_2.

By rescaling the metric on M, we may assume that for some $R > 1$, and any $x \in M$, the Riemannian ball $B(x, R)$ is contained in foliation boxes for each of the five foliations $\mathcal{W}^a$, $a = u, c, s, cu$, or cs. We assume that R is large enough so that all the objects considered in the sequel are small compared with R.

Having fixed such an R, we define the *local leaf of* $\mathcal{W}^a$ *through* x by:

$$\mathcal{W}^a_{loc}(x) = \mathcal{W}^a(x, R).$$

Any foliation box U for one of the foliations $\mathcal{W}^a$ that we consider in the rest of the paper will be small enough so that $\mathcal{W}^a_{loc}(x) \cap U$ is a local leaf of $\mathcal{W}^a$ in U for each $x \in U$.

We may assume that if $d(x, y) \leq 2$, then $\mathcal{W}^{cs}_{loc}(x) \cap \mathcal{W}^u_{loc}(y)$ and $\mathcal{W}^{cu}_{loc}(x) \cap \mathcal{W}^s_{loc}(y)$ are single points.

By (if necessary) slightly increasing ν and $\hat{\nu}$ and slightly decreasing γ and $\hat{\gamma}$ and further rescaling the metric to make the local leaves smaller, we may assume that our metric is still adapted, and that we have the following:

- if $q, q' \in \mathcal{W}^s_{loc}(p)$, then $d(f(q), f(q')) \leq \nu(p)d(q, q')$;
- if $q, q' \in \mathcal{W}^u_{loc}(p)$, then $d(f^{-1}(q), f^{-1}(q')) \leq \hat{\nu}(f^{-1}(p))d(q, q')$;
- if $q, q' \in \mathcal{W}^{cs}_{loc}(p)$, then $d(f(q), f(q')) \leq \hat{\gamma}(p)^{-1}d(q, q')$; and
- if $q, q' \in \mathcal{W}^{cu}_{loc}(p)$, then $d(f^{-1}(q), f^{-1}(q')) \leq \gamma(f^{-1}(p))^{-1}d(q, q')$.

Let U be a foliation box for $\mathcal{F}$, and let τ be a complete transversal to $\mathcal{F}$ in U. Let $Y \subseteq U$ be a measurable set. For a point $q \in \tau$, we define the *fiber* $Y(q)$ *of* Y *over* q to be the intersection of Y with the local leaf of $\mathcal{F}$ in U containing q. The *base* τ_Y *of* Y is the set of all $q \in \tau$ such that the fiber $Y(q)$ is nonempty (note that τ_Y is measurable, since we are working in a foliation box). We will sometimes say "Y fibers over Z" to indicate that $Z = \tau_Y$. If, for some $c \geq 1$, the inequalities

$$c^{-1} \leq \frac{m_{\mathcal{F}}(Y(q))}{m_{\mathcal{F}}(Y(q'))} \leq c$$

hold for all $q, q' \in \tau_Y$, then we say that Y *has c-uniform fibers.* The key estimates in this paper involve volumes of sets with c-uniform fibers.

1.2 Continuity and smoothness of foliations

We need two continuity and smoothness properties of the foliations associated with our dynamically coherent partially hyperbolic diffeomorphism f.

The first property is an almost immediate consequence of the continuity of the partially hyperbolic splitting:

Lemma 1.1 *We can choose $\delta > 0$ small enough and $\beta > 0$ large enough so that the following holds. Let p_0 and p_3 be any points in M with $d(p_0, p_3) < \delta$. Given any permutation (a_1, a_2, a_3) of $\{u, c, s\}$, define p_1 and p_2 by the conditions:*

$$p_i \in \mathcal{W}^{a_i}_{loc}(p_{i-1}), \qquad \textit{for } i = 1, 2, 3.$$

Then $d(p_i, p_{i-1}) \leq \beta d(p_0, p_3)$, for $i = 1, 2, 3$. See Figure 1.2.

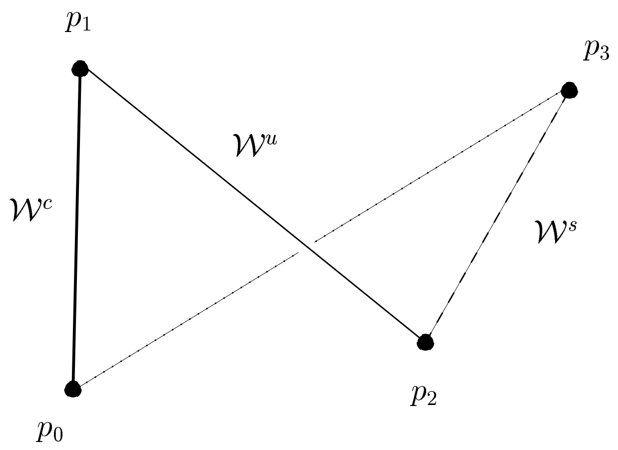


Figure 1: Picture for Lemma 1.1.

For the second property, recall that a map between metric spaces is L-Lipschitz, where L is a positive constant, if the map multiplies distances by at most L.

Proposition 1.2 *There is a constant $L > 0$ with the following properties:*

1. *If $p' \in \mathcal{W}^u_{loc}(p)$, then the holonomy from $\mathcal{W}^c_{loc}(p)$ to $\mathcal{W}^c(p')$ along $\mathcal{W}^u$-leaves is L-Lipschitz.*

2. *If $p' \in \mathcal{W}^s_{loc}(p)$, then the holonomy from $\mathcal{W}^c_{loc}(p)$ to $\mathcal{W}^c(p')$ along $\mathcal{W}^s$-leaves is L-Lipschitz.*

Proof. Both conclusions follow from Theorem B in [PSW]. The first uses the center-bunching inequality $\hat{\nu} < \gamma\hat{\gamma}$; the second uses $\nu < \gamma\hat{\gamma}$. $\diamond$

Remark: The use of Theorem B from [PSW] requires f to be C^2. Everywhere else in this paper it suffices for f to be $C^{1+\alpha}$, for some $\alpha > 0$.

1.3 Measures and absolute continuity

If μ is a measure and A and B are μ-measurable sets with $\mu(B) > 0$, we define the *density of A in B* by:

$$\mu(A : B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

If $S \subseteq M$ is a smooth submanifold, we denote by m_S the volume of the induced Riemannian metric on S. If $\mathcal{F}$ is a foliation with smooth leaves, and A is contained in a single leaf of $\mathcal{F}$ and is measurable in that leaf, then we denote by $m_{\mathcal{F}}(A)$ the induced Riemannian volume of A in that leaf. We use the shorthand m_a for $m_{\mathcal{W}^a}$. Note that m_a is *not* defined as a conditional measure.

When we say that the diffeomorphism f is volume preserving, we mean that f preserves a measure m that is equivalent to the Riemannian volume m_M on M. (This definition is independent of the metric, since the volumes defined by two different metrics on M are always equivalent.) Unless otherwise specified, measurable will mean measurable with respect to m.

Our arguments in this paper use two versions of the property of absolute continuity of a foliation.

The first version of absolute continuity involves holonomy maps between transversals. A foliation $\mathcal{F}$ with smooth leaves is *transversely absolutely continuous with bounded Jacobians* if, for every angle $\alpha \in (0, \pi)$, there exists $C \geq 1$ such that, for every foliation box U of diameter less than R, any two complete transversals τ_1, τ_2 to $\mathcal{F}$ in U of angle at least α with $\mathcal{F}$, and any m_{τ_1}–measurable set A contained in τ_1:

$$C^{-1} m_{\tau_1}(A) \quad \leq \quad m_{\tau_2}(h_{\mathcal{F}}(A)) \leq C m_{\tau_1}(A). \tag{7}$$

The second version involves a Fubini-like property. A foliation $\mathcal{F}$ with smooth leaves is *absolutely continuous with bounded Jacobians* if, for every $\alpha \in (0, \pi)$, there exists $C \geq 1$ such that, for every foliation box U of diameter less than R, any complete transversal τ to $\mathcal{F}$ in U of angle at least α with $\mathcal{F}$, and any measurable set A contained in U, we have the inequality:

$$C^{-1} m(A) \quad \leq \quad \int_{x \in \tau} m_{\mathcal{F}}(A \cap \mathcal{F}_{loc}(x))\, dm_\tau(x) \leq C m(A). \tag{8}$$

If $\mathcal{F}$ is transversely absolutely continuous with bounded Jacobians, then it is absolutely continuous with bounded Jacobians (see [BS] for a proof), but the converse does not hold (see Remark 3.9 in [B]). Note that the minimal C for which (7) holds is not necessarily the same minimal C for which (8) holds.

The foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ for a partially hyperbolic diffeomorphism are transversely absolutely continuous with bounded Jacobians. This was shown in the Anosov case by Anosov [A], in the case of strong partial hyperbolicity by Brin-Pesin and Pugh-Shub [BP, PS1]. In the (general) case of partial hyperbolicity, absolute continuity follows from Pesin theory. A direct proof in this context has been given by Viana [V]. In fact, all of these results show that the Jacobians are continuous functions, and so are bounded, since M is compact. In general, $\mathcal{W}^c$ does not have either absolute continuity property, even when f is dynamically coherent (examples were first constructed by Katok; open sets of examples by Shub-Wilkinson [SW2]).

1.4 Saturated sets and absolute continuity

A set is *saturated by a foliation* $\mathcal{F}$ or $\mathcal{F}$*-saturated* if it is a union of entire leaves of $\mathcal{F}$.

