The centralizer of a C^1 generic diffeomorphism is trivial

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April 16, 2008

Abstract

Answering a question of Smale, we prove that the space of C^1 diffeomorphisms of a compact manifold contains a residual subset of diffeomorphisms whose centralizers are trivial.

Key words: Centralizer, C^1 generic properties.

MSC 2000: 37C05, 37C15, 37C85.

Introduction

In this announcement, we describe the solution in the C^1 topology to a question asked by S. Smale on the genericity of trivial centralizers. The question is posed in the following context. We fix a compact connected manifold M and consider the space $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of M, endowed with the C^r topology. The *centralizer* of $f \in \mathrm{Diff}^r(M)$ is defined as

$$Z^r(f) := \{g \in \mathrm{Diff}^r(M) : fg = gf\}.$$

Clearly $Z^r(f)$ always contains the cyclic group $\langle f \rangle$ of all the powers of f. We say that f has *trivial centralizer* if $Z^r(f) = \langle f \rangle$. Smale asked the following:

Question 1 ([Sm1, Sm2]). *Consider the set of C^r diffeomorphisms of a compact connected manifold M with trivial centralizer.*

1. *Is this set dense in $\mathrm{Diff}^r(M)$?*
2. *Is it* residual *in $\mathrm{Diff}^r(M)$? That is, does it contain a dense G_δ subset?*
3. *Does it contain an open and dense subset of $\mathrm{Diff}^r(M)$?*

For the case $r = 1$ we now have a complete answer to this question.

Theorem A (B-C-W). *For any compact connected manifold M, there is a residual subset of $\mathrm{Diff}^1(M)$ consisting of diffeomorphisms with trivial centralizer.*

Theorem B (B-C-Vago-W). *For any compact manifold M, the set of C^1 diffeomorphisms with trivial centralizer does not contain any open and dense subset.*

Theorem A gives an affirmative answer to the second (and hence the first) part of Question 1: our aim in this text is to present the structure of its proof that will be detailed in [BCW2]. Theorem B gives a negative answer to the third part of Questions 1: with G. Vago we prove in [BCVW] that there exists a family of C^∞ diffeomorphisms with large centralizer that is C^1 dense in a nonempty open subset of $\mathrm{Diff}^1(M)$. For these examples, one has to consider separately the case of the circle, the surfaces and manifolds of dimension greater or equal to 3: in dimension less or equal to two such a diffeomorphism appears as the time-1 map of a flow, whereas in higher dimension each example we build possesses an open set of periodic points.

These results suggest that the topology of the set of diffeomorphisms with trivial centralizer is complicated and motivate the following questions.

Question 2. *1.* Consider the set of diffeomorphisms whose centralizer is trivial. *What is its interior?*

2. Is it a Borel set?
(See [FRW] for a negative answer to this question in the measurable context.)

3. The set $\{(f,g) \in \mathrm{Diff}^1(M) \times \mathrm{Diff}^1(M) : fg = gf\}$ is closed.
What is its local topology? For example, is it locally connected?

Our motivation for considering Question 1 comes from at least two sources. First, the study of C^1-generic diffeomorphisms has seen substantial progress in the last decade, and Question 1 is an elementary test question for the existing techniques. More intrinsically, there are several classical motivations for Question 1. In physics (for example, in Hamiltonian mechanics), one searches for symmetries of a given system in order to reduce the complexity of the orbit space. The group of such symmetries is precisely the centralizer. In a similarly general vein, a central theme in dynamics is to understand the conjugacy classes inside of $\mathrm{Diff}^r(M)$; that is, to find the orbits of the action of $\mathrm{Diff}^r(M)$ on itself by conjugacy. Theorem A implies that the stabilizer in this action of a generic element is trivial.