Proposition 1.3 *Let $\mathcal{F}$ be absolutely continuous with bounded Jacobians, and let U be a foliation box for $\mathcal{F}$ with complete transversal τ. Suppose that $\{Y_n\}_{n\geq 0}$ is a sequence of measurable sets in U with c-uniform fibers, for some $c \geq 1$. Then, for every $\mathcal{F}$-saturated measurable set X, we have the equivalence:*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m_\tau(\tau_X : \tau_{Y_n}) = 1.$$

Remark: The hypothesis that X is $\mathcal{F}$-saturated can be weakened: it suffices for $X \cap U$ to be a union of local leaves of $\mathcal{F}$ in U.

Proof of Proposition 1.3. Let X^* be the complement of X in M. Then X^* is also $\mathcal{F}$-saturated. The proposition can be reformulated in terms of X^*. We have to prove the equivalence:

$$\lim_{n\to\infty} m(X^* : Y_n) = 0 \iff \lim_{n\to\infty} m_\tau(\tau_{X^*} : \tau_{Y_n}) = 0.$$

For each n, let

$$m_n = \inf_{q\in\tau_{Y_n}} m_{\mathcal{F}}(Y_n(q)).$$

Since the fibers of Y_n are c-uniform, it follows that $m_n > 0$ for all n, and:

$$m_n \leq m_{\mathcal{F}}(Y_n(q)) \leq c m_n,$$

for all $q \in \tau_{Y_n}$. Absolute continuity implies that there exists a $C \geq 1$ such that

$$C^{-1} m(Y_n) \leq \int_{q\in\tau} m_{\mathcal{F}}(Y_n(q))\, dm_\tau(q) \leq C m(Y_n).$$

Together, these inequalities imply that, for any $q \in \tau_{Y_n}$,

$$C^{-1} m_n m(Y_n) \leq m_\tau(\tau_{Y_n}) \leq C c m_n m(Y_n). \tag{9}$$

Since X^* is $\mathcal{F}$-saturated, the $\mathcal{F}$-fiber of $X^* \cap Y_n$ over a point $q \in \tau_{Y_n}$ is either empty or equal to $Y_n(q)$. Thus $X^* \cap Y_n$ also has c-uniform fibers, and, as above, we obtain:

$$C^{-1} m_n m(X^* \cap Y_n) \leq m_\tau(\tau_{X^* \cap Y_n}) \leq C c m_n m(X^* \cap Y_n). \tag{10}$$

Noting that $\tau_{X^* \cap Y_n} = \tau_{X^*} \cap \tau_{Y_n}$ and dividing the inequalities in (10) by those in (9), we obtain:

$$(C^2 c)^{-1} m(X^* : Y_n) \leq m_\tau(\tau_{X^*} : \tau_{Y_n}) \leq C^2 c m(X^* : Y_n).$$

The result follows easily from this. $\diamond$

A set A is *essentially $\mathcal{F}$-saturated* if there exists a measurable $\mathcal{F}$-saturated set A', which we call an *essential $\mathcal{F}$-saturate* of A, with $m(A \,\Delta\, A') = 0$.

Corollary 1.4 *Let $\mathcal{F}$, U and τ be as in Proposition 1.3. Let $\{Y_n\}$ and $\{Z_n\}$ be sequences of measurable subsets of U with c-uniform fibers. Suppose that $\tau_{Y_n} = \tau_{Z_n}$, for all n. Then, for any essentially $\mathcal{F}$-saturated set $X \subseteq U$, we have the equivalence:*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \iff \lim_{n\to\infty} m(X : Z_n) = 1.$$

Proof. Let X' be an essential $\mathcal{F}$-saturate of X. Using Proposition 1.3, we have the equivalences:

$$\begin{aligned}
\lim_{n\to\infty} m(X : Y_n) = 1 \quad &\iff \quad \lim_{n\to\infty} m(X' : Y_n) = 1 \\
&\iff \quad \lim_{n\to\infty} m_\tau(\tau_{X'} : \tau_{Y_n}) = 1 \\
&\iff \quad \lim_{n\to\infty} m_\tau(\tau_{X'} : \tau_{Z_n}) = 1 \\
&\iff \quad \lim_{n\to\infty} m(X' : Z_n) = 1 \\
&\iff \quad \lim_{n\to\infty} m(X : Z_n) = 1.
\end{aligned}$$

$\diamond$

1.5 Notational conventions and a distortion estimate

In the rest of this paper we adhere to the convention that if q is a point in M and j is an integer, then q_j denotes the point $f^j(q)$, with $q_0 = q$. If $\alpha : M \to \mathbf{R}$ is a positive function, and $j \geq 1$ is an integer, let

$$\alpha_j(p) = \alpha(p)\alpha(p_1)\cdots\alpha(p_{j-1}),$$

and

$$\alpha_{-j}(p) = \alpha(p_{-j})^{-1}\alpha(p_{-j+1})^{-1}\cdots\alpha(p_{-1})^{-1}.$$

We set $\alpha_0(p) = 1$. Observe that α_j is a multiplicative cocycle; in particular, we have $(\alpha\beta)_j = \alpha_j\beta_j$ and $\alpha_{-j}(p)^{-1} = \alpha_j(p_{-j})$. Note also that if α is a constant function, then $\alpha_n = \alpha^n$.

Using this notation, it is easy to formulate a generalization of the estimates in Section 1.1 on how f changes the distance between nearby points.

Lemma 1.5 *The following hold for all $n \geq 0$. If $q, q' \in \mathcal{W}^s_{loc}(p)$, then*

$$d(q_n, q'_n) \leq \nu_n(p)d(q, q').$$

If $q, q' \in \mathcal{W}^u_{loc}(p)$, then

$$d(q_{-n}, q'_{-n}) \leq \hat{\nu}_{-n}(p)^{-1}d(q, q').$$

If $q_j, q'_j \in \mathcal{W}^{cs}_{loc}(p_j)$ for $0 \leq j \leq n-1$, then

$$d(q_n, q'_n) \leq \hat{\gamma}_n(p)^{-1}d(q, q').$$

If $q_{-j}, q'_{-j} \in \mathcal{W}^{cu}_{loc}(p_{-j})$ for $1 \leq j \leq n$, then

$$d(q_{-n}, q'_{-n}) \leq \gamma_{-n}(p)d(q, q').$$

The hypotheses in the last two parts of this lemma are stronger because f may not map $\mathcal{W}^{cs}_{loc}(p)$ into $\mathcal{W}^{cs}_{loc}(f(p))$, and similarly, f^{-1} may not map $\mathcal{W}^{cu}_{loc}(p)$ into $\mathcal{W}^{cu}_{loc}(f^{-1}(p))$.

The next proposition will be used to compare values of Hölder cocycles at nearby points.

Proposition 1.6 *Let $\alpha : M \to \mathbf{R}$ be a positive Hölder continuous function, with exponent $\theta > 0$. Then there exists a constant $H > 0$ such that, for all $p, q \in M$ and all $B > 0$, if:*

$$\sum_{i=0}^{n-1} d(p_i, q_i)^\theta \leq B,$$

for some $n \geq 1$, then

$$e^{-HB} \leq \frac{\alpha_n(p)}{\alpha_n(q)} \leq e^{HB}.$$

Similarly, if

$$\sum_{i=1}^{n} d(p_{-i}, q_{-i})^\theta \leq B,$$

for some $n \geq 1$, then

$$e^{-HB} \leq \frac{\alpha_{-n}(p)}{\alpha_{-n}(q)} \leq e^{HB}.$$

Proof. We prove the first part of the proposition. The second part is proved similarly.

The function $\log \alpha$ is also Hölder continuous with exponent θ. Let $H > 0$ be the Hölder constant of $\log \alpha$, so that for all $x, y \in M$:

$$|\log \alpha(x) - \log \alpha(y)| \leq H d(x, y)^\theta.$$

The desired inequalities are equivalent to:

$$|\log \alpha_n(p) - \log \alpha_n(q)| \leq HB.$$

Expanding $\log \alpha_n$ as a series, we obtain:

$$\begin{aligned} |\log \alpha_n(p) - \log \alpha_n(q)| &\leq \sum_{i=0}^{n-1} |\log \alpha(p_i) - \log \alpha(q_i)| \\ &\leq H \sum_{j=0}^{n-1} d(p_j, q_j)^\theta. \\ &\leq HB, \end{aligned}$$

since

$$\sum_{i=0}^{n-1} d(p_i, q_i)^\theta \leq B,$$

by the hypothesis of the Proposition. $\diamond$

As usual $P = O(Q)$ means that there is a constant $C > 0$ such that $|P| \leq CQ$. Usually P and Q will depend on an integer n and one or more points in M. The constant C must be independent of n and the choice of the points.

A sequence Y_n of measurable sets is *regular* if there exist $C > 0$ and $k \geq 1$ such that, for all $n \geq 0$,

$$m(Y_{n+k}) \geq Cm(Y_n).$$

Two sequences of sets Y_n and Z_n are *comparable* if there exists a $k \geq 1$ such that, for all $n \geq k$, we have

$$Y_{n+k} \subseteq Z_n \subseteq Y_{n-k}.$$

Comparability is an equivalence relation. The following lemma is an easy consequence of the definitions.

Lemma 1.7 *Let Y_n and Z_n be comparable sequences of measurable sets, with Y_n regular. Then Z_n is also regular. If the sets Y_n have positive measure, then so do the Z_n, and, for any measurable set X,*

$$\lim_{n\to\infty} m(X : Y_n) = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} m(X : Z_n) = 1.$$

2 The main theorem

The properties of accessibility and essential accessibility can be reformulated using the notion of saturation. Accessibility means that a set which is both $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated must be either empty or all of M. Essential accessibility means that a measurable set which is both $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated must have either 0 or full measure.