Knowing the centralizer of a diffeomorphism gives answers to more concrete questions as well, such as the embeddability of a diffeomorphism in a flow and the existence of roots of a diffeomorphism. The study of diffeomorphisms and flows are closely related, and indeed, every diffeomorphism appears as the return map of a smooth flow to a cross-section, and the time-1 map of a flow is a diffeomorphism. These two studies have many differences as well, and it is natural to ask when a given diffeomorphism can be embedded as the time-1 map of a flow (the centralizer of such a diffeomorphism must contain either $\mathbb{R}$ or the circle $\mathbb{R}/\mathbb{Z}$). A weaker question is to ask whether a diffeomorphism f admits a root; that is, if one can write $f = g^k$, for some integer $k > 1$. If f admits such a root, then its centralizer is not trivial, although it might still be discrete.

Question 1 can also be viewed as a problem about the group structure of $\mathrm{Diff}^1(M)$, from a generic vantage point. An easy transversality argument (written in [G, Proposition 4.5] for circle homeomorphisms) gives a description of the group generated by a generic family of diffeomorphisms: *for a generic $(f_1, \dots, f_p) \in (\mathrm{Diff}^r(M))^p$ with $p \geq 2$ and $r \geq 0$, the group $\langle f_1, \dots, f_p\rangle$ is free.* Restated in these terms, Theorem A says that for a generic f, if G is any abelian subgroup of $\mathrm{Diff}^1(M)$

containing f, then $G = \langle f \rangle$. The same conclusion holds if G is assumed to be nilpotent, for then the center of G, and thus G itself, must equal $\langle f \rangle$. One can ask whether the same conclusions hold for other properties of G, such as solvability. Question 1 could be generalized in the following way.

Question 3. *Fix a reduced word $w(f, g_1, \ldots, g_k)$ in $\mathit{Diff}^1(M)$. How small can the set $\{\mathbf{g} \in \left(\mathit{Diff}^1(M)\right)^k : w(f, \mathbf{g}) = id\}$ be for the generic $f \in \mathit{Diff}^1(M)$?*

The history of Question 1 goes back to the work of N. Kopell [Ko], who gave a complete answer for $r \geq 2$ and the circle $M = S^1$: the set of diffeomorphisms with trivial centralizer contains an open and dense subset of $\mathrm{Diff}^r(S^1)$. For $r \geq 2$ on higher dimensional manifolds, there are partial results with additional dynamical assumptions, such as hyperbolicity [PY1, PY2] and partial hyperbolicity [Bu1]. In the C^1 setting, Togawa proved that generic Axiom A diffeomorphisms have trivial centralizer. In an earlier work [BCW1], we showed that for $\dim(M) \geq 2$, the C^1 generic conservative (volume-preserving or symplectic) diffeomorphism has trivial centralizer in $\mathrm{Diff}^1(M)$. A more precise list of previous results can be found in [BCW1].

The rest of the paper describes some of the main novelties in the proof of Theorem A and the structure of its proof.

Local and global: the structure of the proof of Theorem A

The proof of Theorem A breaks into two parts, a "local" one and a "global" one. The local part proves that for the generic f, if g commutes with f, then $g = f^\alpha$ on an open and dense subset $W \subset M$, where $\alpha\colon W \to \mathbb{Z}$ is a locally constant function. The global part consists in proving that for generic f, α is constant. This is also the general structure of the proofs of the main results in [Ko, PY1, PY2, To1, To2, Bu2]. In contrast, in the context of the C^1 flow embedding problem studied by J. Palis [P], there are local obstructions, like the existence of transverse heteroclinic orbits, which prevent a diffeomorphism from being embedded in a flow.

a) The local strategy

In describing the local strategy, let us first make a very rough analogy with the symmetries of a Riemanniann manifold. If you want to prevent a Riemanniann metric from having global isometries, it is enough to perturb the metric in order to get a point which is locally isometric to no others, and which does not admit any local isometries. Hence the answer to the global problem is indeed given by a purely local perturbation, and the same happens for the flow embedding problem: if a diffeomorphism f does not agree in some place with the time-1 map of a flow, then neither does the global diffeomorphism.

The situation of the centralizer problem is quite different: the centralizer of f may be locally trivial at some place, but f may still admit a large centralizer supported in another place. Coming back to our analogy with isometries, our strategy consists in producing local perturbations covering a open and dense subset of orbits, avoiding non-trivial local symmetries on that set. This step consists in "individualizing" a dense collection of orbits, arranging that the behavior of the diffeomorphism in a neighborhood of one orbit is different from the behavior in a neighborhood of any other. Hence any commuting diffeomorphism must preserve each of these orbits.

This individualization of orbits happens whenever a property of unbounded distortion (UD) holds between certain orbits of f, a property which we describe precisely in the next section. In the first step of our proof we show that the (UD) property holds for a residual set of f. This gives local rigidity of the centralizer of a generic f, which gives the locally constant function α.

b) The global strategy

The global strategy goes like this. Assuming that we already proved the first step, we have that any diffeomorphism g commuting with the generic f is of the form $g = f^\alpha$ where α is locally constant and defined on a dense open subset. Furthermore, α is uniquely defined on the non-periodic points for f. Assuming that the periodic points of f are isolated, it is now enough to verify that the function α is bounded. This will be the case if the derivative Df^n takes large values on each orbit of f, *for each large* n: the bound on Dg would then forbid α from taking arbitrarily large values. Notice that this property is global in nature: we require large derivative of f^n on each orbit, for each large n.

Because it holds for every orbit (not just a dense set of orbits) and every large n, this large derivative (LD) property is not generic, although we prove that it is dense. This lack of genericity affects the structure of our proof: it is not possible to obtain both (UD) and (LD) properties just by intersecting two residual sets. There are two more steps in the argument. First, we show that *among the diffeomorphisms satisfying (UD)*, the property (LD) is dense. This allows us to conclude that the set of diffeomorphisms with trivial centralizer is C^1-dense, answering the first part of Question 1.

c) From dense to residual

At this point in the proof, we have obtained a C^1-dense set of diffeomorphisms with trivial centralizer. There is some subtlety in how we obtain a residual subset from a dense subset. An obvious way to do this would be to prove that the set of diffeomorphisms with trivial centralizer is a G_δ, i.e., a countable intersection of open sets. It is not however clear from the definition that this set is even a Borel set, let alone a G_δ. Instead we use a semicontinuity argument. To make this argument work, we must consider centralizers defined inside of a larger space of homeomorphisms, the bi-Lipschitz homeomorphisms. The compactness of the space of bi-Lipschitz homeomorphisms with bounded norm is used in a crucial way. The details are described below. The conclusion is that if a C^1-dense set of diffeomorphisms has trivial centralizer inside of the space of bi-Lipschitz homeomorphisms, then this property holds on a C^1 residual set.

1 Background on C^1-generic dynamics

The space $\mathrm{Diff}^1(M)$ is a Baire space in the C^1 topology. A *residual* subset of a Baire space is one that contains a countable intersection of open-dense sets; the Baire category theorem implies that a residual set is dense. We say that a property holds for the *C^1-generic diffeomorphism* if it holds on a residual subset of $\mathrm{Diff}^1(M)$.

For example, the Kupka-Smale Theorem asserts (in part) that for a C^1-generic diffeomorphism f, the periodic orbits of f are all hyperbolic. It is easy to verify that, furthermore, the C^1-generic diffeomorphism f has the following property: if x, y are periodic points of f with period m and n respectively, and if their orbits are distinct, then the set of eigenvalues of $Df^m(x)$ and of $Df^n(y)$ are disjoint. If this property holds, we say that the *periodic orbits of f have distinct eigenvalues.*

The nonwandering set $\Omega(f)$ is the set of all points x such that every neighborhood U of x meets some iterate of U:

$$U \cap \bigcup_{k>0} f^k(U) \neq \emptyset.$$

The elements of $\Omega(f)$ are called *nonwandering points.* By the canonical nature of its construction, the compact set $\Omega(f)$ is preserved by any homeomorphism g that commutes with f.

In [BC] it is shown that for a C^1-generic diffeomorphism f, each connected component O of the interior of $\Omega(f)$ is contained in the closure of the stable manifold of a periodic point $p \in O$. Conceptually, this result means that for C^1 generic f, the interior of $\Omega(f)$ and the *wandering set* $\mathcal{M} \setminus \Omega(f)$ share certain nonrecurrent features, as we now explain.