The central result of this paper is:

Theorem 2.1 *Let f be C^2, partially hyperbolic, dynamically coherent, and center bunched. Let A be a measurable set that is both essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated. Then the set of Lebesgue density points of A is $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated.*

Recall that x is a *Lebesgue density point* of the measurable set A if

$$\lim_{r \to 0} m(A : B(x,r)) = 1,$$

where $B(x,r)$ is the Riemannian ball of radius r about x. This definition is independent of choice of metric. If A is a measurable set, the set $\widehat{A}$ of Lebesgue density points of A satisfies $m(A \,\Delta\, \widehat{A}) = 0$.

The central result of Pugh and Shub in [PS2] is a version of Theorem 2.1, which involves a different notion of density point (defined in [PS2]) and a slightly different hypothesis:

For $a = u$ or s, if A is essentially $\mathcal{W}^a$-saturated, then the set of julienne density points of A is $\mathcal{W}^a$-saturated.

In contrast, Theorem 2.1 requires that A be *both* essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated in order to conclude anything.

Theorem 0.1 follows easily from Theorem 2.1 by a version of the Hopf argument and a result of Brin and Pesin (see Section 2 of [BPSW] for more details).

Proof of Theorem 0.1. To prove that f is ergodic, it suffices to show that the Birkhoff averages of continuous functions are almost everywhere constant. Let φ be a continuous function, and let

$$\hat{\varphi}_s(p) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(f^i(p)) \quad \text{and } \hat{\varphi}_u(p) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(f^{-i}(p))$$

be the forward and backward Birkhoff averages of φ under f. The function $\hat{\varphi}_s$ is constant along $\mathcal{W}^s$-leaves, and $\hat{\varphi}_u$ is constant along $\mathcal{W}^u$-leaves. It follows that for any $a \in \mathbf{R}$, the sets

$$A_s(a) = \hat{\varphi}_s^{-1}(-\infty, a] \quad \text{and} \quad A_u(a) = \hat{\varphi}_u^{-1}(-\infty, a].$$

are $\mathcal{W}^s$-saturated and $\mathcal{W}^u$-saturated, respectively.

The Birkhoff Ergodic Theorem implies that $\hat{\varphi}_s = \hat{\varphi}_u$ almost everywhere. Consequently $m(A_s(a) \,\Delta\, A_u(a)) = 0$, so that the set $A(a) = A_u(a) \cap A_s(a)$ has $A_u(a)$ as an essential $\mathcal{W}^u$-saturate and $A_s(a)$ as an essential $\mathcal{W}^s$-saturate. Thus $A(a)$ satisfies the hypotheses of Theorem 2.1: it is both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated.

It follows from Theorem 2.1 that the set $\widehat{A}(a)$ of Lebesgue density points of $A(a)$ is both $\mathcal{W}^u$-saturated and $\mathcal{W}^s$-saturated. Essential accessibility implies that $\widehat{A}(a)$ has 0 or full measure. But $m(A(a) \,\Delta\, \widehat{A}(a)) = 0$, so $A(a)$ itself has

0 or full measure. Since a was arbitrary, it follows that $\hat{\varphi}_s$ and $\hat{\varphi}_u$ are almost everywhere constant, and so f is ergodic.

To prove that f has the Kolmogorov property, it suffices to show that all sets in the Pinsker subalgebra $\mathcal{P}$ have 0 or full measure. According to Proposition 5.1 of [BP], if f is partially hyperbolic, then any set in $P \in \mathcal{P}$ is both essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated. It again follows from Theorem 2.1 and essential accessibility that P has 0 or full measure. $\diamond$

In order to prove Theorem 2.1 it suffices to show that the set of Lebesgue density points of A is $\mathcal{W}^s$-saturated; applying this result with f replaced by f^{-1} then shows that the set of Lebesgue density points of A is also $\mathcal{W}^u$-saturated.

Following Pugh and Shub, we consider for each $p \in M$ a sequence of sets, called center-unstable juliennes, that lie in the center-unstable manifold $\mathcal{W}^{cu}(p)$ and shrink exponentially as $n \to \infty$ while becoming increasingly thin in the $\mathcal{W}^u$-direction. We choose Hölder continuous functions τ and σ such that

$$\nu < \tau < \sigma\gamma \quad \text{and} \quad \sigma < \min\{\hat{\gamma}, 1\}.$$

This is possible because of the center bunching assumption. The reader should think of τ as being just a little bigger than ν and σ as just a little bit less than $\min\{\hat{\gamma}, 1\}$. The reader might also choose to keep in mind the global case where the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ are constants, and where τ and σ can be chosen to be constant. In this case the cocycles τ_n and σ_n are just the constants τ^n and σ^n.

We then define

$$B_n^c(p) = \mathcal{W}^c(p, \sigma_n(p)),$$

$$J_n^u(p) = f^{-n}(\mathcal{W}^u(p_n, \tau_n(p)))$$

and

$$J_n^{cu}(p) = \bigcup_{q \in B_n^c(p)} J_n^u(q).$$

This definition is the most convenient for our arguments. The distortion estimates in Section 1.5 imply that the varying radii $\tau_n(q)$ for $q \in B_n^c(p)$ used in the definition of $J_n^{cu}(p)$ could be replaced by the fixed radius $\tau_n(p)$ without affecting the conclusions of the main results (Propositions 2.3 and 2.4) below. More precisely, the sequences of sets $J_n^{cu}(p)$ and $\bigcup_{q \in B_n^c(p)} f^{-n}(\mathcal{W}^u(q_n, \tau_n(p)))$

are comparable. This is a straightforward consequence of the following lemma.

Lemma 2.2 *Let $\alpha : M \to \mathbf{R}$ be a positive, Hölder continuous function. Then there is a constant $C \geq 1$ such that*

$$C^{-1} \leq \frac{\alpha_n(y)}{\alpha_n(x)} \leq C,$$

whenever $x, y \in M$ satisfy any of the following:

1. $y \in \mathcal{W}^s_{loc}(x)$,
2. $y \in \mathcal{W}^c(x, \sigma_n(x))$, *or*
3. $y \in f^{-n}(\mathcal{W}^u_{loc}(x_n))$.

Proof. This is an easy consequence of Proposition 1.6. In case 2., we use the facts that $\sigma < \min\{\hat{\gamma}, 1\}$, and

$$d(x_j, y_j) \leq \hat{\gamma}_j(x)^{-1} d(x, y)$$

for $j = 0, \ldots, n-1$, which follows from Lemma 1.5.$\diamond$

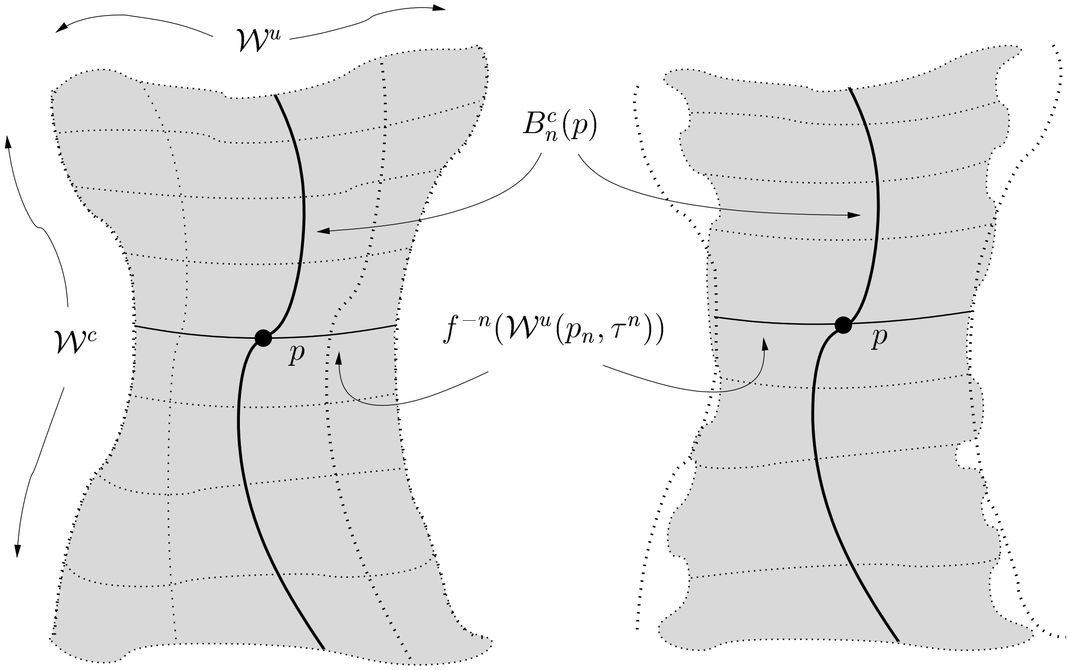
Our cu-juliennes are closely related to, but not exactly the same as, those of Pugh and Shub. In the case where σ and τ are constant functions, their center-unstable julienne is the foliation product of $\mathcal{W}^c(p, \sigma^n)$ and $f^{-n}(\mathcal{W}^u(p_n, \tau^n))$; see Figure 2. In this case, the image under f^n of our $J^{cu}_n(p)$ appears in [PS2] as a tubelike approximation to the Pugh-Shub center-unstable postjulienne of rank n. The results of [PS2] show that the cu-juliennes defined here and in [PS2] are comparable. Thus our cu-juliennes could be replaced by the Pugh-Shub cu-juliennes in Propositions 2.3 and 2.4.

As in [PS2], the cu-juliennes have a quasi-conformality property: they are approximately preserved by holonomy along the stable foliation.