While points in the interior of $\Omega(f)$ all have nonwandering dynamics, if one instead considers the restriction of f to a stable manifold of a periodic orbit $W^s(p) \setminus \mathcal{O}(p)$, the dynamics are no longer recurrent; in the induced topology on the submanifold $W^s(p) \setminus \mathcal{O}(p)$, every point has a *wandering neighborhood* V whose iterates are all disjoint from V. Furthermore, the sufficiently large future iterates of such a wandering neighborhood are contained in a neighborhood of a periodic orbit. While the forward dynamics on the wandering set are not similarly "localized" as they are on a stable manifold, they still share this first feature: on the wandering set, every point has a wandering neighborhood (this time the neighborhood is in the topology on M).

Thus, the results in [BC] imply that for the C^1 generic f, we have the following picture: there is an f-invariant open and dense subset W of M, consisting of the union of the interior of $\Omega(f)$ and the complement of $\Omega(f)$, and densely in W the dynamics of f can be decomposed into components with "wandering strata." We exploit this fact in our local strategy, outlined in the next section.

2 Conditions for the local strategy: the unbounded distortion (UD) properties

In the local strategy, we control the dynamics of the C^1 generic f on the open and dense set $W = \mathrm{Int}(\Omega(f)) \cup (M \setminus \Omega(f))$. We describe here the main analytic properties we use to control these dynamics.

We say that diffeomorphism f satisfies the *unbounded distortion property on the wandering set (UD$^{M\setminus\Omega}$)* if there exists a dense subset $\mathcal{X} \subset M \setminus \Omega(f)$ such that, for any $K > 0$, any $x \in \mathcal{X}$ and any $y \in M \setminus \Omega(f)$ not in the orbit of x, there exists $n \geq 1$ such that:

$$|\log|\det Df^n(x)| - \log|\det Df^n(y)|| > K.$$

A diffeomorphism f satisfies the *unbounded distortion property on the stable manifolds(UD^s)* if for any hyperbolic periodic orbit $\mathcal{O}$, there exists a dense subset $\mathcal{X} \subset W^s(\mathcal{O})$ such that, for any $K > 0$, any $x \in \mathcal{X}$ and any $y \in W^s(\mathcal{O})$ not in the orbit of x, there exists $n \geq 1$ such that:

$$|\log|\mathrm{Det} Df^n_{|W^s(\mathcal{O})}(x)| - \log|\mathrm{Det} Df^n_{|W^s(\mathcal{O})}(y)|| > K.$$

Our first main perturbation result in [BCW2] is:

Theorem 2.1 (Unbounded distortion). *The diffeomorphisms in a residual subset of Diff$^1(M)$ satisfy the (UD$^{M\setminus\Omega}$) and the (UD^s) properties.*

A variation of an argument due to Togawa [To1, To2] detailed in [BCW1] shows the (UD^s) property holds for a C^1-generic diffeomorphism. To prove Theorem 2.1, we are thus left to prove that the (UD$^{M\setminus\Omega}$) property holds for a C^1-generic diffeomorphism. This property is significantly more difficult to establish C^1-generically than the (UD^s) property. The reason is that points on the stable manifold of a periodic point all have the same future dynamics, and these dynamics are "constant" for all large iterates: in a neighborhood of the periodic orbit, the dynamics of f are effectively linear. In the wandering set, by contrast, the orbits of distinct points can be completely unrelated after sufficiently many iterates.

Nonetheless, the proofs that the (UD$^{M\setminus\Omega}$) and (UD^s) properties are C^1 residual share some essential features, and both rely on the essentially non-recurrent aspects of the dynamics on both the wandering set and the stable manifolds.

3 Condition for the global strategy: the large derivative (LD) property

Here we describe the analytic condition on the C^1-generic f we use to extend the local conclusion on the centralizer of f to a global conclusion.