Proposition 2.3 *There exists a positive integer k such that if p and p' are any points in M with $p' \in \mathcal{W}^s(p, 1)$, then the holonomy map $h^s : \mathcal{W}^{cu}_{loc}(p) \to \mathcal{W}^{cu}(p')$ induced by the stable foliation $\mathcal{W}^s$ has the property that*

$$J^{cu}_{n+k}(p') \subseteq h^s(J^{cu}_n(p)) \subseteq J^{cu}_{n-k}(p'),$$

for all $n \geq k$.



Center-unstable julienne $J^{cu}_n(p)$ in [PS2]

$J^{cu}_n(p)$ in this paper.

Figure 2: Two types of center-unstable juliennes, when τ and σ are constant.

The other crucial property of the cu-juliennes is that, *for the sets that appear in the proof of Theorem 2.1*, Lebesgue density points are precisely cu-julienne density points.

Proposition 2.4 *Let X be a measurable set that is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated. Then $p \in M$ is a Lebesgue density point of X if and only if:*

$$\lim_{n\to\infty} m_{cu}(X : J_n^{cu}(p)) = 1.$$

Remark: Pugh and Shub show that Lebesgue almost every point of *any* measurable set is a cu-julienne density point. In their argument they prove a Vitali covering lemma for their juliennes. This argument accounts for their definition of cu-juliennes as a foliation product and for the stronger bunching hypothesis in their main result. We do know whether their result, specifically Theorem 7.1 of [PS2], still holds under our weaker bunching hypothesis (although it can be shown that their hypothesis can be weakened from $\nu < \gamma^{2+2/\theta_0}$ to $\nu < \gamma^{1/\theta_0}$).

Proof of Theorem 2.1. Let A^s be an essential $\mathcal{W}^s$-saturate of A. Since $m(A \,\Delta\, A^s) = 0$, the Lebesgue density points of A are precisely the same as those of A^s, which by Proposition 2.4 are precisely the cu-julienne density points of A^s. On the other hand, it follows easily from the transverse absolute continuity of $\mathcal{W}^s$ and Proposition 2.3 that the set of cu-julienne density points of A^s is $\mathcal{W}^s$-saturated. Thus the set of Lebesgue density points for A is $\mathcal{W}^s$-saturated. As we noted above, to see that this set is $\mathcal{W}^u$-saturated, we just consider f^{-1} instead of f. $\diamond$

The proof of Proposition 2.3 is the same as in [PS2]. For completeness we reproduce the argument in the next section. The proof of Proposition 2.4 is in the final section; what is new in this paper is to be found there.

3 Julienne quasiconformality

We outline Pugh and Shub's proof of Proposition 2.3. It will suffice to show that k can be chosen so that

$$h^s(J_n^{cu}(p)) \subseteq J_{n-k}^{cu}(p'), \tag{11}$$

for all $n \geq k$, whenever p and p' satisfy the hypotheses of the proposition. The hypotheses of the proposition treat p and p' symmetrically, so we can then reverse their roles to obtain:

$$\overline{h}^s(J_n^{cu}(p')) \subseteq J_{n-k}^{cu}(p),$$

for all $n \geq k$, where $\overline{h}^s : \mathcal{W}_{loc}^{cu}(p') \to \mathcal{W}^{cu}(p)$ is the holonomy induced by the stable foliation. Since $\overline{h}^s$ and h^s are inverses, we then obtain:

$$J_n^{cu}(p') \subseteq h^s(J_{n-k}^{cu}(p)),$$

for all $n \geq k$, which is even slightly stronger than the claim in the statement of Proposition 2.3.

In order to prove that k can be chosen so that (11) holds, we need two lemmas.

Lemma 3.1 *There exists a positive integer k_1 such that:*

$$h^s(B_n^c(p)) \subseteq B_{n-k_1}^c(p'),$$

for all $n \geq k_1$, whenever $p' \in \mathcal{W}^s(p, 1)$.

Proof. Since h^s is L-Lipschitz by Proposition 1.2, the image of $\mathcal{W}^c(p, \sigma_n(p))$ under h^s is contained in $\mathcal{W}^c(p', L\sigma_n(p)) \subseteq \mathcal{W}^c(p', \sigma_{n-k_1}(p'))$, for any k_1 large enough so that $\sigma_{-k_1} > L$. $\diamond$

Lemma 3.2 *There exists a positive integer k_2 such that the following holds for every integer $n \geq k_2$ and every $x \in M$. If $x' \in \mathcal{W}^s(x, 2)$ and $y \in f^{-n}(\mathcal{W}^u(x_n, \tau_n(x)))$, then*

$$y' \in J_{n-k_2}^{cu}(x'),$$

where y' is the image of y under stable holonomy from $\mathcal{W}_{loc}^{cu}(x)$ to $\mathcal{W}^{cu}(x')$.

A picture of the setup in Lemma 3.2 is found in Figure 3.

Proof of Lemma 3.2. Let z' be the unique point in $\mathcal{W}_{loc}^u(y') \cap W_{loc}^c(x')$. Then z'_n is the unique point in $\mathcal{W}_{loc}^u(y'_n) \cap W_{loc}^c(x'_n)$. It will suffice to prove that $d(y'_n, z'_n) = O(\tau_n(z'))$ and $d(x', z') = O(\sigma_n(x'))$. Since σ, τ and ν are Hölder continuous, it follows from Lemma 2.2 that if α is any of these functions,

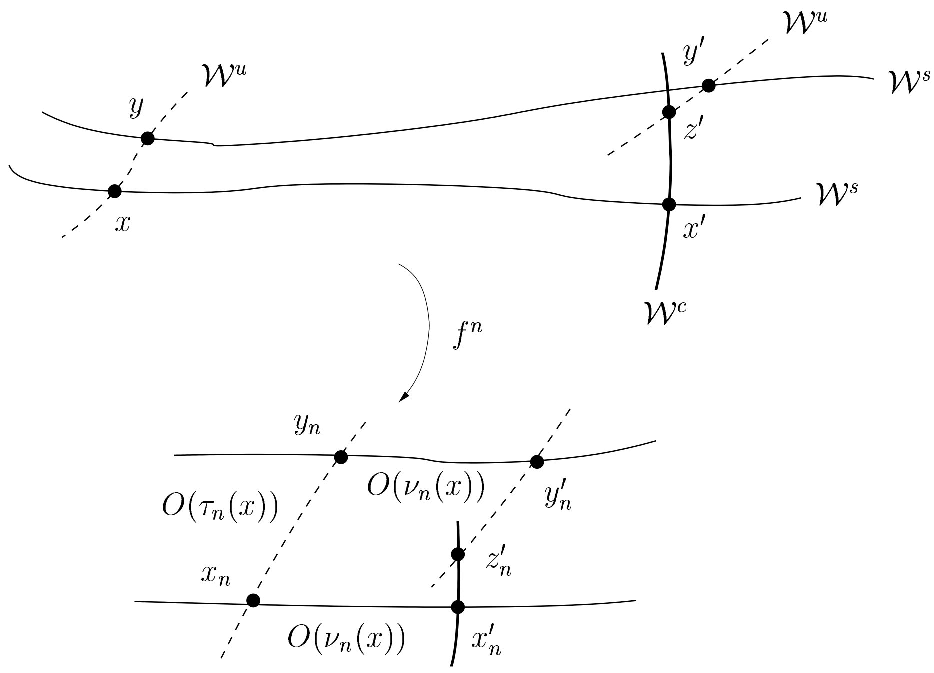


Figure 3: Picture for the proof of Lemma 3.2.

then $\alpha_n(a) = O(\alpha_n(b))$, where a and b are any of the points x, y, x, y', and z'.

We have $d(x_n, y_n) \leq \tau_n(x)$ because $y \in f^{-n}(\mathcal{W}^u(x_n, \tau_n(x)))$. By Lemma 1.5, we also have that $d(x_n, x'_n) = O(\nu_n(x))$ and $d(y_n, y'_n) = O(\nu_n(y)) = O(\nu_n(x))$, since $d(x, x')$ and $d(y, y')$ are both $O(1)$. Since $\nu < \tau$, it follows that $d(x'_n, y'_n) = O(\tau_n(x))$. It now follows from Lemma 1.1 that $d(x'_n, z'_n)$ and $d(y'_n, z'_n)$ are both $O(\tau_n(x)) = O(\tau_n(z'))$.

Since x'_n and z'_n lie in the same center leaf at distance $O(\tau_n(x)) = O(\tau_n(x'))$, Lemma 1.5 now implies that $d(x', z') = O(\gamma_{-n}(x')\tau_n(x'))$. But τ and σ were chosen so that $\tau < \gamma\sigma$. Hence $\gamma_{-n}(x')\tau_n(x') < \sigma_n(x')$ and $d(x', z') = O(\sigma_n(x'))$, as desired. $\diamond$

Proof of Proposition 2.3. As noted above, it suffices to prove the inclusion (11). For $q \in B^c_n(p)$, let $q' = h^s(q)$. Then $q' \in B^c_{n-k_1}(p')$ by Lemma 3.1. Hence $d(q, q') < 2$ and we can apply Lemma 3.2 to obtain

$$h^s(J^{cu}_n(p)) \subseteq \bigcup_{q' \in B^c_{n-k_1}(p')} J^{cu}_{n-k_2}(q') = \bigcup_{z \in Q} J^u_{n-k_2}(z),$$

where

$$Q = \bigcup_{q' \in B^c_{n-k_1}(p')} B^c_{n-k_2}(q').$$

For $k \geq k_2$, we have:

$$\bigcup_{z \in Q} J^u_{n-k_2}(z) \subseteq \bigcup_{z \in Q} J^u_{n-k}(z).$$

It therefore suffices to find $k \geq k_2$ so that $Q \subseteq B^c_{n-k}(p')$. This latter inclusion holds if:

$$\sigma_{n-k_1}(p') + \sigma_{n-k_2}(q') \leq \sigma_{n-k}(p'),$$

which is equivalent to:

$$\sigma_{-k_1}(p'_n) + \sigma_{-k_2}(q'_n) \leq \sigma_{-k}(p'_n).$$

We complete the proof by choosing k large enough so that $\sigma_{-k} \geq \sigma_{-k_1} + \sup \sigma_{-k_2}$ (which automatically gives $k \geq k_2$). $\diamond$

Some related results will be needed in the next section. Define the extended cu-julienne $E_n(p)$ by

$$E_n(p) = \bigcup_{p' \in \mathcal{W}^s(p, \sigma_n(p))} J^{cu}_n(p').$$

Let $F_n(p)$ be the foliation product of $J^{cu}_n(p)$ and $\mathcal{W}^s(p, \sigma_n(p))$. The set $F_n(p)$ consists of points q such that $\mathcal{W}^s_{loc}(q)$ passes through $J^{cu}_n(p)$ and $\mathcal{W}^{cu}_{loc}(q)$ passes through $\mathcal{W}^s(p, \sigma_n(p))$. Both E_n and F_n fiber over $\mathcal{W}^s(p, \sigma_n(p))$. In the case of E_n the fibers are the juliennes $J^{cu}_n(p')$, $p' \in \mathcal{W}^s(p, \sigma_n(p))$. In the case of F_n the fiber over $p' \in \mathcal{W}^s(p, \sigma_n(p))$ is the image of $J^{cu}_n(p)$ under holonomy along $\mathcal{W}^s$ from $\mathcal{W}^{cu}_{loc}(p)$ to $\mathcal{W}^{cu}(p')$.

The following is an immediate consequence of Lemma 3.2:

Corollary 3.3 *Let k_2 be as in Lemma 3.2. Then*

$$E_{n+k_2}(p) \subseteq F_n(p) \subseteq E_{n-k_2}(p)$$

for any $p \in M$ and any $n \geq k_2$. Therefore the sequences $E_n(p)$ and $F_n(p)$ are comparable.

We can also think of $F_n(p)$ as fibering over $J_n^{cu}(p)$. The fibers are the images of $\mathcal{W}^s(p, \sigma_n(p))$ under holonomy along center-unstable leaves. The fiber over $q \in J_n^{cu}(p)$ is comparable to $\mathcal{W}^s(q, \sigma_n(p))$. This is made precise in the next lemma.

Lemma 3.4 *Let*

$$G_n(p) = \bigcup_{q \in J_n^{cu}(p)} \mathcal{W}^s(q, \sigma_n(p)).$$

Then there is a positive integer k_3 such that

$$G_{n+k_3}(p) \subseteq F_n(p) \subseteq G_{n-k_3}(p)$$

for any $p \in M$ and any $n \geq k_3$. The sequences $F_n(p)$ and $G_n(p)$ are comparable.

Proof.

Suppose q' lies in the boundary of the fiber of $F_n(p)$ that lies in $\mathcal{W}^s_{loc}(q)$ for some $q \in J_n^{cu}(p)$. Then $q' \in J_n^{cu}(p')$ for a point p' that lies in the boundary of $\mathcal{W}^s(p, \sigma^n(p))$. The diameters of $J_n^{cu}(p)$ and $J_n^{cu}(p')$ are both $O(\sigma_n(p)) = O(\sigma_n(p'))$, and $d(p, p') = \sigma_n(p)$. Hence, if k_3 is large enough, we will have

$$\sigma_{n+k_3}(p) \leq d(q, q') \leq \sigma_{n-k_3}(p).$$

Thus all points on the boundary of the fiber of $F_n(p)$ in $\mathcal{W}^s_{loc}(q)$ lie outside $\mathcal{W}^s(q, \sigma_{n+k_3}(p))$ and inside $\mathcal{W}^s(q, \sigma_{n-k_3}(p))$. $\diamond$

Figure 3 is a schematic illustration of the relationship between the sets $E_n(p)$, $F_n(p)$ and $G_n(p)$. All three sets contain $J_n^{cu}(p)$ and $\mathcal{W}^s(p, \sigma_n(p))$. The set $E_n(p)$ fibers over $\mathcal{W}^s(p, \sigma_n(p))$ with fibers of the form $J_n^{cu}(\cdot)$. The set $G_n(p)$ fibers over $J_n^{cu}(p)$ with fibers of the form $\mathcal{W}^s(\cdot, \sigma_n(p))$. The foliation product $F_n(p)$ of $J_n^{cu}(p)$ and $\mathcal{W}^s(p, \sigma_n(p))$ is, in some sense, intermediate between $E_n(p)$ and $G_n(p)$. Corollary 3.3 and Lemma 3.4 tell us that the sequences $E_n(p)$, $F_n(p)$ and $G_n(p)$ are all comparable.

4 Lebesgue density and cu-julienne density

We now come to the proof of Proposition 2.4. We must show that if a measurable set X is both $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated, then a point $p \in M$ is a Lebesgue density point of X if and only if

$$\lim_{n \to \infty} m_{cu}(X : J_n^{cu}(p)) = 1.$$

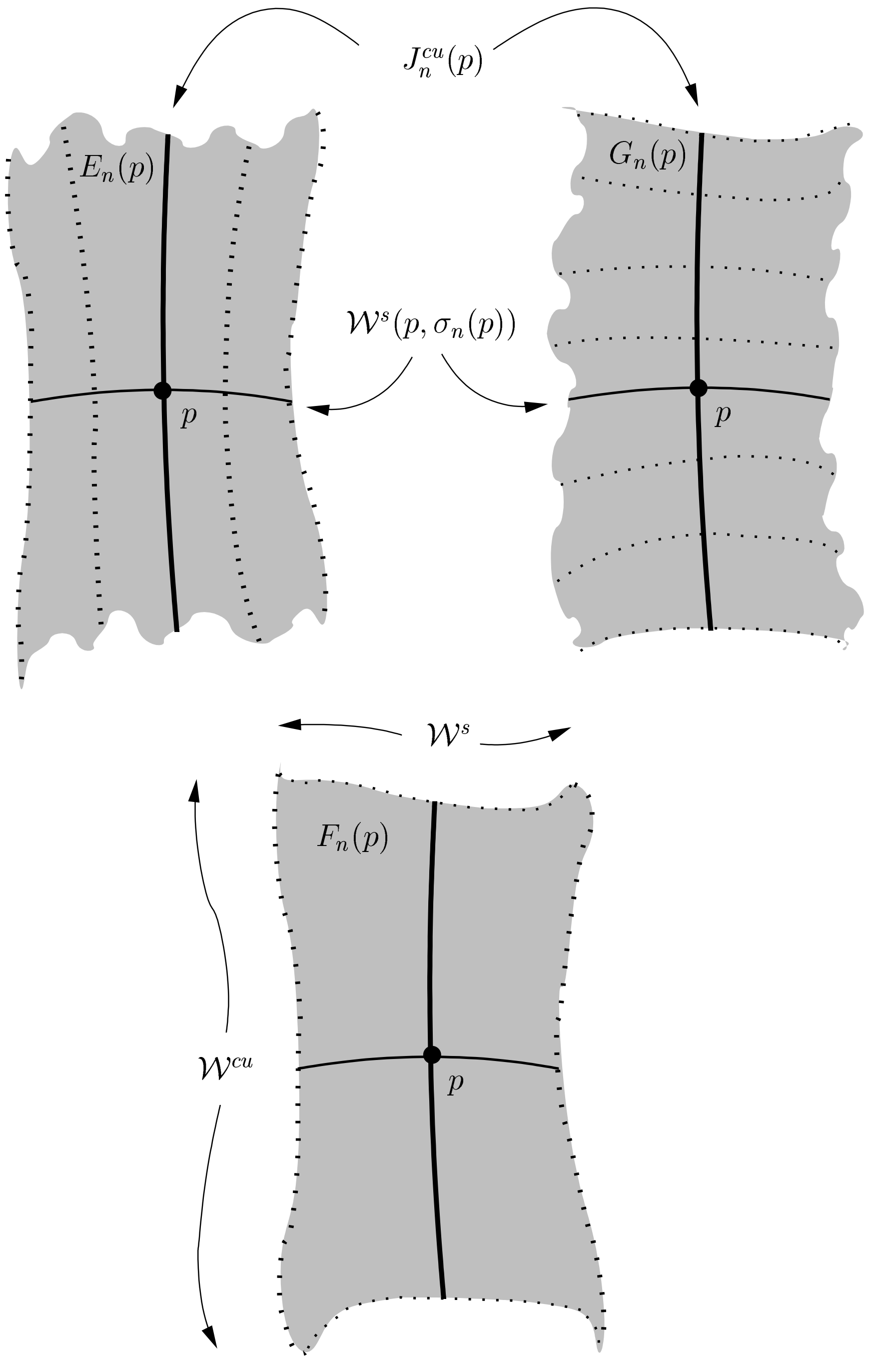


Figure 4: Comparison between $E_n(p)$, $F_n(p)$ and $G_n(p)$.