A diffeomorphism f satisfies the *large derivative property (LD) on a set X* if, for any $K > 0$, there exists $n(K) \geq 1$ such that for any $x \in X$ and $n \geq n(K)$, there exists $j \in \mathbb{Z}$ such that:

$$\sup\{\|Df^n(f^j(x))\|, \|Df^{-n}(f^{j+n}(x))\|\} > K.$$

Rephrased informally, the (LD) property on X means that the derivative Df^n "tends to ∞" *uniformly* on all orbits passing through X. We emphasize that the large derivative property is a property of the *orbits* of points in X, and if it holds for X, it also holds for all iterates of X.

The second main perturbation result in [BCW2] is:

Theorem 3.1 (Large derivative). *Let f be a diffeomorphism whose periodic orbits are hyperbolic. Then, there exists a diffeomorphism g arbitrarily close to f in $\mathrm{Diff}^1(M)$ such that the property (LD) is satisfied on $M \setminus \mathrm{Per}(f)$.*

Moreover,

- *f and g are conjugate via a homeomorphism Φ, i.e. $g = \Phi f \Phi^{-1}$;*
- *for any periodic orbit $\mathcal{O}$ of f, the derivatives of f on $\mathcal{O}$ and of g on $\Phi(\mathcal{O})$ are conjugate (in particular the periodic orbits of g are hyperbolic);*
- *if f satisfies the ($UD^{M\setminus\Omega}$) property, then so does g;*
- *if f satisfies the (UD^s) property, then so does g.*

As a consequence of Theorems 2.1 and 3.1 we obtain:

Corollary 3.2. *There exists a dense subset $\mathcal{D}$ of $\mathrm{Diff}^1(M)$ such that any $f \in \mathcal{D}$ satisfies the following properties:*

- *the periodic orbits are hyperbolic and have distinct eigenvalues;*
- *any component O of the interior of $\Omega(f)$ contains a periodic point whose stable manifold is dense in O;*
- *f has the ($UD^{M\setminus\Omega}$) and the (UD^s) properties;*
- *f has the (LD) property on $M \setminus \mathrm{Per}(g)$.*

The proofs of Theorems 2.1 and 3.1 are intricate, incorporating the topological towers developed in [BC] with novel perturbation techniques. We say more about the proofs in Section 6.

4 Checking that the centralizer is trivial

We now explain why properties (UD) and (LD) together imply that the centralizer is trivial.

Proposition 4.1. *Any diffeomorphism f in the C^1-dense subset $\mathcal{D} \subset \mathrm{Diff}^1(M)$ given by Corollary 3.2 has a trivial centralizer $Z^1(f)$.*

Proof of Proposition 4.1. Consider a diffeomorphism $f \in \mathcal{D}$. Let $g \in Z^1(f)$ be a diffeomorphism commuting with f, and let $K > 0$ be a Lipschitz constant for g and g^{-1}. Let $W = \mathrm{Int}(\Omega(f)) \cup (M \setminus \Omega(f))$ be the f-invariant, open and dense subset of M whose properties are discussed in Section 1.

Our first step is to use the "local hypotheses" ($\mathrm{UD}^{M\setminus\Omega}$) and ($\mathrm{UD}^s$) to construct a function $\alpha\colon W \to \mathbb{Z}$ that is constant on each connected component of W and satisfies $g = f^\alpha$. We then use the "global hypothesis" (LD) to show that α is bounded on W, and therefore extends to a constant function on M.

We first contruct α on the wandering set $M \setminus \Omega(f)$. The basic properties of Lipschitz functions and the relation $f^n g = g f^n$ imply that for any $x \in M$, and any $n \in \mathbb{Z}$, we have

$$|\log \det(Df^n(x)) - \log \det(Df^n(g(x)))| \leq 2d \log K, \tag{1}$$

where $d = \dim M$. On the other hand, f satisfies the $\mathrm{UD}^{M\setminus\Omega(f)}$ property, and hence there exists a dense subset $\mathcal{X} \subset M \setminus \Omega(f)$, each of whose points has unbounded distortion with respect to any point in the wandering set not on the same orbit. That is, for any $x \in \mathcal{X}$, and $y \in M \setminus \Omega(f)$ not on the orbit of x, we have:

$$\limsup_{n\to\infty} |\log|\det Df^n(x)| - \log|\det Df^n(y)|| = \infty.$$

Inequality (1) then implies that x and $y = g(x)$ lie on the same orbit, for all $x \in \mathcal{X}$, hence $g(x) = f^{\alpha(x)}(x)$. Using the continuity of g and the fact that the points in $M \setminus \Omega(f)$ admit wandering neighborhoods whose f-iterates are pairwise disjoint, we deduce that the map $\alpha\colon \mathcal{X} \to \mathbb{Z}$ is constant in the neighborhood of any point in $M \setminus \Omega(f)$. Hence the function α extends on $M \setminus \Omega(f)$ to a function that is constant on each connected component of $M \setminus \Omega(f)$. Furthermore, $g = f^\alpha$ on $M \setminus \Omega(f)$.

We now define the function α on the interior $\mathrm{Int}(\Omega(f))$ of the nonwandering set. Since the periodic orbits of $f \in \mathcal{D}$ have distinct eigenvalues and since g preserves the rate of convergence along the stable manifolds, the diffeomorphism g preserves each periodic orbit of f. Using the (UD^s) condition, one can extend the argument above for the wandering set to the stable manifolds of each periodic orbit (see also [BCW1, Lemma 1.2]). We obtain that for any periodic point p, the diffeomorphism g coincides with a power f^α on each connected component of $W^s(p) \setminus \{p\}$. For $f \in \mathcal{D}$, each connected component O of the interior of $\Omega(f)$ contains a periodic point p whose stable manifold is dense in O. It follows that g coincides with some power f^α of f on each connected component of the interior of $\Omega(f)$.

We have seen that there is a locally constant function $\alpha\colon W \to \mathbb{Z}$ such that $g = f^\alpha$ on the f invariant, open and dense subset $W \subset M$. We now turn to the global strategy. Notice that, since f and g commute, the function α is constant along the non-periodic orbits of f. Now $f \in \mathcal{D}$ satisfies the (LD) property. Consequently there exists $N > 0$ such that, for every non-periodic point x, and for every $n \geq N$ there is a point $y = f^i(x)$ such that either $\|Df^n(y)\| > K$ or $\|Df^{-n}(y)\| > K$. This implies that the function $|\alpha|$ is bounded by N: otherwise, α would be greater than N on the invariant open set W of M. This open set contains a non-periodic point x and an iterate $y = f^i(x)$ such that either $\|Df^\alpha(y)\| > K$ or $\|Df^{-\alpha}(y)\| > K$. This contradicts the fact that g and g^{-1} are K-Lipschitz.

We have just shown that $|\alpha|$ is bounded by some integer N. Let Per_{2N} be the set of periodic points of f whose period is less than $2N$, and for $i \in \{-N, \dots, N\}$ consider the set

$$P_i = \{x \in M \setminus \mathrm{Per}_{2N},\ g(x) = f^i(x)\}.$$

This is a closed invariant subset of $M \setminus \mathrm{Per}_{2N}$. What we proved above implies that $M \setminus \mathrm{Per}_{2N}$ is the union of the sets P_i, $|i| \leq N$. Moreover any two sets P_i, P_j with $i \neq j$ are disjoint since a point in $P_i \cap P_j$ would be $|i - j|$ periodic for f.

If $\dim(M) \geq 2$, since M is connected and Per_{2N} is finite, the set $M \setminus \mathrm{Per}_{2N}$ is connected. It follows that only one set P_i is non-empty, implying that $g = f^i$ on M. This concludes the proof in this case.

If $\dim(M) = 1$, one has to use that g is a diffeomorphism and is not only Lipschitz: this shows that on the two sides of a periodic orbit of f, the map g coincides with the same iterate of f. This proves again that only one set P_i is nonempty. □

5 From dense to residual: compactness and semicontinuity

The previous results show that the set of diffeomorphisms having a trivial centralizer is dense in $\mathrm{Diff}^1(M)$, but this is not enough to conclude the proof of the Main Theorem. Indeed the dense subset $\mathcal{D}$ in Corollary 3.2 is *not* a residual subset if $\dim(M) \geq 2$: in the appendix we exhibit a nonempty open set in which C^1-generic diffeomorphisms do not satisfy the (LD)-property.