The proof involves several sequences of sets containing p. Some have already been introduced; the others are defined below. Our strategy is to establish the following chain of equivalences:

$$\begin{aligned}
p \text{ is a Lebesgue density point of } X &\iff \lim_{n\to\infty} m(X : B_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m(X : C_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m(X : D_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m(X : E_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m(X : F_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m(X : G_n(p)) = 1 \\
&\iff \lim_{n\to\infty} m_{cu}(X : J_n^{cu}(p)) = 1.
\end{aligned}$$

The final equivalence requires $\mathcal{W}^s$-saturation of X, and the equivalence

$$\lim_{n\to\infty} m(X : D_n(p)) = 1 \iff \lim_{n\to\infty} m(X : E_n(p)) = 1$$

requires essential $\mathcal{W}^u$-saturation of X; the other equivalences require only measurability of X. We have already seen in the previous section that the sequences $E_n(p)$, $F_n(p)$ and $G_n(p)$ are all comparable. We show in this section that $G_n(p)$ is regular, in the sense of Lemma 1.7. We shall see below that the sequences $B_n(p)$, $C_n(p)$ and $D_n(p)$ are all comparable and $B_n(p)$ is regular.

The forward implication in the first equivalence is obvious from the definition of $B_n(p)$:

$$B_n(p) = B(p, \sigma_n(p)).$$

The backward implication follows from this definition and the fact that the ratio $\sigma_{n+1}(p)/\sigma_n(p) = \sigma(p_n)$ of successive radii is less than 1, and is bounded away from both 0 and 1 independently of n. From this we also see that $B_n(p)$ is regular.

We now define $C_n(p)$ and $D_n(p)$. Like the sets $E_n(p)$ and $F_n(p)$ introduced in the previous section, both $C_n(p)$ and $D_n(p)$ fiber over $\mathcal{W}^s(p, \sigma_n(p))$. Over a point $p' \in \mathcal{W}^s(p, \sigma_n(p))$, the fiber of $C_n(p)$ is

$$\mathcal{W}^u(\mathcal{W}^c(p', \sigma_n(p)), \sigma_n(p)),$$

and the fiber of $D_n(p)$ is

$$\mathcal{W}^u(\mathcal{W}^c(p', \sigma_n(p')), \sigma_n(p)).$$

It is easily shown using Lemma 1.1 that the sequences $B_n(p)$ and $C_n(p)$ are comparable, for all $p \in M$. It follows immediately that:

$$\lim_{n\to\infty} m(X : B_n(p)) = 1 \Longleftrightarrow \lim_{n\to\infty} m(X : C_n(p)) = 1.$$

The equivalence

$$\lim_{n\to\infty} m(X : C_n(p)) = 1 \Longleftrightarrow \lim_{n\to\infty} m(X : D_n(p)) = 1$$

is proved using Lemma 2.2, which implies that $\sigma_n(p') = O(\sigma_n(p''))$, for any $p', p'' \in \mathcal{W}^s_{loc}(p)$. It follows from this that the sequences $\mathcal{W}^c(p', \sigma_n(p))$ and $\mathcal{W}^c(p', \sigma_n(p'))$ are comparable for any $p' \in W^s_{loc}(p)$. Hence, the sequences $C_n(p)$ and $D_n(p)$ are comparable, for any $p \in M$.

We now consider the equivalence:

$$\lim_{n\to\infty} m(X : D_n(p)) = 1 \Longleftrightarrow \lim_{n\to\infty} m(X : E_n(p)) = 1.$$

This is proved by applying Corollary 1.4, with $\mathcal{F} = \mathcal{W}^u$ and $\tau = \mathcal{W}^{cs}_{loc}(p)$, to the sequences of sets $D_n(p)$ and $E_n(p)$. Both sets $D_n(p)$ and $E_n(p)$ fiber over the same base $D^{cs}_n(p)$, where:

$$D^{cs}_n(p) = \bigcup_{p' \in \mathcal{W}^s(p, \sigma_n(p))} \mathcal{W}^c(p', \sigma_n(p')).$$

Our desired equivalence follows immediately from Corollary 1.4 provided the fibers of $D_n(p)$ and $E_n(p)$ are c-uniform for some c. The fibers of $D_n(p)$ are the balls $\mathcal{W}^u(q, \sigma_n(p))$, for $q \in D^{cs}_n(p)$, which are clearly c-uniform for any large enough c. The fibers of $E_n(p)$ are $J^u_n(q) = f^{-n}\mathcal{W}^u(q_n, \tau_n(q))$, for $q \in D^{cs}_n(p)$. Uniformity of these fibers is the content of the next lemma.

Lemma 4.1 *There exists $c \geq 1$ such that for all $q, q' \in D^{cs}_n(p)$, and all $n \geq 0$, we have:*

$$c^{-1} \leq \frac{m_u(J^u_n(q))}{m_u(J^u_n(q'))} \leq c.$$

Proof. We use the uniformity of the balls $\mathcal{W}^u(q_n, \tau_n(q))$ and the distortion estimate from Section 1.5. As an immediate consequence of Lemma 2.2 we have:

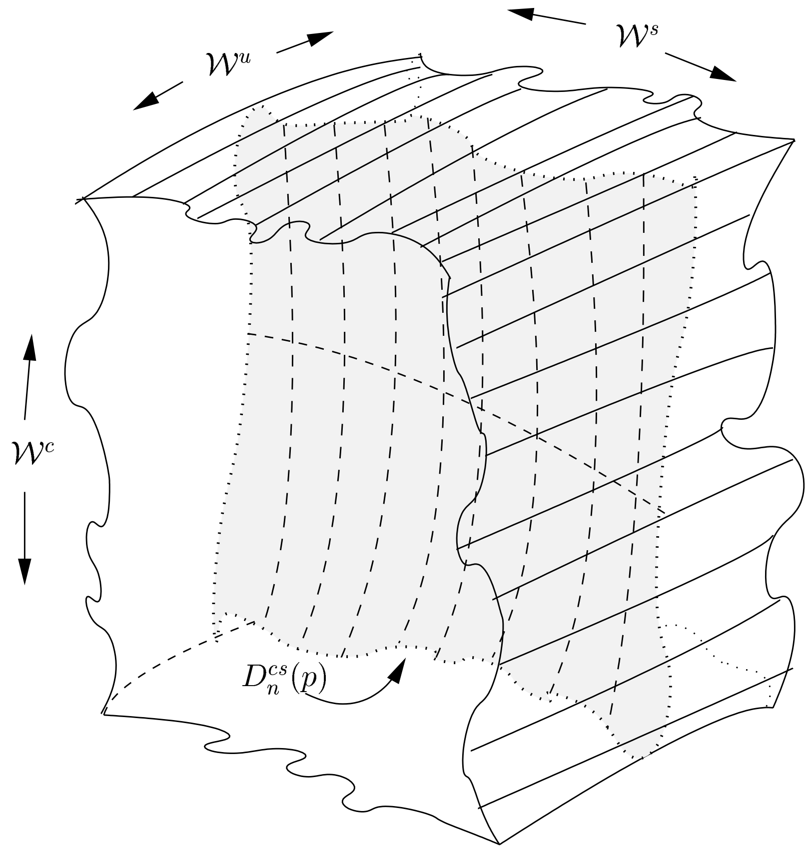


Figure 5: Cubelike object $D_n(p)$.

Lemma 4.2 *Let $\alpha : M \to \mathbf{R}$ be positive and Hölder continuous. Then for all $n \geq 0$ and any $q, q' \in D_n^{cs}(p)$, we have:*

$$\alpha_n(q) = O(\alpha_n(q')).$$

Applying this lemma with $\alpha = \tau$, it is then clear that we may choose $S \geq 1$ so that for all $n \geq 0$ and $q, q' \in D_n^{cs}(p)$, we have

$$S^{-1} \leq \frac{m_u(\mathcal{W}^u(q_n, \tau_n(q)))}{m_u(\mathcal{W}^u(q'_n, \tau_n(q')))} \leq S. \tag{12}$$

We next observe that the jacobian $\mathrm{Jac}(Tf^{-n}|_{E^u})$ is nearly constant when

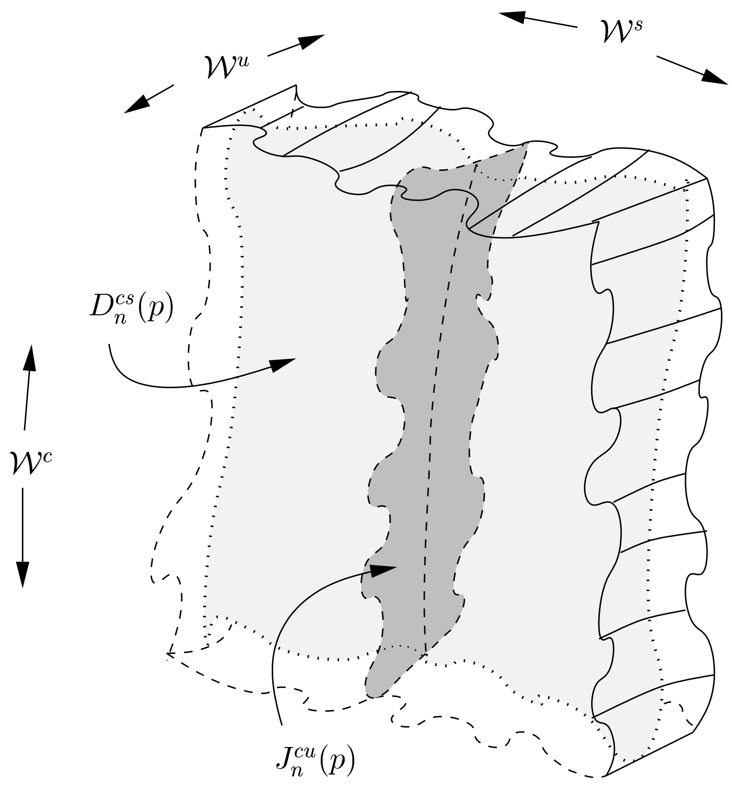


Figure 6: Extended julienne $E_n(p)$.