Fix a metric structure on M. A homeomorphism $f : M \to M$ is *K-bi-Lipschitz* if both f and f^{-1} are Lipschitz, with Lipschitz norm bounded by K. A homeomorphism that is K-bi-Lipschitz for some K is called a *bi-Lipschitz homeomorphism*, or *lipeomorphism*. We denote by $\mathrm{Lip}^K(M)$ the set of K-bi-Lipschitz homeomorphisms of M and by $\mathrm{Lip}(M)$ the set of bi-Lipschitz homeomorphisms of M. The Arzèla-Ascoli theorem implies that $\mathrm{Lip}^K(M)$ is compact in the uniform (C^0) topology. Note that $\mathrm{Lip}(M) \supset \mathrm{Diff}^1(M)$. For $f \in \mathrm{Lip}(M)$, the set $Z^{\mathrm{Lip}}(f)$ is defined analogously to the C^r case:

$$Z^{\mathrm{Lip}}(f) := \{g \in \mathrm{Lip}(M) : fg = gf\}.$$

In dimension 1, the Main Theorem was a consequence of Togawa's work [To2]. In higher dimension, the Main Theorem is a direct corollary of:

Theorem 5.1. *If $\dim(M) \geq 2$, the set of diffeomorphisms f with trivial centralizer $Z^{Lip}(f)$ is residual in $\mathrm{Diff}^1(M)$.*

The proof of Theorem 5.1 has two parts.

Proposition 5.2. *If $\dim(M) \geq 2$, any diffeomorphism f in the C^1-dense subset $\mathcal{D} \subset \mathrm{Diff}^1(M)$ given by Corollary 3.2 has a trivial centralizer $Z^{Lip}(f)$.*

The proof of this proposition from Theorems A and B is the same as the proof of Proposition 4.1 (see also Lemma 1.2 in [BCW1]).

Proposition 5.3. *Consider the set $\mathcal{T}$ of diffeomorphisms $f \in \mathrm{Diff}^1(M)$ having a trivial centralizer $Z^{Lip}(f)$. Then, if $\mathcal{T}$ is dense in $\mathrm{Diff}^1(M)$, it is also residual.*

Remark: The proof of Proposition 5.3 also holds in the C^r topology $r \geq 2$ on any manifold M on which the C^r-generic diffeomorphism has at least one hyperbolic periodic orbit (for example, on the circle, or on manifolds of nonzero Euler characteristic). On the other hand, Theorem 5.1 is false for general manifolds in the C^2

topology, at least for the circle. In fact, a simple folklore argument (see the proof of Theorem B in [N]) implies that for *any* Kupka-Smale diffeomorphism $f \in \mathrm{Diff}^2(S^1)$, the set $Z^{\mathrm{Lip}}(f)$ is *infinite dimensional.* It would be interesting to find out what is true in higher dimensions.

Proof of Proposition 5.3. For any compact metric space X we denote by $\mathcal{K}(X)$ the set of non-empty compact subsets of X, endowed with the Hausdorff distance d_H. We use the following classical fact.

Proposition 5.4. *Let $\mathcal{B}$ be a Baire space, let X be a compact metric space, and let $h : \mathcal{B} \to \mathcal{K}(X)$ be an upper-semicontinuous function. Then the set of continuity points of h is a residual subset of $\mathcal{B}$.*

In other words, if h has the property that for all $b \in \mathcal{B}$,

$$b_n \to b \implies \limsup b_n = \bigcap_n \overline{\bigcup_{i>n} h(b_i)} \subseteq h(b),$$

then there is a residual set $\mathcal{R}_h \subset \mathcal{B}$ such that, for all $b \in \mathcal{R}_h$,

$$b_n \to b \implies \lim d_H(b_n, b) = 0.$$

To prove Proposition 5.3, we note that for a fixed $K > 0$, the set $Z^{\mathrm{Lip}}(f) \cap \mathrm{Lip}^K(M)$ is a closed subset (in the C^0 topology) of the compact metric space $\mathrm{Lip}^K(M)$. This is a simple consequence of the facts that $Z^{\mathrm{Lip}}(f)$ is defined by the relation $fgf^{-1}g^{-1} = \mathrm{id}$, and that composition and inversion are continuous. Thus there is well-defined map h_K from $\mathrm{Diff}^1(M)$ to $\mathcal{K}(\mathrm{Lip}^K(M))$, sending f to $h_K(f) = Z^{\mathrm{Lip}}(f) \cap \mathrm{Lip}^K(M)$. It is easy to see that h_K is upper-semicontinuous: if f_n converges to f in $\mathrm{Diff}^1(M)$ and $g_n \in h_K(f_n)$ converges uniformly to g then g belongs to $h_K(f)$.

Let $\mathcal{R}_K \subset \mathrm{Diff}^1(M)$ be the set of points of continuity of h_K; it is a residual subset of $\mathrm{Diff}^1(M)$, by Proposition 5.4. Let $\mathcal{R}_{Hyp} \subset \mathrm{Diff}^1(M)$ be the set of diffeomorphisms such that each $f \in \mathcal{R}_{Hyp}$ has at least one hyperbolic periodic orbit (the C^1 Closing Lemma [Pu] implies that $\mathcal{R}_{Hyp}$ is residual). Finally, let

$$\mathcal{R} = \mathcal{R}_{Hyp} \cap \bigcap_{K=1}^{\infty} \mathcal{R}_K.$$

Assuming that $\mathcal{T}$ is dense in $\mathrm{Diff}^1(M)$, we claim that the set $\mathcal{R}$ is contained in $\mathcal{T}$, implying that $\mathcal{T}$ is residual. To see this, fix $f \in \mathcal{R}$, and let $f_n \to f$ be a sequence of diffeomorphisms in $\mathcal{T}$ converging to f in the C^1 topology. Let $g \in Z^{\mathrm{Lip}}(M)$ be a K-bi-Lipschitz homeomorphism satisfying $fg = gf$. Since h_K is continuous at f, there is a sequence $g_n \in Z^{\mathrm{Lip}}(f_n)$ of K-bi-Lipschitz homeomorphisms with $g_n \to g$ in the C^0 topology. The fact that $f_n \in \mathcal{T}$ implies that the centralizer $Z^{\mathrm{Lip}}(f_n)$ is trivial, so there exist integers m_n such that $g_n = f^{m_n}$.

If the sequence (m_n) is bounded, then passing to a subsequence, we obtain that $g = f^m$, for some integer m. If the sequence (m_n) is not bounded, then we obtain a contradiction as follows. Let x be a hyperbolic periodic point of f, of period p. For n large, the map f_n has a periodic orbit x_n of period p, and the derivatives $Df_n^p(x_n)$

tend to the derivative $Df^p(x)$. But then $|\log \|Df_n^{m_n}\||$ tends to infinity as $n \to \infty$. This contradicts the fact that the diffeomorphisms $f_n^{m_n} = g_n$ and $f_n^{-m_n} = g_n^{-1}$ are both K-Lipschitz, concluding the proof. $\square$

6 Conclusion

To complete the proof of Theorem A, it remains to prove Theorems 2.1 and 3.1. Both of these results split in two parts. The first part is a local perturbation tool, which changes the derivative of f in a very small neighborhood of a point, the neighborhood being chosen so small that f looks like a linear map on many iterates of this neighborhood. In the second part, we perform perturbations provided by the first part at different places in such a way that the derivative of every (wandering or non-periodic) orbit will be changed in the desirable way. For the (UD) property on the wandering set, the existence of open sets disjoint from all its iterates are very helpful, allowing us to spread the perturbation out over time. For the (LD) property, we need to control every non-periodic orbit. The existence of *topological towers* with very large return time, constructed in [BC], are the main tool, allowing us again to spread the perturbations out over a long time interval.

Acknowledgement. We thank Andres Navas for calling our attention to the Lipschitz centralizer and for pointing out to us Ghys's paper [G]. This work was supported by NSF grants DMS-0401326 and DMS-0701018.

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