restricted to the set

$$f^n(E_n(p)) = \bigcup_{q \in D_n^{cs}(p)} \mathcal{W}^u(q_n, \tau_n(q)).$$

More precisely, we have:

Lemma 4.3 *There exists $C \geq 1$ such that, for all $n \geq 1$, and all $x, y \in E_n(p)$,*

$$C^{-1} \leq \frac{\mathrm{Jac}(Tf^{-n}|_{E^u})(x_n)}{\mathrm{Jac}(Tf^{-n}|_{E^u})(y_n)} \leq C.$$

Proof. The desired inequalities are equivalent to:

$$C^{-1} \leq \frac{\mathrm{Jac}(Tf^n|_{E^u})(x)}{\mathrm{Jac}(Tf^n|_{E^u})(y)} \leq C.$$

These inequalities follow from Proposition 1.6 with $\alpha = \mathrm{Jac}(Tf|_{E^u})$, once we show that there exists a positive constant $\kappa < 1$ such that

$$\max_{0 \le j \le n-1} \mathrm{diam}(f^j(E_n(p))) = O(\kappa^n),$$

for all $n \ge 1$. To find κ, note first that

$$f^j(E_n(p)) \subset \bigcup_{q \in D_n^{cs}(p)} f^{j-n}(\mathcal{W}^u(q_n, \tau_n(q))) \subset \bigcup_{q \in D_n^{cs}(p)} \mathcal{W}^u(q_j, \tau_n(q))).$$

If $q \in D^{cs}(p)$, then $q \in \mathcal{W}^c(p', \sigma_n(p'))$, for some $p' \in \mathcal{W}^s(p, \sigma_n(p))$. By Lemma 4.2, we have $\sigma_n(p') = O(\sigma_n(p))$, and hence $d(p,q) = O(\sigma_n(p))$. It follows that the diameter of $D_n^{cs}(p)$ is $O(\sigma_n(p))$.

Our choice of σ ensures that $\hat{\gamma}^{-1}\sigma < 1$. It now follows from Lemma 1.5 and an inductive argument that, for $q, q' \in D_n^{cs}(p)$, we have

$$d(q_j, q'_j) \le \hat{\gamma}_j(p)^{-1}\, d(q, q') = O(\hat{\gamma}_n(p)^{-1}\sigma_n(p)),$$

for $0 \le j \le n-1$.

We finish the proof by setting $\kappa = \max\{\hat{\gamma}^{-1}\sigma, \tau\}$. We have shown that $f^j(D_n^{cs}(p))$, which is the base of the set $f^j(E_n(p))$, is contained in a ball in $\mathcal{W}^{cs}_{loc}(p_j)$ of radius $O((\hat{\gamma}^{-1}\sigma)_n(p)) = O(\kappa^n)$. The fiber of $f^j(E_n(p))$ over a point $q_j \in f^j(D_n^{cs}(p))$ is contained in $\mathcal{W}^u(q_j, \tau_n(q)) \subseteq \mathcal{W}^u(q_j, \kappa^n)$. $\diamond$

Let $q \in D_n^{cs}(p)$, and let $X \subseteq J_n^u(q)$ be a measurable set (such as $J_n^u(q)$ itself). Then:

$$\begin{aligned} m_u(X) &= m_u(f^{-n}(f^n(X)) \\ &= \int_{f^n(X)} \mathrm{Jac}(Tf^{-n}|_{E^u})(x)\, dx. \end{aligned}$$

From this and Lemma 4.3 we then obtain:

Lemma 4.4 *There exists a $K > 0$ such that, for all $n \ge 0$, for any $q, q' \in D_n^{cs}(p)$, and any measurable sets $X \subset J_n^u(q), X' \subset J_n^u(q')$, we have:*

$$K^{-1}\frac{m_u(f^n(X))}{m_u(f^n(X'))} \le \frac{m_u(X)}{m_u(X')} \le K\frac{m_u(f^n(X))}{m_u(f^n(X'))}.$$

Lemma 4.1 now follows from (12) and Lemma 4.4 with $X = J^u_n(q)$ and $X' = J^u_n(q')$.$\diamond$

The final equivalence,

$$\lim_{n\to\infty} m(X : G_n(p)) = 1 \Longleftrightarrow \lim_{n\to\infty} m_{cu}(X : J^{cu}_n(p)) = 1,$$

is an application of Proposition 1.3, with $\mathcal{F} = \mathcal{W}^s$ and $Y_n = G_n(p)$. The set $G_n(p)$ fibers over $J^{cu}_n(p)$ (which lies in the transversal $\tau = W^{cu}_{loc}(p)$) with fibers $\mathcal{W}^s(q, \sigma_n(p))$, which are clearly c-uniform, for some $c \geq 1$.

Lemma 4.5 *The sequence $G_n(p)$ is regular for each $p \in M$.*

Proof. Recall that:

$$G_n(p) = \bigcup_{q\in J^{cu}_n(p)} \mathcal{W}^s(q, \sigma_n(p)),$$

where

$$J^{cu}_n(p) = \bigcup_{q\in B^c_n(p)} J^u_n(q).$$

As we saw above, the ratio $\sigma_{n+1}(p)/\sigma_n(p) = \sigma(p_n)$ is uniformly bounded below away from 0. Consequently, the ratio

$$\frac{m_s(\mathcal{W}^s(q, \sigma_{n+1}(p)))}{m_s(\mathcal{W}^s(q, \sigma_n(p)))}$$

is bounded away 0, uniformly in p, q, and n. It follows from this and the absolute continuity of $\mathcal{W}^s$ that it will suffice to show that

$$\frac{m_{cu}(J^{cu}_{n+1}(p))}{m_{cu}(J^{cu}_n(p))}$$

is bounded away from 0, uniformly in $n \geq 0$.

Again using the fact that σ_{n+1}/σ_n is uniformly bounded away from 0, we obtain that the ratio

$$\frac{m_c(B^c_{n+1}(p))}{m_c(B^c_n(p))}$$

is bounded away from 0, uniformly in p and n. By the Lipschitzness of $\mathcal{W}^u$ inside $\mathcal{W}^{cu}$, it thus suffices to show that:

$$\frac{m_u(J^u_{n+1}(q))}{m_u(J^u_n(q))} \geq C, \tag{13}$$

for some $C > 0$ and all $n \geq 0$ and $q \in B^c_n(p)$. In fact, we shall show that (13) holds for some $C > 0$, and all $n \geq 0$, $q \in M$.

To obtain (13), we will apply Lemma 4.4 with $q = q'$, $X = J^u_{n+1}(q)$, and $X' = J^u_n(q)$. This gives us:

$$\frac{m_u(J^u_{n+1}(q))}{m_u(J^u_n(q))} \quad \geq \quad K^{-1}\frac{m_u(f^n(J^u_{n+1}(q)))}{m_u(f^n(J^u_n(q)))}.$$

But $f^n(J^u_{n+1}(q)) = f^{-1}(\mathcal{W}^u(q_{n+1}, \tau_{n+1}(q)))$ and $f^n(J^u_n(q)) = \mathcal{W}^u(q_n, \tau_n(q))$, and hence:

$$\frac{m_u(f^n(J^u_{n+1}(q)))}{m_u(f^n(J^u_n(q)))} \quad = \quad \frac{m_u(f^{-1}(\mathcal{W}^u(q_{n+1}, \tau_{n+1}(q))))}{m_u(\mathcal{W}^u(q_n, \tau_n(q)))}.$$

This ratio is uniformly bounded below away from 0, since f^{-1} is a diffeomorphism, the leaves of $\mathcal{W}^u$ are uniformly smooth, and the ratio $\tau_{n+1}(q)/\tau_n(q) = \tau(q_n)$ is uniformly bounded below away from 0. ⋄

Finally, Lemma 1.7, Corollary 3.3, Lemma 3.4, and Lemma 4.5 give us:

$$\begin{aligned}\lim_{n\to\infty} m(X : E_n(p)) = 1 \quad &\Longleftrightarrow \quad \lim_{n\to\infty} m(X : F_n(p)) = 1 \\ &\Longleftrightarrow \quad \lim_{n\to\infty} m(X : G_n(p)) = 1.\end{aligned}$$

This completes the proof of Proposition 2.4. ⋄

Remark: It is tempting to try to shorten the proof of Proposition 2.4 by showing directly that

$$\lim_{n\to\infty} m(X : D_n(p)) = 1 \Longleftrightarrow \lim_{n\to\infty} m_{cu}(X : J^{cu}_n(p)) = 1.$$

This could be done if $X \cap \mathcal{W}^{cu}_{loc}(p)$ were essentially $\mathcal{W}^u$-saturated. The proof would be similar to the above arguments, but easier, because $\mathcal{W}^u$ is C^1 inside a $\mathcal{W}^{cu}$-leaf. However, since the foliation $\mathcal{W}^{cu}$ might not be absolutely continuous, we have no way of knowing that $X \cap \mathcal{W}^{cu}(p)$ is essentially $\mathcal{W}^u$-saturated.

5 Appendix: Center Bunching and Dynamical Coherence

As mentioned in the Introduction, there is an example, attributed by Smale to A. Borel, of a partially hyperbolic diffeomorphism for which the center bundle is C^2 but is not closed under the formation of Lie brackets. The example is also described in [KH]. It is presented there and in Smale's original paper [S] as a linear Anosov diffeomorphism on a nilmanifold that is not a torus. Wilkinson [W] observed that if one creates a center subbundle for this diffeomorphism by grouping together invariant weak subbundles, one obtains a partially hyperbolic diffeomorphism whose center subbundle is not integrable. This is described in more detail in [P].

We now give a brief description of these examples. Let H be the Heisenberg group, which is the subgroup of $GL(3, \mathbf{R})$ consisting of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then H is a non-abelian simply-connected nilpotent Lie group, whose Lie algebra $\mathfrak{h}$ is generated by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which satisfy the relations:

$$[X, Z] = [Y, Z] = 0; \ [X, Y] = Z.$$

We identify $(\xi, \eta, \zeta) \in \mathbf{R}^3$ with $\xi X + \eta Y + \zeta Z \in \mathfrak{h}$. Since H is nilpotent and simply-connected, any lattice in H is cocompact and, up to finite index, the image of a lattice in h under the exponential map. We note that there are partially hyperbolic automorphisms of compact quotients of H, which are dynamically coherent. Let $A \in SL(2, \mathbf{Z})$ be a hyperbolic matrix, and let $\Gamma_0 = \exp(\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z})$. It is easily checked that

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

is an automorphism of $\mathfrak{h}$ that preserves the integer lattice $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. It induces an automorphism f_A of the compact manifold H/Γ_0 that is partially hyperbolic. The center foliation is a nontrivial fibration of H/Γ_0 by circles that fibers over the torus $\mathbf{T}^2$. The projection of f_A onto this torus is the automorphism induced by A. The diffeomorphism f_A is accessible, because the stable and unstable directions lie in the span of (the left-invarariant vector fields corresponding to) X and Y, and $[X, Y] = Z$. Note that there are no Anosov automorphisms of H/Γ_0.

Now consider the group $G = H \times H$. Its lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ is generated by $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2\}$ whose nontrivial bracket relations are:

$$[X_1, Y_1] = Z_1; \quad [X_2, Y_2] = Z_2.$$

We identify $(\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $\xi_1 X_1 + \eta_1 Y_1 + \zeta_1 Z_1 + \xi_2 X_2 + \eta_2 Y_2 + \zeta_2 Z_2 \in \mathfrak{g}$ (the reason for rearranging the terms in the first $\mathbf{R}^3$ factor will become clear in a minute).

As above, consider $A \in SL(2, \mathbf{Z})$ and let $\lambda > 1$ and $\lambda^{-1} < 1$ be the eigenvalues of A. Then λ and λ^{-1} are units in the ring of algebraic integers. The field $\mathbf{Q}(\lambda)$ is a quadratic extension of $\mathbf{Q}$; its Galois involution σ interchanges λ and λ^{-1}. Now let $\widetilde{\Gamma}$ be the irreducible lattice in $\mathfrak{g}$ consisting of vectors of the form:

$$(u_1, u_2, u_3, \sigma(u_1), \sigma(u_2), \sigma(u_3)),$$

with $u_1, u_2, u_3 \in \mathbf{Z}[\lambda]$, the ring of algebraic integers in $\mathbf{Q}(\lambda)$. Since G is simply connected and nilpotent, $\Gamma = \exp(\widetilde{\Gamma})$ is a cocompact lattice in G. For any real numbers a and b, the linear map

$$B : (\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2) \mapsto (\lambda^{a+b}\zeta_1, \lambda^b \eta_1, \lambda^a \xi_1, \lambda^{-a}\xi_2, \lambda^{-b}\eta_2, \lambda^{-a-b}\zeta_2)$$

is an automorphism of $\mathfrak{g}$. If a and b are integers, this automorphism preserves $\widetilde{\Gamma}$ and thus induces a diffeomorphism $f_B : G/\Gamma \to G/\Gamma$. The diffeomorphism f_B is partially hyperbolic if one of $a, b, a+b$ is nonzero and Anosov if all three are nonzero. Smale describes in [S] the cases $a = 1, b = 2$ and $a = 1, b = -3$. More general algebraic constructions of Anosov diffeomorphisms along these lines can be found in [L].

We now now assume that $a + b > b \geq a \geq 0$. In this case f_B is partially hyperbolic, with E^u spanned by Z_1, E^s spanned by Z_2, and E^c spanned by X_1, X_2, Y_1, and Y_2. Then f_B is not dynamically coherent, since E^c is not closed under formation of Lie brackets; in fact,

$$[E^c, E^c] = \mathfrak{g}.$$

Note that there is always another way to view f_B as partially hyperbolic with an integrable center. If $a = 0$, then $a + b = b > 0$, and we can take E^c to be the bundle spanned by X_1 and X_2; if $a > 0$, then f_B is Anosov and we can take E^c to be the trivial bundle. Recently, Nicolas Gourmelon showed that it is possible to deform f_B in the case $a > 0$ to obtain a stably ergodic, volume-preserving, partially hyperbolic diffeomorphism that is neither dynamically coherent nor Anosov.

Now we examine the center bunching hypothesis for f_B when $a+b > b \geq a \geq 0$. We have

$$\nu = \hat{\nu} = \lambda^{-a-b}, \quad \text{and} \quad \gamma = \hat{\gamma} = \lambda^{-b}.$$

Consequently,

$$\nu = \hat{\nu} = \lambda^{-a-b} \geq \lambda^{-2b} = \gamma\hat{\gamma},$$

with equality holding if and only if $a = b$. Thus f_B is never center bunched. It falls just short of being center bunched in the best possible case, when $a = b$. In fact, as the next theorem shows, this type of construction will never produce a center bunched diffeomorphism that is not dynamically coherent.

Theorem 5.1 *Suppose that $f : M \to M$ is C^2 and partially hyperbolic, and satisfies the symmetry conditions $\nu = \hat{\nu}$ and $\gamma = \hat{\gamma}$. If f is center bunched and the partially hyperbolic splitting is C^2, then f is dynamically coherent.*

Proof. We show that $E^c \oplus E^s$ is integrable. The proof that $E^u \oplus E^c$ is integrable is very similar. If $E^c \oplus E^s$ is not integrable, then there exist a constant $C > 0$, a point $p \in M$, and a sequence of C^1 paths $\kappa_j : [0, 1] \to M$ such that

1. $\kappa_j(0) = p$ and $\kappa_j(1) \in W^u(p)$,
2. $\dot{\kappa}_j(t) \in (E^c \oplus E^s)(\kappa_j(t))$, for all $t \in (0, 1)$, and
3. $d(\kappa_j(0), \kappa_j(1)) \geq C\delta_j^2$, where $\delta_j := \text{length}(\kappa_j) \to 0$ as $j \to \infty$.

Condition 3. is where we use the fact that $E^s \oplus E^u$ is C^2.

For j sufficiently large, we can choose n_j so that

$$d(f^{n_j}(\kappa_j(0)), f^{n_j}(\kappa_j(1))) \geq 1,$$

and

$$d(f^i(\kappa_j(0)), f^i(\kappa_j(1))) < 1,$$

for $i < n_j$. Consequently, there exists an $M > 0$ such that, for all j:

$$d(f^{n_j}(\kappa_j(0)), f^{n_j}(\kappa_j(1))) \leq M.$$

Lemma 1.5 implies that

$$\begin{aligned} M &\geq d(f^{n_j}(\kappa_j(0)), f^{n_j}(\kappa_j(1))) \\ &\geq \hat{\nu}_{n_j}(p)^{-1} d(\kappa_j(0), \kappa_j(1)) \\ &\geq \hat{\nu}_{n_j}(p)^{-1} C\delta_j^2, \end{aligned}$$

so that

$$\delta_j \leq \sqrt{\frac{M\hat{\nu}_{n_j}(p)}{C}}.$$

Since $\delta_j \to 0$ as $j \to \infty$, we also have that $n_j \to \infty$ as $j \to \infty$.

It is not difficult to show, using Lemma 2.2, that there is a constant $K \geq 1$ such that

$$K^{-1} \leq \frac{\hat{\gamma}_{n_j}(\kappa_j(s))}{\hat{\gamma}_{n_j}(\kappa_j(t))} \leq K,$$

for all $s, t \in [0, 1]$ and $j \geq 0$. Since κ_j is tangent to $E^c \oplus E^s$, we thus obtain:

$$\begin{aligned} \text{length}(f^{n_j}(\kappa_j)) &\leq K\hat{\gamma}_{n_j}(p)^{-1}\text{length}(\kappa_j) \\ &\leq K\hat{\gamma}_{n_j}(p)^{-1}\delta_j \\ &\leq K\sqrt{\frac{M}{C} \cdot \frac{\hat{\nu}_{n_j}(p)}{\hat{\gamma}_{n_j}(p)^2}} \end{aligned}$$

Center bunching and symmetry imply that $\hat{\nu} < (\gamma\hat{\gamma}) = \hat{\gamma}^2$, so it follows that

$$\text{length}(f^{n_j}(\kappa_j)) \to 0 \quad \text{as} \quad j \to \infty.$$

But this contradicts the fact that

$$d(f^{n_j}(\kappa_j(0)), f^{n_j}(\kappa_j(1))) \geq 1.$$

◇

Two natural questions raised by the above discussion are:

Question: Let $f : M \to M$ be partially hyperbolic and center bunched. Is f then dynamically coherent?
and:

Question: Let $f : M \to M$ be a partially hyperbolic affine transformation of a compact homogeneous space G/Γ. If f is accessible, then is f also dynamically coherent?

If f fails to be dynamically coherent, we say that f is dynamically incoherent. It is not known whether the algebraic examples discussed here are stably dynamically incoherent. In fact the following question is open:

Question: Is dynamical incoherence a C^1-stable property?

If the answer to this question is "yes" then by the results in [DW], in a C^1 neighborhood of any non dynamically coherent diffeomorphism, there is a C^1-dense set of stably accessible and stably non dynamically coherent diffeomorphisms.

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